

**Lecture Notes**  
**Methods of Mathematical Physics**  
**MATH 535**

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# Chapter 1

## Normed Vector Spaces

### 1.1 Vector Spaces

**Definition 1.1.1** (Vector Space.) *A vector space over a field  $\mathbb{F}$  of real or complex numbers is a nonempty set  $E$  with two operations:*

1. *Addition,  $+$  :  $E \times E \rightarrow E$ , and*
2. *Multiplication by scalars,  $\cdot$  :  $\mathbb{F} \times E \rightarrow E$ ,*

*satisfying the following conditions:  $\forall x, y, z \in E$  and  $\forall \alpha, \beta \in \mathbb{F}$*

1.  $x + y = y + x$ ,
2.  $(x + y) + z = x + (y + z)$ ,
3.  $\exists 0 \in E$  such that  $x + 0 = x$ ,
4.  $\forall x \in E$ ,  $\exists(-x) \in E$  such that  $x + (-x) = 0$ ,
5.  $\alpha(\beta x) = (\alpha\beta)x$ ,
6.  $(\alpha + \beta)x = \alpha x + \beta x$ ,
7.  $\alpha(x + y) = \alpha x + \alpha y$ ,
8.  $1 \cdot x = x$ .

- **Remarks.**

- A vector space over  $\mathbb{R}$  is a **real vector space**.
- A vector space over  $\mathbb{C}$  is a **complex vector space**.
- Elements of  $\mathbb{F}$  are **scalars**.
- Elements of  $E$  are **vectors**.
- The element  $0$  is **zero vector**.
- The zero vector is unique.
- For any  $x, y \in E$  there is a unique  $z \in E$  such that  $x + z = y$ . Such a vector is denoted by  $z = y - x$ .

Proof: Exercise.

- **Properties.**

- Let  $\lambda \in \mathbb{F}$ ,  $x \in E$ , and  $\lambda x = 0$ .

If  $\lambda \neq 0$ , then  $x = 0$ . If  $x \neq 0$ , then  $\lambda = 0$ .

Proof: Exercise.

- For any  $x \in \mathbb{F}$ ,

$$0x = 0 \text{ and } (-1)x = -x.$$

- **Example.**  $\mathbb{F}^n$  ( $\mathbb{R}^n$  and  $\mathbb{C}^n$ ).
- **Example (Function Spaces).** The vector space  $F(X, E)$  of all functions  $f : X \rightarrow E$  from a set  $X$  into a vector space  $E$ .
- $\mathbb{F}^n$  can be defined as  $F(\mathbb{Z}_n, \mathbb{F})$ .
- A **vector subspace** (or a subspace) of a vector space  $E$  is a subset  $E_1 \subseteq E$  of  $E$  which is itself a vector space.
- A **proper vector subspace** of a vector space  $E$  is a proper subset  $E_1 \subset E$  of  $E$  which is itself a vector space.

- **Example.** Let  $\Omega \subset \mathbb{R}^n$  be an open subset in  $\mathbb{R}^n$ . The space of all functions from  $\Omega$  into  $\mathbb{C}$  is a vector space. The following are subspaces of this vector space:

1.  $C(\Omega)$  (continuous functions)
2.  $C^k(\Omega)$  (functions with continuous partial derivatives of order  $k$ )
3.  $C^\infty(\Omega)$  (smooth functions)
4.  $P(\Omega)$  (polynomials)

- **Example (Sequence Spaces).** Let  $\mathbb{N}$  be the set of positive integers. The space  $F(\mathbb{N}, \mathbb{F})$  of all functions from  $\mathbb{N}$  into  $\mathbb{F}$  is the vector space of sequences of scalars. The following are subspaces of this vector space:

1. bounded sequences,
2. convergent sequences,

- **Example ( $l^p$ -Spaces).** Let  $p \geq 1$ . The space of infinite sequences of complex numbers,  $(z_n)$ , such that

$$\sum_{n=1}^{\infty} |z_n|^p < \infty$$

**Proof:** Use Minkowski inequality.

- Notation: Let

$$\|x\|_p = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$$

and let  $xy$  denote the sequence  $(x_n y_n)$ .

**Theorem 1.1.1 Hölder's Inequality.** *Let  $p > 1$ ,  $q > 1$  and*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

*Let  $(x_n)$  and  $(y_n)$  be any two sequences of complex numbers. Then*

$$\|xy\|_1 \leq \|x\|_p \|y\|_q .$$

**Proof:**

1. Let  $\eta = \xi^{p-1}$  be a curve in the  $\eta - \xi$ -plane for  $\eta, \xi > 0$ .

2. Let  $a, b > 0$ .

3. Let

$$S_1 = \int_0^a d\xi \xi^{p-1} = \frac{a^p}{p}$$

be the area between the curve and the  $\xi$ -axis for  $0 \leq \xi \leq a$ .

4. Let

$$S_2 = \int_0^b d\eta \eta^{q-1} = \frac{b^q}{q}$$

be the area between the curve and the  $\eta$ -axis for  $0 \leq \eta \leq b$ .

5. We have

$$ab \leq S_1 + S_2.$$

6. Thus

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

7. Let

$$a = \frac{|x_j|}{\|x\|_p}, \quad b = \frac{|y_j|}{\|y\|_q},$$

8. Then

$$\frac{|x_j|}{\|x\|_p} \frac{|y_j|}{\|y\|_q} \leq \frac{1}{p} \frac{|x_j|^p}{\|x\|_p^p} + \frac{1}{q} \frac{|y_j|^q}{\|y\|_q^q}$$

9. Finally

$$\frac{\|xy\|_1}{\|x\|_p \|y\|_q} = \frac{\sum_{j=1}^n |x_j y_j|}{\|x\|_p \|y\|_q} \leq \frac{1}{p} + \frac{1}{q} = 1.$$

■

**Theorem 1.1.2 Minkowski's Inequality** Let  $p \geq 1$ . Let  $(x_n)$  and  $(y_n)$  be any two sequences of complex numbers. Then

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

**Proof:**

1. If  $p = 1$  it is true by triangle inequality.

2. If  $p > 1$ , then

$$(\|x+y\|_p)^p = \sum_{n=1}^{\infty} |x_n+y_n|^p \leq \sum_{n=1}^{\infty} |x_n| |x_n+y_n|^{p-1} + \sum_{n=1}^{\infty} |y_n| |x_n+y_n|^{p-1}$$

3. Further, by Hölder inequality

$$(\|x+y\|_p)^p \leq (\|x\|_p + \|y\|_p) \left( \sum_{n=1}^{\infty} |x_n+y_n|^{q(p-1)} \right)^{\frac{1}{q}}$$

4. We have  $q(p-1) = p$ . Thus

$$(\|x+y\|_p)^p \leq (\|x\|_p + \|y\|_p) (\|x+y\|_p)^{\frac{p}{q}}$$

This gives the inequality. ■

- **Examples (Cartesian Product of Vector Spaces).** Let  $\{E_j\}_{j=1}^n$  be a collection of vector spaces over a field  $\mathbb{F}$ . The **Cartesian product** (or product) of vector spaces  $E_j$  is the space

$$\begin{aligned} E &= E_1 \times \cdots \times E_n \\ &= \{(x_1, \dots, x_n) \mid x_j \in E_j, 1 \leq j \leq n\}. \end{aligned}$$

### 1.1.1 Homework

- Exercises: [1,2,4(c),6].

## 1.2 Linear Independence, Basis, Dimension

**Definition 1.2.1 Linear Combination.** Let  $E$  be a vector space over a field  $\mathbb{F}$  and  $\mathcal{A} = \{x_j\}_{j=1}^k$  be a finite collection of vectors from  $E$ . The **linear combination** of these vectors is a vector

$$x = \alpha_1 x_1 + \cdots + \alpha_k x_k$$

with scalars  $\alpha_j \in \mathbb{F}$ ,  $1 \leq j \leq k$ .

- **Example.**

**Definition 1.2.2 Linear Independence.** A finite collection of vectors  $\mathcal{A} = \{x_j\}_{j=1}^k$  is **linearly independent** if

$$\alpha_1 x_1 + \cdots + \alpha_k x_k = 0$$

implies  $\alpha_j = 0$ ,  $1 \leq j \leq k$ .

An infinite collection  $\mathcal{A}$  of vectors is linearly independent if every finite subcollection of  $\mathcal{A}$  is linearly independent.

A collection  $\mathcal{A}$  of vectors is **linearly dependent** if it is not linearly independent.

- A collection  $\mathcal{A}$  of vectors is linearly independent if no vector of  $\mathcal{A}$  is a linear combination of a finite number of vectors from  $\mathcal{A}$ .
- **Example.**  $\mathbb{R}^n$
- Linear independence depends of the field of scalars. Compare  $\mathbb{C}_{\mathbb{R}}$  and  $\mathbb{C}_{\mathbb{C}}$ .
- Let  $\mathcal{A} \subset E$  be a subset of a vector space  $E$ . The **span of  $\mathcal{A}$** ,  $\text{span } \mathcal{A}$ , is the set of all finite linear combinations of vectors from the subset  $\mathcal{A}$ .
- Span of a subset  $\mathcal{A}$  of a vector space  $E$  is a vector subspace of the vector space  $E$  **spanned by  $\mathcal{A}$** .
- Span of  $\mathcal{A}$  is the smallest subspace of the vector space  $E$  containing  $\mathcal{A}$ .
- **Example.**  $\mathbb{R}^3$

- |   |
|---|
| <b>Definition 1.2.3</b> <b>Basis.</b> <i>A collection <math>\mathcal{B}</math> of vectors of a vector space <math>E</math> is a <b>basis</b> of <math>E</math> if <math>\mathcal{B}</math> is linearly independent and <math>\text{span } \mathcal{B} = E</math>.</i> |
|---|
- A vector space  $E$  is **finite-dimensional** if it has a finite basis.
- A vector space  $E$  is **infinite-dimensional** if it does not have a finite basis.
- If the vector space  $E$  is finite-dimensional, then the number of vectors in any basis is the same. Proof: Exercise.
- The **dimension**,  $\dim E$ , of a finite-dimensional vector space  $E$  is the number of vectors in a basis.
- **Example.**  $\mathbb{R}^n$ ,  $C(\Omega)$ ,  $\mathbb{C}_{\mathbb{R}}^n$ ,  $\mathbb{C}_{\mathbb{C}}^n$
- The dimension of the vector space depends on the field of scalars.

### 1.2.1 Homework

- Exercises: [11,12,13,14]

### 1.3 Normed Spaces

**Definition 1.3.1 Norm.** Let  $E$  be a vector space over a field  $\mathbb{F}$ . A **norm** is a function  $\| \cdot \|: E \rightarrow \mathbb{R}$  that assigns to every vector  $x$  a real number  $\| x \|$  and satisfies the following conditions:

- 1.  $\| x \| \geq 0, \quad \forall x \in E;$
  2.  $\| x \| = 0$  if and only if  $x = 0;$
  3.  $\| \lambda x \| = |\lambda| \| x \|, \quad \forall x \in E, \lambda \in \mathbb{F};$
  4.  $\| x + y \| \leq \| x \| + \| y \|, \quad \forall x, y \in E.$

• **Examples.**

1. Euclidean Norm in  $\mathbb{R}^n$ .

$$\| x \| = \sqrt{x_1^2 + \cdots + x_n^2}$$

- 2.

$$\| x \| = |x_1| + \cdots + |x_n|$$

- 3.

$$\| x \| = \max_{1 \leq i \leq n} |x_i|$$

4. Norm in  $\mathbb{C}^n$

$$\| x \| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$$

5. **Norm of Uniform Convergence.** Let  $\Omega \subset \mathbb{R}^n$  be a closed bounded subset of  $\mathbb{R}^n$  and  $C(\Omega)$  be the space of continuous functions on  $\Omega$ . Norm in  $C(\Omega)$

$$\| f \|_\infty = \max_{x \in \Omega} |f(x)|$$

6. Norm in  $l^p$

$$\| x \|_p = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$$

7. Find the limit  $p \rightarrow \infty$ .

- **Definition 1.3.2 Normed Space.** *A vector space with a norm is called a **normed space**.*

- One can define different norms on the same vector space.
- A normed space is a pair  $(E, \|\cdot\|)$ , where  $E$  is a vector space and  $\|\cdot\|$  is a norm on  $E$ .
- Some vector spaces have standard norms.
- A vector subspace of a normed space is a normed space with the same norm.
- The norm can be used to define convergence.

- **Definition 1.3.3 Convergence in a Normed Space.** *Let  $(E, \|\cdot\|)$  be a normed space and  $(x_n)$  be a sequence of vectors in  $E$ . The sequence  $(x_n)$  converges to  $x \in E$  if for every  $\varepsilon > 0$  there exists a positive integer  $M \in \mathbb{N}$  such that for every  $n \geq M$  we have*

$$\|x_n - x\| < \varepsilon.$$

*Then we write  $x = \lim_{n \rightarrow \infty} x_n$  or  $x_n \rightarrow x$ .*

- $x_n \rightarrow x$  simply means that  $\|x_n - x\| \rightarrow 0$  in  $\mathbb{R}$ .
- **Properties of convergence in normed space.**
- A convergent sequence has a unique limit.
- If  $x_n \rightarrow x$  and  $\lambda_n \rightarrow \lambda$ , then  $\lambda_n x_n \rightarrow \lambda x$ .
- If  $x_n \rightarrow x$  and  $y_n \rightarrow y$ , then  $x_n + y_n \rightarrow x + y$ .
- Not every convergence in a vector space can be defined by a norm.
- **Example (Uniform Convergence).** Let  $C(\Omega)$  be the space of all continuous functions on a closed bounded set  $\Omega \subset \mathbb{R}^n$  and let  $(f_n) \in C(\Omega)$  be a sequence of functions in  $C(\Omega)$ . The sequence  $(f_n)$  **converges uniformly** to  $f$  if for every  $\varepsilon > 0$  there exists a positive integer  $M = M(\varepsilon) \in \mathbb{N}$  such that for all  $x \in \Omega$  and for all  $n \geq M$  we have

$$|f(x) - f_n(x)| < \varepsilon.$$

The norm of uniform convergence defines the uniform convergence, i.e. the sequence  $(f_n)$  converges uniformly to  $f$  if and only if

$$\|f_n - f\|_\infty = \max_{x \in \Omega} |f_n(x) - f(x)| \rightarrow 0.$$

- **Example (Pointwise Convergence).** Let  $C([0, 1])$  be the space of continuous functions on the interval  $[0, 1]$  and let  $(f_n)$  be a sequence of functions in  $C([0, 1])$ . The sequence  $(f_n)$  **converges pointwise** to  $f$  if for all  $x \in [0, 1]$  and for every  $\varepsilon > 0$  there exists a positive integer  $M = M(\varepsilon, x) \in \mathbb{N}$  such that for all  $n \geq M$  we have

$$|f(x) - f_n(x)| < \varepsilon.$$

The pointwise convergence simply means that for every  $x \in [0, 1]$  the sequence  $(f_n(x))$  converges to  $f(x)$ , i.e.

$$f_n(x) \rightarrow f(x) \quad \text{or} \quad |f_n(x) - f(x)| \rightarrow 0.$$

- There is no norm on  $C([0, 1])$  which defines the pointwise convergence.  
**Proof:** (by contradiction). Construct a sequence  $(f_n)$  of functions such that

1.  $\|f_n\| = 1$  for all  $n \in \mathbb{N}$  and
2.  $f_n(x) \rightarrow 0$  as  $n \rightarrow \infty \forall x \in [0, 1]$ .

**Definition 1.3.4 Equivalence of Norms.** *Two norms on the same vector space  $E$  are equivalent if they define the same convergence.*

- *That is, the norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent if for any sequence  $(x_n)$  in  $E$  and  $x \in E$ ,*

$$\|x_n - x\|_1 \rightarrow 0 \text{ if and only if } \|x_n - x\|_2 \rightarrow 0.$$

- **Example.**  $\mathbb{R}^2$ .

**Theorem 1.3.1** Two norms  $\| \cdot \|_1$  and  $\| \cdot \|_2$  in a vector space  $E$  are equivalent if and only if there exist positive real numbers  $\alpha, \beta \in \mathbb{R}_+$  such that

$$\alpha \| x \|_1 \leq \| x \|_2 \leq \beta \| x \|_1 \quad \text{for all } x \in E.$$

**Proof:**

1. This condition implies the equivalence of norms (obvious).
2. Let the norms be equivalent.
3. Assume that there is no  $\alpha$  such that

$$\alpha \| x \|_1 \leq \| x \|_2 \quad \text{for all } x \in E.$$

4. Then there exists a sequence  $(x_n)$  such that

$$\frac{1}{n} \| x_n \|_1 > \| x_n \|_2.$$

5. Let

$$y_n = \frac{1}{\sqrt{n}} \frac{x_n}{\| x_n \|_2}.$$

6. Then

$$\| y_n \|_2 = \frac{1}{\sqrt{n}} \quad \text{and} \quad \| y_n \|_1 \geq \sqrt{n}.$$

7. Contradiction. ■

- Every normed space  $(E, \| \cdot \|)$  is a metric space  $(E, d)$  with the metric

$$d(x, y) = \| x - y \|.$$

**Definition 1.3.5** A metric space  $(E, d)$  is a set  $E$  with a metric  $d$ . A metric  $d$  on a set  $E$  is a function  $d : E \times E \rightarrow \mathbb{R}$  satisfying the following axioms:  $\forall x, y, z \in E$

1.  $d(x, y) \geq 0$ ,
2.  $d(x, y) = 0$  if and only if  $x = y$ ,
3.  $d(x, z) \leq d(x, y) + d(y, z)$ .

**Definition 1.3.6** A **topological space**  $(E, \mathcal{T})$  is a set  $E$  with a **topology**  $\mathcal{T}$ . A topology  $\mathcal{T}$  on a set  $E$  is a collection  $\mathcal{T}$  of subsets of  $E$  (called **open sets**) that contains  $E$  and  $\emptyset$  and is closed under union and finite intersection.

Topology satisfies the following axioms:

- 1.  $E, \emptyset \in \mathcal{T}$ ,
  2.  $\cup_{\alpha \in A} O_\alpha \in \mathcal{T}$  for any subcollection of open sets  $\{O_\alpha\}_{\alpha \in A}$ ,
  3.  $\cap_{k=1}^n O_k \in \mathcal{T}$  for a finite subcollection of open sets  $\{O_k\}_{k=1}^n$ .
- The convergence defined by the norm  $\|\cdot\|$  is the same as the convergence defined by the metric  $d(x, y) = \|x - y\|$ .
- The metric defines a topology in  $E$  (open and closed sets).
- The basic topological notions can be defined without a metric.

**Definition 1.3.7 Open Balls, Closed Balls, Spheres.** Let  $E$  be a normed space,  $x \in E$  and  $r \in \mathbb{R}_+$  a positive real number. We define the following sets:

**Open ball**

$$B(x, r) = \{y \in E \mid \|x - y\| < r\}$$

- **Closed ball**

$$\bar{B}(x, r) = \{y \in E \mid \|x - y\| \leq r\}$$

**Sphere**

$$S(x, r) = \{y \in E \mid \|x - y\| = r\}$$

Here  $x$  is the **center** and  $r$  is the **radius**.

- **Examples.**  $\mathbb{R}^2$ ,  $C([0, 1])$ ,  $\|\cdot\|_\infty$ .

**Definition 1.3.8 Open and Closed Sets.** A subset  $S \subseteq E$  of a normed space  $E$  is **open** if for every  $x \in S$  there exist  $\varepsilon > 0$  such that  $B(x, \varepsilon) \subseteq S$ .

- A subset  $S \subseteq E$  of a normed space  $E$  is **closed** if its complement  $E \setminus S$  is open.

- **Example.** Let  $\Omega$  be a closed bounded set in  $\mathbb{R}^n$  and  $C(\Omega)$  be the space of continuous functions on  $\Omega$  with the norm of uniform convergence  $\|\cdot\|_\infty$ . Let  $f \in C(\Omega)$  such that  $f(x) > 0$  for all  $x \in \Omega$ . The set

$$\{g \in C(\Omega) \mid g(x) < f(x), \quad \forall x \in \Omega\}$$

is open  $C(\Omega)$ , and the sets

$$\{g \in C(\Omega) \mid g(x) \leq f(x), \quad \forall x \in \Omega\}$$

and

$$\{g \in C(\Omega) \mid g(x_0) = \lambda \quad (x_0 \in \Omega, \lambda \in \mathbb{C})\}$$

are closed in  $C(\Omega)$ .

**Theorem 1.3.2** 1. The union of any number of open sets is open.

2. The intersection of a finite number of open sets is open.

3. The union of a finite number of closed sets is closed.

4. The intersection of any number of closed sets is closed.

5. The empty set and the whole space are both open and closed.

**Proof:** Exercise.

**Theorem 1.3.3** A subset  $S$  of a normed space  $E$  is closed if and only if every sequence of elements of  $S$  convergent in  $E$  has its limit in  $S$ .

That is, if  $x_n \in S$  and  $x_n \rightarrow x$ , then  $x \in S$ .

**Proof:**

1. (I) Suppose  $S$  is closed,  $x_n \in S$ ,  $x_n \rightarrow x$  and  $x \notin S$ .
2.  $\exists \varepsilon > 0$  such that  $B(x, \varepsilon) \subseteq E \setminus S$ .
3. But  $\|x_n - x\| \rightarrow 0$  (contradiction).
4. (II) Suppose that for any  $x_n \in S$  if  $x_n \rightarrow x$ , then  $x \in S$  but  $S$  is not closed.
5. Then  $E \setminus S$  is not open. So  $\exists x \in E \setminus S$  such that every ball  $B(x, \varepsilon)$ ,  $\forall \varepsilon > 0$ , contains elements of  $S$ .

6.  $\exists x_n \in S$  such that  $x_n \in B(x, \frac{1}{n})$ .
7. Then  $x_n \rightarrow x \in S$ , which contradicts  $x \in E \setminus S$ .

■

**Definition 1.3.9 Closure.** Let  $S$  be a subset of a normed space  $E$ . The **closure** of  $S$  (denoted by  $\bar{S}$  or  $\text{cl } S$ ) is the intersection of all closed sets containing  $S$ .

- The closure of a set is a closed set.
- The closure of a set is the smallest closed set which contains  $S$ .

**Theorem 1.3.4** Let  $S$  be a subset of a normed space  $E$ . The closure of  $S$  is the set of limits of all convergent sequences of elements of  $S$ .

That is

$$\text{cl } S = \{x \in E \mid \exists x_n \in S \text{ such that } x_n \rightarrow x\}$$

**Proof:** Exercise.

- **Examples (Weierstrass Approximation Theorem).** The closure of the set of all polynomials on  $[a, b]$  is the whole space  $C([a, b])$ .

**Definition 1.3.10 Dense Subsets.** A subset  $S$  of a normed space  $E$  is **dense** in  $E$  if  $\text{cl } S = E$ .

- **Examples.**

1. The set of all polynomials on  $[a, b]$  is dense in  $C([a, b])$ .
2. The set of all sequences of complex numbers which have only a finite number of nonzero terms is dense in  $l^p$  for any  $p \geq 1$ .

**Theorem 1.3.5** Let  $S$  be a subset of a normed space  $E$ . The following conditions are equivalent:

- 1.  $S$  is dense in  $E$ .
  2. For every  $x \in E$  there exist  $x_n \in S$  such that  $x_n \rightarrow x$ .
  3. Every nonempty open subset of  $E$  contains an element of  $S$ .

**Proof:** Exercise.

**Definition 1.3.11 Compact Sets.** A subset  $S$  of a normed space  $E$  is **compact** in  $E$  if every sequence in  $S$  contains a convergent subsequence whose limit belongs to  $S$ .

- **Examples.**  $\mathbb{R}^n, \mathbb{C}^n$

**Theorem 1.3.6** Compact sets are closed and bounded.

*Proof:*

1. (I) Let  $S$  be compact,  $x_n \in S$  and  $x_n \rightarrow x$ .
2.  $\exists$  a subsequence  $x_{n_k}$  which converges to some  $y \in S$ .
3. Since  $x_{n_k} \rightarrow x$ , then  $x = y \in S$ .
4. So,  $S$  is closed.
5. (II) Suppose  $S$  is not bounded.
6.  $\exists$  a sequence  $x_n \in S$  such that  $\|x_n\| > n$  for all  $n \in \mathbb{N}$ .
7.  $(x_n)$  does not contain a convergent subsequence.
8. So,  $S$  is not compact. ■

- **Example (Noncompact Closed and Bounded Set.)** Let  $C([0, 1])$  be the space of continuous functions on  $[0, 1]$ . The closed unit ball  $\bar{B}(0, 1)$  is a closed and bounded set. Let  $x_n(t) = t^n \in \bar{B}(0, 1)$  be the sequence of functions of unit norm. Then  $(x_n)$  does not have a convergent subsequence. So, the closed unit ball  $\bar{B}(0, 1)$  is not compact.

### 1.3.1 Finite Dimensional Normed Spaces

**Lemma 1.3.1** Let  $X \subset E$  be a finite dimensional vector subspace of a normed space  $(E, \|\cdot\|)$ . Let  $n = \dim X$  and  $\{e_i\}_{i=1}^n$  be a basis in  $X$ . Define a norm  $\|\cdot\|_1$  on  $X$  as follows. For any  $x = \sum_{i=1}^n \alpha_i e_i \in X$ , let

$$\|x\|_1 = \sum_{i=1}^n |\alpha_i|.$$

Then there is a real number  $c \in \mathbb{R}$  such that for any  $x \in X$  we have

$$\|x\| \geq c \|x\|_1.$$

**Proof:**

1. This is equivalent to the following: for any  $\|x\|_1 = 1$  there is a  $c \in \mathbb{R}$  such that

$$\|x\| \geq c.$$

2. Suppose this is false. Then there exists a sequence  $(y_m) \in X$  such that

$$\|y_m\|_1 = 1, \quad \text{and} \quad (y_m) \rightarrow 0.$$

3. Let

$$y_m = \sum_{i=1}^n \alpha_{i,(m)} e_i.$$

4. Since  $\|y_m\|_1 = 1$ , then  $\forall i, m, |\alpha_{i,(m)}| \leq 1$ .  
 5. For each  $i$ , the sequence  $(\alpha_{i,(m)})_{m \in \mathbb{N}}$  is bounded.  
 6. Then, the sequence  $(\alpha_{1,(m)})$  has a convergent subsequence.  
 7. By relabeling the sequences, there exists a sequence  $(y_m) \in X$  such that

$$\|y_m\|_1 = 1, \quad (y_m) \rightarrow 0, \quad \text{and} \quad (\alpha_{1,(m)}) \rightarrow \alpha_1.$$

8. Repeating this procedure  $n$  times we get a sequence  $(y_m) \in X$  such that

$$\|y_m\|_1 = 1, \quad (y_m) \rightarrow 0, \quad \text{and} \quad (\alpha_{i,(m)}) \rightarrow \alpha_i.$$

9. Therefore,

$$\|y_m\|_1 = 1, \quad (y_m) \rightarrow 0, \quad y_m \rightarrow y = \sum_{i=1}^n \alpha_i e_i.$$

10. Since

$$\|y\|_1 = 1,$$

then  $y \neq 0$ .

11. We also have

$$\|y_m\| \rightarrow \|y\| \neq 0,$$

which contradicts  $y_m \rightarrow 0$ .

■

- **Theorem 1.3.7** *Any two norms on a finite-dimensional vector space are equivalent.*

**Proof:**

1. Let  $\{e_i\}_{i=1}^n$  be a basis in a finite-dimensional normed space  $E$ .
2. Let  $x = \sum_{i=1}^n \alpha_i e_i \in E$ .
3. Then for a norm  $\|\cdot\|_1$ ,  $\exists c \in \mathbb{R}$  such that

$$\|x\|_1 \geq c \sum_{i=1}^n |\alpha_i|$$

4. For another norm  $\|\cdot\|_2$  by triangle inequality

$$\|x\|_2 \leq \sum_{i=1}^n |\alpha_i| \|e_i\|_2 \leq k \sum_{i=1}^n |\alpha_i|,$$

where  $k = \max_{1 \leq i \leq n} \|e_i\|_2$ .

5. Thus

$$\alpha \|x\|_2 \leq \|x\|_1,$$

where  $\alpha = c/k$ .

6. Similarly, the second inequality.

■

- **Theorem 1.3.8 (Completeness.)** *Every finite dimensional subspace of a normed space is complete. Every finite dimensional normed space is complete.*

- **Theorem 1.3.9 (Closedness.)** *Every finite dimensional subspace of a normed space is closed.*

### 1.3.2 Homework

- Exercises: [18,21,22,23,24,25,26]

## 1.4 Banach Spaces

**Definition 1.4.1 Cauchy Sequence.** A sequence of vectors  $(x_n)$  in a normed space is a **Cauchy sequence** if for every  $\varepsilon > 0$  there exists  $M \in \mathbb{N}$  such that for all  $n, m \geq M$ ,

$$\|x_m - x_n\| < \varepsilon.$$

**Theorem 1.4.1** The following statements are equivalent:

1.  $(x_n)$  is a Cauchy sequence.
2. Let  $(p_n)$  and  $(q_n)$  be increasing sequences of positive integers. Then

$$\|x_{p_n} - x_{q_n}\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

3. Let  $(p_n)$  be an increasing sequence of positive integers. Then

$$\|x_{p_{n+1}} - x_{p_n}\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

**Proof:**

1. We have (a) implies (b) and (b) implies (c). So, we have to prove (c) implies (a).
2. By contradiction. Suppose that (c) holds but  $(x_n)$  is not Cauchy.
3. Get a contradiction to (c).

■

- Every convergent sequence is Cauchy. Proof: Exercise.
- Not every Cauchy sequence in a normed space  $E$  converges to a vector in  $E$ .
- **Example.** Incompleteness.

**Lemma 1.4.1** Let  $(x_n)$  be a Cauchy sequence of vectors in a normed space. Then the sequence  $(\|x_n\|)$  or real numbers converges.

**Proof:**

1. We have  $|\|x_m\| - \|x_n\|| \leq \|x_m - x_n\|$ . Thus  $(\|x_n\|)$  is Cauchy.

■

- Every Cauchy sequence is bounded.

Proof: Exercise.

- **Definition 1.4.2 Banach Space.** *A normed space  $E$  is complete (or Banach space) if every Cauchy sequence in  $E$  converges to an element in  $E$ .*

- **Examples.**

1.  $\mathbb{R}^n$  and  $\mathbb{C}^n$  (with any norm) are complete.
2. The space  $C(\Omega)$  of continuous functions on a closed bounded subset  $\Omega \subset \mathbb{R}^n$  with the norm of uniform convergence  $\|\cdot\|_\infty$  is complete.
3. The space  $l^p$  of complex sequences with the norm  $\|\cdot\|_p$  is complete.

- **Theorem 1.4.2 Completeness of  $l^2$ .** *The space of complex sequences with the norm  $\|\cdot\|_2$  is complete.*

**Proof:**

1. Let  $(a_n) \in l^2$  be Cauchy. Let  $a_n = (\alpha_1^{(n)}, \alpha_2^{(n)}, \dots)$ .
2. Let  $\varepsilon > 0$ . Then  $\exists M$  such that for  $n, m \geq M$ ,

$$\|a_n - a_m\|^2 = \sum_{k=1}^{\infty} |\alpha_k^{(n)} - \alpha_k^{(m)}|^2 < \varepsilon^2.$$

3. So, for any  $k \in \mathbb{N}$ , for  $n, m \geq M$ ,

$$|\alpha_k^{(n)} - \alpha_k^{(m)}| < \varepsilon$$

4. So,  $(\alpha_k^{(n)})$  is Cauchy sequence of complex numbers for each  $k \in \mathbb{N}$ .

5. Let  $\alpha_k = \lim_{n \rightarrow \infty} \alpha_k^{(n)}$  and  $a = (\alpha_1, \alpha_2, \dots)$ .

6. Claim:  $a \in l^2$  and  $a_n \rightarrow a$ .

7. We have, for any  $n \geq M$

$$\|a_n - a\|^2 < \varepsilon^2.$$

8. By Minkowski inequality

$$\|a\| = \|a - a_M + a_M\| \leq \|a - a_M\| + \|a_M\| < \infty$$

So,  $a \in l^2$ .

9. Also

$$\lim_{n \rightarrow \infty} \|a_n - a\| = 0$$

So,  $a_n \rightarrow a$ . ■

**Theorem 1.4.3 Completeness of  $C([a, b])$ .** *The space of complex-valued continuous functions on an interval  $[a, b]$  with the norm*

$$\|f\|_\infty = \max_{[a, b]} |f(x)|$$

*is complete.*

**Proof:**

1. Let  $(f_n) \in C([a, b])$  be Cauchy.
2. Let  $\varepsilon > 0$ . Then  $\exists M$  such that for  $n, m \geq M$ ,

$$\|f_n - f_m\| = \max_{[a, b]} |f_n(x) - f_m(x)| < \varepsilon.$$

3. So, for any  $x \in [a, b]$ , for  $n, m \geq M$ ,

$$|f_n(x) - f_m(x)| < \varepsilon$$

4. So,  $(f_n(x))$  is Cauchy sequence of complex numbers for each  $x \in [a, b]$ .

5. Let

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \text{ for } x \in [a, b].$$

6. Claim:  $f \in C([a, b])$  and  $f_n \rightarrow f$ .

7. We have, for any  $n \geq M$  and  $\forall x \in [a, b]$

$$|f_n(x) - f(x)| < \varepsilon.$$

8. Let  $x, x_0 \in [a, b]$ . Then,  $\exists \delta > 0$  such that if  $|x - x_0| < \delta$ , then

$$|f_M(x) - f_M(x_0)| < \varepsilon$$

9. Then if  $|x - x_0| < \delta$ , then

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_M(x) + f_M(x) - f_M(x_0) + f_M(x_0) - f(x_0)| \\ &\leq |f(x) - f_M(x)| + |f_M(x) - f_M(x_0)| + |f_M(x_0) - f(x_0)| < 3\varepsilon. \end{aligned}$$

So,  $f$  is continuous,  $f \in C([a, b])$ .

10. We have, for any  $n \geq M$

$$\|f_n - f\| < \varepsilon$$

Thus  $f_n \rightarrow f$ . ■

**Definition 1.4.3** **Convergent and Absolutely Convergent Series.** A series  $\sum_{n=1}^{\infty} x_n$  converges in a normed space  $E$  if the sequence of partial sums  $s_n = \sum_{k=1}^n x_k$  converges in  $E$ .

- That is, there is  $x \in E$  such that  $\|s_n - x\| \rightarrow 0$  as  $n \rightarrow \infty$ .

If  $s_n \rightarrow x$ , then  $\sum_{n=1}^{\infty} x_n = x$ .

If  $\sum_{n=1}^{\infty} \|x_n\| < \infty$ , then the series **converges absolutely**.

- An absolutely convergent series does not need to converge.

**Theorem 1.4.4** A normed space is complete if and only if every absolutely convergent series converges.

**Proof:**

1. (I) Let  $E$  be a Banach space (complete normed space).

2. Let  $(x_n)$  be an absolutely convergent sequence in  $E$  such that

$$\sum_{n=1}^{\infty} \|x_n\| < \infty$$

and  $s_n = \sum_{k=1}^n x_k$ .

3. Claim:  $(s_n)$  is Cauchy.

4. Let  $\varepsilon > 0$ . Then  $\exists M$  such that

$$\sum_{n=M+1}^{\infty} \|x_n\| < \varepsilon$$

5. Then  $\forall m, n \geq M$ ,

$$\|s_n - s_m\| = \left\| \sum_{j=n+1}^m x_j \right\| \leq \sum_{j=M+1}^{\infty} \|x_j\| < \varepsilon$$

So,  $(s_n)$  is Cauchy in  $E$ .

6. Thus,  $\exists x \in E$  such that  $s_n \rightarrow x$ , or  $\sum_{n=1}^{\infty} x_n = x$ .

7. (II) Conversely, assume that every absolutely convergent series converges in  $E$ .

8. Claim:  $E$  is complete.

9. Let  $(x_n)$  be Cauchy in  $E$ .

10. Then there exists a strictly increasing sequence of positive integers  $(p_k) \in \mathbb{N}$  such that for all  $m, n \geq p_k$

$$\|x_m - x_n\| < 2^{-k}$$

11. Consider the telescopic series

$$\sum_{k=1}^{\infty} (x_{p_{k+1}} - x_{p_k})$$

12. Since

$$\sum_{k=1}^{\infty} \|x_{p_{k+1}} - x_{p_k}\| \leq \sum_{k=1}^{\infty} 2^{-k} < \infty.$$

it converges absolutely.

13. Therefore, by assumption it converges.

14. Thus, the partial sums

$$s_n = \sum_{k=1}^n (x_{p_{k+1}} - x_{p_k}) = -x_{p_1} + x_{p_{n+1}} \rightarrow s \in E$$

converge to an  $s \in E$ .

15. Therefore, the sequence

$$x_{p_n} = x_{p_1} + s_{n-1} \rightarrow x = x_{p_1} + s$$

converges to an  $x \in E$ .

16. Finally,

$$\|x_n - x\| \leq \|x_n - x_{p_n}\| + \|x_{p_n} - x\| \rightarrow 0$$

■

- |   |
|---|
| <b>Theorem 1.4.5</b> <i>A closed vector subspace of a Banach space is a Banach space.</i> |
|---|

***Proof:***

1. Let  $E$  be a Banach space.
2. Let  $F$  be a closed vector subspace of  $E$ .
3. Let  $(x_n)$  be a Cauchy sequence in  $F$ .
4. Then  $(x_n)$  is Cauchy in  $E$  and  $x_n \rightarrow x \in E$ .
5. Since  $F$  is closed,  $x \in F$ .

■

### 1.4.1 Homework

- Exercises: [31,33,34,36]

## 1.5 Linear Mappings

- Let  $L : E_1 \rightarrow E_2$  be a mapping from a vector space  $E_1$  into a vector space  $E_2$ .
- If  $x \in E_1$ , then  $L(x)$  is the **image** of the vector  $x$ .
- If  $A \subset E_1$  is a subset of  $E_1$ , then the set

$$L(A) = \{y \in E_2 \mid y = L(x) \text{ for some } x \in A\}$$

is the **image of the set**  $A$ .

- If  $B \subset E_2$  is a subset of  $E_2$ , then the set

$$L^{-1}(B) = \{x \in E_1 \mid L(x) \in B\}$$

is the **inverse image of the set**  $B$ .

- A mapping  $L : D(L) \rightarrow E_2$  may be defined on a proper subset (called the **domain**)  $D(L) \subset E_1$  of the vector space  $E_1$ .
- The image of the domain,  $L(D(L))$ , of a mapping  $L$  is the **range** of  $L$ . That is the range of  $L$  is

$$R(L) = \{y \in E_2 \mid y = L(x) \text{ for some } x \in D(L)\}.$$

- The **null space**  $N(L)$  (or the **kernel**  $\text{Ker}(L)$ ) of a mapping  $L$  is the set of all vectors in the domain  $D(L)$  which are mapped to zero, that is

$$N(L) = \{x \in D(L) \mid L(x) = 0\}.$$

- The **graph**  $\Gamma(L)$  of a mapping  $L$  is the set of ordered pairs  $(x, L(x))$ , that is

$$\Gamma(L) = \{(x, y) \in E_1 \times E_2 \mid x \in D(L) \text{ and } y = L(x)\}.$$

**Definition 1.5.1 Continuous Mappings.** A mapping  $f : E_1 \rightarrow E_2$  from a normed space  $E_1$  into a normed space  $E_2$  is **continuous at**  $x_0 \in E_1$  if any sequence  $(x_n)$  in  $E_1$  converging to  $x_0$  is mapped to a sequence  $f(x_n)$  in  $E_2$  that converges to  $f(x_0)$ .

- That is  $f$  is continuous at  $x_0$  if

$$\|x_n - x_0\| \rightarrow 0 \text{ implies } \|f(x_n) - f(x_0)\| \rightarrow 0.$$

A mapping  $f : E_1 \rightarrow E_2$  is **continuous** if it is continuous at every  $x \in E_1$ .

- **Proposition.** The norm  $\|\cdot\| : E \rightarrow \mathbb{R}$  in a normed space  $E$  is a continuous mapping from  $E$  into  $\mathbb{R}$ .

Proof: If  $\|x_n - x\| \rightarrow 0$ , then

$$|\|x_n\| - \|x\|| \leq \|x_n - x\| \rightarrow 0$$

**Theorem 1.5.1** Let  $f : E_1 \rightarrow E_2$  be a mapping from a normed space  $E_1$  into a normed space  $E_2$ . The following conditions are equivalent:

- 1.  $f$  is continuous.
  2. The inverse image of any open set of  $E_2$  is open in  $E_1$ .
  3. The inverse image of any closed set of  $E_2$  is closed in  $E_1$ .

**Proof:** Exercise. ■

**Definition 1.5.2 Linear Mappings.** A mapping  $L : E_1 \rightarrow E_2$  is **linear** if  $\forall x, y \in E_1, \forall \alpha, \beta \in \mathbb{F}$ ,

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y).$$

- Let  $S \subset E_1$  be a subset of a vector space  $E_1$ . A mapping  $L : S \rightarrow E_2$  is **linear** if  $\forall x, y \in S$  and  $\forall \alpha, \beta \in \mathbb{F}$  such that  $\alpha x + \beta y \in S$ ,

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y).$$

- **Proposition.** If  $S$  is not a vector subspace of  $E_1$ , then there is a *unique extension* of  $L : S \rightarrow E_2$  to a linear mapping  $\tilde{L} : \text{span } S \rightarrow E_2$  from the vector subspace  $\text{span } S$  to  $E_2$ .

Proof: The extension  $\tilde{L}$  is defined by linearity. For any  $x \in \text{span } S$  such that  $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$  with  $x_i \in S$  and  $\alpha_i \in \mathbb{F}$ , we define

$$\tilde{L}(x) = \alpha_1 L(x_1) + \cdots + \alpha_n L(x_n).$$

- Thus, one can assume that the domain of a linear mapping is a vector space.
- **Proposition.** The range, the null space and the graph of a linear mapping are vector spaces.

Proof: Exercise.

- For any linear mapping  $L$ ,  $L(0) = 0$ . Thus,  $0 \in N(L)$  and the null space  $N(L)$  is always nonempty.

- **Theorem 1.5.2** *A linear mapping  $L : E_1 \rightarrow E_2$  from a normed space  $E_1$  into a normed space  $E_2$  is continuous if and only if it is continuous at a point.*

**Proof:**

1. Assume  $L$  is continuous at  $x_0 \in E_1$ .
2. Let  $x \in E_1$  and  $(x_n) \rightarrow x$ .
3. Then  $(x_n - x + x_0) \rightarrow x_0$ .
4. Thus

$$\|L(x_n) - L(x)\| = \|L(x_n - x + x_0) - L(x_0)\| \rightarrow 0$$

■

- **Definition 1.5.3 Bounded Linear Mappings.** *A linear mapping  $L : E_1 \rightarrow E_2$  from a normed space  $E_1$  into a normed space  $E_2$  is **bounded** if there is a real number  $K \in \mathbb{R}$  such that for all  $x \in E_1$ ,*

$$\|L(x)\| \leq K \|x\| .$$

- **Theorem 1.5.3** *A linear mapping  $L : E_1 \rightarrow E_2$  from a normed space  $E_1$  into a normed space  $E_2$  is continuous if and only if it is bounded.*

**Proof:**

1. (I). Assume that  $L$  is bounded.
2. Claim:  $L$  is continuous at 0.
3. Indeed,  $x_n \rightarrow 0$  implies

$$\|L(x_n)\| \leq K \|x_n\| \rightarrow 0$$

4. Hence,  $L$  is continuous.
5. (II). Assume that  $L$  is continuous.
6. By contradiction, assume that  $L$  is unbounded.
7. Then, there is a sequence  $(x_n)$  in  $E_1$  such that

$$\|L(x_n)\| > n \|x_n\|$$

8. Let

$$y_n = \frac{x_n}{n \|x_n\|}, \quad n \in \mathbb{N}$$

9. Then

$$\|y_n\| = \frac{1}{n}, \quad \text{and} \quad \|L(y_n)\| > 1$$

10. Then  $y_n \rightarrow 0$  but  $L(y_n) \not\rightarrow 0$ .
11. Thus,  $L$  is not continuous at zero. ■

- **Remark.** For linear mappings, continuity and uniform continuity are equivalent.
- The set  $L(E_1, E_2)$  of all linear mappings from a vector space  $E_1$  into a vector space  $E_2$  is a vector space with the addition and multiplication by scalars defined by

$$(L_1 + L_2)(x) = L_1(x) + L_2(x), \quad \text{and} \quad (\alpha L)(x) = \alpha L(x).$$

- The set  $B(E_1, E_2)$  of all bounded linear mappings from a normed space  $E_1$  into a normed space  $E_2$  is a vector subspace of the space  $L(E_1, E_2)$ .

**Theorem 1.5.4** *The space  $B(E_1, E_2)$  of all bounded linear mappings  $L : E_1 \rightarrow E_2$  from a normed space  $E_1$  into a normed space  $E_2$  is a normed space with norm defined by*

$$\| L \| = \sup_{x \in E_1, x \neq 0} \frac{\| L(x) \|}{\| x \|} = \sup_{x \in E_1, \|x\|=1} \| L(x) \| .$$

**Proof:**

1. Obviously,  $\| L \| \geq 0$ .
2.  $\| L \| = 0$  if and only if  $L = 0$ .
3. Claim:  $\| L \|$  satisfies triangle inequality.
4. Let  $L_1, L_2 \in B(E_1, E_2)$ .
5. Then

$$\begin{aligned} \| L_1 + L_2 \| &= \sup_{\|x\|=1} \| L_1(x) + L_2(x) \| \\ &\leq \sup_{\|x\|=1} \| L_1(x) \| + \sup_{\|x\|=1} \| L_2(x) \| \\ &= \| L_1 \| + \| L_2 \| \end{aligned}$$

■

- For any bounded linear mapping  $L : E_1 \rightarrow E_2$

$$\| L(x) \| \leq \| L \| \| x \|, \quad \forall x \in E_1 .$$

- $\| L \|$  is the *least* real number  $K$  such that

$$\| L(x) \| \leq K \| x \| \text{ for all } x \in E_1 .$$

- The norm defined by  $\| L \| = \sup_{x \in E_1, \|x\|=1} \| L(x) \|$  is called the **operator norm**.
- Convergence with respect to the operator norm is called the **uniform convergence of operators**.

- The **strong convergence** in  $B(E_1, E_2)$  is defined as follows:

**Definition 1.5.4** A sequence of bounded linear mappings  $L_n \in B(E_1, E_2)$  **converges strongly** to  $L \in B(E_1, E_2)$  if for every  $x \in E_1$  we have

$$\| L_n(x) - L(x) \| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

- **Proposition.** Uniform convergence implies strong convergence.

Proof: Follows from

$$\| L_n(x) - L(x) \| \leq \| L_n - L \| \| x \|$$

- Converse is not true.

**Theorem 1.5.5** Let  $E_1$  be a normed space and  $E_2$  be a Banach space. Then  $B(E_1, E_2)$  is a Banach space.

**Proof:**

1. Claim:  $B(E_1, E_2)$  is complete.
2. Let  $(L_n)$  be a Cauchy sequence in  $B(E_1, E_2)$ .
3. Then  $\forall x \in E_1$ , as  $m, n \rightarrow \infty$

$$\| L_m(x) - L_n(x) \| \leq \| L_m - L_n \| \| x \| \rightarrow 0$$

4. So,  $\forall x \in E_1$ ,  $(L_n(x))$  is Cauchy sequence in  $E_2$ .
5. Therefore,  $L_n(x) \rightarrow y(x) \in E_2$ .
6. Define  $L : E_1 \rightarrow E_2$  by

$$L(x) = \lim_{n \rightarrow \infty} L_n(x).$$

7. Claim:  $L \in B(E_1, E_2)$ .
8. Since  $(L_n)$  is Cauchy,  $\exists M$  such that  $\forall n \in \mathbb{N}$ ,

$$\| L_n \| \leq M$$

9. Hence

$$\| L(x) \| = \left\| \lim_{n \rightarrow \infty} L_n(x) \right\| = \lim_{n \rightarrow \infty} \| L_n(x) \| \leq M \| x \| .$$

10. So,  $L$  is bounded, that is  $L \in B(E_1, E_2)$ .

11. Claim:  $\|L_n - L\| \rightarrow 0$ .

12. Let  $\varepsilon > 0$ .

13.  $\exists k \in \mathbb{N}$  such that  $\forall n, m \geq k$ ,

$$\|L_m - L_n\| < \varepsilon$$

14. Thus,

$$\|L_m(x) - L_n(x)\| \leq \|L_m - L_n\| < \varepsilon \|x\| .$$

15. So, as  $n \rightarrow \infty$  for any  $m \geq k$

$$\|L_m(x) - L(x)\| < \varepsilon \|x\| .$$

16. Thus,  $\forall \varepsilon > 0, \exists k \in \mathbb{N}$  such that  $\forall m \geq k$ ,

$$\|L_m - L\| \leq \varepsilon$$

■

**Theorem 1.5.6** *Let  $E_1$  be a normed space and  $E_2$  be a Banach space. Let  $S \subset E_1$  be a subspace of  $E_1$  and  $L : S \rightarrow E_2$  be a continuous linear mapping from  $S$  into  $E_2$ . Then  $L$  has a unique extension to a continuous linear mapping  $\tilde{L} : \bar{S} \rightarrow E_2$  defined on the closure of the domain of the mapping  $L$ .*

*If  $S$  is dense in  $E_1$ , then  $L$  has a unique extension to a continuous linear mapping  $\tilde{L} : E_1 \rightarrow E_2$ .*

**Proof:**

1. Let  $x \in \bar{S}$ .

2.  $\exists (x_n) \in S, x_n \rightarrow x$ .

3. Then

$$\|L(x_m) - L(x_n)\| \leq \|L\| \|x_m - x_n\| \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

4. Thus,  $(L(x_n))$  is a Cauchy sequence in  $E_2$ .

5. So,  $L(x_n) \rightarrow z \in E_2$ .

6. Define the extension  $\tilde{L} : \bar{S} \rightarrow E_2$  of  $L$  by  $\tilde{L}(x) = z$ , that is

$$\tilde{L}(x) = \lim_{n \rightarrow \infty} L(x_n).$$

7. Claim:  $\tilde{L}(x)$  does not depend on the sequence  $x_n$  but only on its limit.

8. Let  $y_n \in S$ ,  $y_n \rightarrow x$ .

9. Then

$$y_n - x_n \rightarrow 0, \quad \text{and} \quad L(y_n - x_n) \rightarrow 0.$$

10. Hence

$$L(y_n) = L(y_n - x_n) + L(x_n) \rightarrow z$$

11. Since  $L$  is continuous, for any  $x \in S$ ,  $\tilde{L}(x) = L(x)$ .

12. Also,  $\tilde{L}$  is a linear mapping.

13. Claim:  $\tilde{L}$  is continuous.

14. Let  $x \in \bar{S}$  and  $x_n \in S$  such that  $x_n \rightarrow x$ .

15. Then

$$\begin{aligned} \|\tilde{L}(x)\| &= \left\| \lim_{n \rightarrow \infty} L(x_n) \right\| = \lim_{n \rightarrow \infty} \|L(x_n)\| \\ &\leq \|L\| \lim_{n \rightarrow \infty} \|x_n\| \\ &= \|L\| \|x\|. \end{aligned}$$

16. Thus  $\tilde{L}$  is bounded and  $\|\tilde{L}\| = \|L\|$ . ■

**Theorem 1.5.7** *Let  $E_1$  and  $E_2$  be normed spaces,  $S \subset E_1$  be a subspace of  $E_1$  and  $L : S \rightarrow E_2$  be a continuous linear mapping from  $S$  into  $E_2$ . Then the null space  $N(L)$  is a closed subspace of  $E_1$ .*

*If the domain  $S$  is a closed subspace of  $E_1$ , then the graph  $\Gamma(L)$  of  $L$  is a closed subspace of  $E_1 \times E_2$ .*

**Proof:** Exercise. ■

- A bounded linear mapping  $L : E \rightarrow \mathbb{F}$  from a normed space  $E$  into the scalar field  $\mathbb{F}$  is called a **functional**.
- The space  $B(E, \mathbb{F})$  of functionals is called the **dual space** and denoted by  $E'$  or  $E^*$ .
- The dual space is always a Banach space.

Proof: since  $\mathbb{F}$  is complete.

**Theorem 1.5.8 Diagonal Theorem.** *Let  $E$  be a normed space. Let  $X : \mathbb{N} \times \mathbb{N} \rightarrow E$  be a mapping defined by an infinite matrix  $(x_{ij})$ ,  $i, j \in \mathbb{N}$ , of elements of  $E$  such that*

1.  $\forall j \in \mathbb{N}, \lim_{i \rightarrow \infty} x_{ij} = 0$ , and
2. every increasing sequence  $(p_j)$  of positive integers has a subsequence  $(q_j)$  such that

$$\lim_{i \rightarrow \infty} \sum_{j=1}^{\infty} x_{q_i q_j} = 0$$

Then

$$\lim_{i \rightarrow \infty} x_{ii} = 0.$$

**Proof:** Read in the textbook. ■

**Theorem 1.5.9 Banach-Steinhaus Theorem (Uniform Boundedness Principle).** *Let  $\mathcal{T}$  be a family of bounded linear mappings from a Banach space  $X$  into a normed space  $Y$ . If for every  $x \in X$  there exists a constant  $C_x$  such that  $\|T(x)\| \leq C_x$  for all  $T \in \mathcal{T}$ , then there exists a constant  $M > 0$  such that*

$$\|T\| \leq M \text{ for all } T \in \mathcal{T}$$

**Proof:** Read in the textbook. ■

### 1.5.1 Homework

- Exercises: [37,38,39,42]

## 1.6 Completion of Normed Spaces

**Definition 1.6.1** Let  $(E, \|\cdot\|)$  be a normed space. A normed space  $(\tilde{E}, \|\cdot\|_1)$  is a **completion** of  $(E, \|\cdot\|)$  if

1. There exists a linear injection  $\Phi : E \rightarrow \tilde{E}$ ,
2. For every  $x \in E$

$$\|x\| = \|\Phi(x)\|_1,$$

3.  $\Phi(E)$  is dense in  $\tilde{E}$ ,
4.  $\tilde{E}$  is complete.

- The space  $\tilde{E}$  is defined as follows.
- Let  $(x_n)$  and  $(y_n)$  be Cauchy sequences in  $E$ .
- The sequences  $(x_n)$  and  $(y_n)$  are **equivalent**,

$$(x_n) \sim (y_n), \text{ if } \lim \|x_n - y_n\| = 0.$$

- The set of equivalent Cauchy sequences equivalent to a Cauchy sequence  $(x_n)$  is the **equivalence class** of  $(x_n)$

$$[(x_n)] = \{(y_n) \in E \mid (y_n) \sim (x_n)\}$$

- The set of all equivalent classes is

$$\tilde{E} = E / \sim = \{[(x_n)] \mid (x_n) \text{ is Cauchy sequence in } E\}$$

- The addition and multiplication by scalars in  $\tilde{E}$  are defined by

$$[(x_n)] + [(y_n)] = [(x_n + y_n)], \quad \lambda[(x_n)] = [(\lambda x_n)]$$

- The norm in  $\tilde{E}$  is defined by the limit

$$\|[(x_n)]\|_1 = \lim_{n \rightarrow \infty} \|x_n\|,$$

which exists for every Cauchy sequence.

- This definition is consistent since for any two equivalent Cauchy sequences  $(x_n)$  and  $(y_n)$

$$\| [(x_n)] \|_1 = \| [(y_n)] \|_1$$

Proof: Exercise.

- The linear bijection  $\Phi : E \rightarrow \tilde{E}$  is defined by a constant sequence

$$\Phi(x) = [(x_n)] \text{ such that } x_n = x, \forall n \in \mathbb{N}.$$

- Then  $\Phi$  is one-to-one.

Proof: Exercise.

- Obviously,  $\forall x \in E, \| x \| = \| \Phi(x) \|_1$ .

- Claim:  $\Phi(E)$  is dense in  $\tilde{E}$ .

Proof: Since every element  $[(x_n)]$  of  $\tilde{E}$  is the limit of a sequence  $(\Phi(x_n))$ .

- Claim:  $\tilde{E}$  is complete.

Proof:

- Let  $(y_n)$  be a Cauchy sequence in  $\tilde{E}$ .

- Then  $\exists (x_n)$  such that

$$\| \Phi(x_n) - y_n \|_1 < \frac{1}{n}.$$

- Claim:  $(x_n)$  is Cauchy sequence in  $E$ .

Proof:

$$\begin{aligned} \| x_n - x_m \| &= \| \Phi(x_n) - \Phi(x_m) \|_1 \\ &\leq \| \Phi(x_n) - y_n \|_1 + \| y_n - y_m \|_1 + \| y_m - \Phi(x_m) \|_1 \\ &\leq \| y_n - y_m \|_1 + \frac{1}{n} + \frac{1}{m}. \end{aligned}$$

- Let  $y = [(x_n)]$ .

- Claim:

$$\|y_n - y\|_1 \rightarrow 0.$$

Proof:

$$\begin{aligned} \|y_n - y\|_1 &\leq \|y_n - \Phi(x_n)\|_1 + \|\Phi(x_n) - y\|_1 \\ &< \|\Phi(x_n) - y\|_1 + \frac{1}{n} \rightarrow 0. \end{aligned} \quad (1.1)$$

- **Definition 1.6.2 Homeomorphism.** *Two topological spaces  $E_1$  and  $E_2$  are **homeomorphic** if there exists a bijection  $\Psi : E_1 \rightarrow E_2$  from  $E_1$  onto  $E_2$  such that both  $\Psi$  and  $\Psi^{-1}$  are continuous.*

- **Definition 1.6.3 Isomorphism of Normed Spaces.** *Two normed spaces  $(E_1, \|\cdot\|_1)$  and  $(E_2, \|\cdot\|_2)$  are **isomorphic** if there exists a linear homeomorphism  $\Psi : E_1 \rightarrow E_2$  from  $E_1$  onto  $E_2$ .*

- Any two completions of a normed space are isomorphic.

Proof: Read elsewhere.

### 1.6.1 Homework

- Exercises: [ ]

## 1.7 Contraction Mappings and the Banach Fixed Point Theorem

- **Examples.**

**Definition 1.7.1 Contraction Mapping.** *Let  $E$  be a normed space and  $A \subset E$  be a subset of  $E$ . A mapping  $f : A \rightarrow E$  from  $A$  into  $E$  is a **contraction mapping** if there exists a real number  $\alpha$ , such that  $0 < \alpha < 1$  and  $\forall x, y \in A$*

$$\| f(x) - f(y) \| \leq \alpha \| x - y \| .$$

- A contraction mapping is continuous.

Proof: Exercise.

- If  $\forall x, y \in A$

$$\| f(x) - f(y) \| \leq \| x - y \| ,$$

then it is not necessarily a contraction since the contraction constant  $\alpha$  may not exist.

- **Examples.**

**Theorem 1.7.1 Banach Fixed Point Theorem.** *Let  $E$  be a Banach space and  $A \subset E$  be a closed subset of  $E$ . Let  $f : A \rightarrow A$  be a contraction mapping from  $A$  into  $A$ . Then there exists a unique  $z \in A$  such that  $f(z) = z$ .*

**Proof:**

1. We have  $\forall x, y \in A$

$$\| f(x) - f(y) \| \leq \alpha \| x - y \|$$

2. Let  $x_0 \in A$  and

$$x_n = f(x_{n-1}), \quad n \in \mathbb{N}.$$

3. Claim:  $(x_n)$  is a Cauchy sequence in  $A$ .

4. We estimate,  $\forall n \in \mathbb{N}$

$$\| x_{n+1} - x_n \| \leq \alpha^n \| x_1 - x_0 \| .$$

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5. So,  $\forall n, m \in \mathbb{N}, m < n$ , by triangle inequality

$$\|x_n - x_m\| \leq \sum_{i=m}^{n-1} \alpha^i \|x_1 - x_0\| \leq \frac{\alpha^m}{1 - \alpha} \|x_1 - x_0\|$$

6. Thus,

$$\|x_n - x_m\| \rightarrow 0 \text{ as } m \rightarrow \infty$$

7. Next, there exists  $z = \lim x_n \in A$ .

8. Claim:  $z$  is the unique point such that  $f(z) = z$ .

9. We estimate

$$\begin{aligned} \|f(z) - z\| &\leq \|f(z) - x_n\| + \|x_n - z\| \\ &= \|f(z) - f(x_{n-1})\| + \|x_n - z\| \\ &\leq \alpha \|z - x_{n-1}\| + \|x_n - z\| \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

10. Therefore,  $f(z) = z$ .

11. Suppose there exists  $w \in F$  such that  $f(w) = w$ .

12. Then

$$\|z - w\| = \|f(z) - f(w)\| \leq \alpha \|z - w\|.$$

13. This implies  $z = w$ . ■

### • **Examples.**

- If the contraction constant  $\alpha = 1$ , then the Fixed Point Theorem is no longer valid.

### 1.7.1 Homework

- Exercises: [44,45,46,47]



# Chapter 2

## Lebesgue Integral

### 2.1 Step Functions

- **Characteristic function** of a set  $A \subset X$  is a mapping

$$\chi_A : X \rightarrow \{0, 1\}$$

defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

- For a non-zero function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , the closure of the set of all points in  $\mathbb{R}^n$  for which  $f(x) \neq 0$  is called the **support of  $f$** , i.e.

$$\text{supp } f = \overline{\{x \in \mathbb{R}^n \mid f(x) \neq 0\}}.$$

- Clearly,

$$\text{supp } \chi_A = \bar{A}.$$

- Let  $I$  be a semi-open interval in  $\mathbb{R}^n$  defined by

$$I = \{x \in \mathbb{R}^n \mid a_k \leq x_k < b_k, \quad k = 1, \dots, n\}$$

for some  $a_k < b_k$ .

- A finite linear combination of characteristic functions of semi-open intervals

$$f = \sum_{k=1}^N \alpha_k \chi_{I_k}$$

is called a **step function**.

- The collection of all step functions is a vector space.
- The absolute value of a step function is a step function.
- For any two functions  $f$  and  $g$ , the functions  $h = \min(f, g)$  and  $z = \max(f, g)$  are step functions.

## 2.2 Lebesgue Integrable Functions

- The **measure** of the set  $I$  is defined to be

$$\mu(I) = (b_1 - a_1) \cdots (b_n - a_n).$$

- The **Lebesgue integral** of a characteristic function of the set  $I$  is defined by

$$\int \chi_I = \mu(I).$$

- The **Lebesgue integral** of a step function is defined by linearity

$$\int \sum_{k=1}^N \alpha_k \chi_{I_k} = \sum_{k=1}^N \alpha_k \mu(I_k).$$

- A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is **Lebesgue integrable** if there exists a sequence of step functions  $\{f_k\}$  such that

$$f \simeq \sum_{k=1}^{\infty} f_k,$$

which means that two conditions are satisfied:

$$\begin{aligned} a) \quad & \sum_{k=1}^{\infty} \int |f_k| < \infty \\ b) \quad & f(x) = \sum_{k=1}^{\infty} f_k(x), \quad \forall x \in \mathbb{R}^n \text{ such that } \sum_{k=1}^{\infty} |f_k(x)| < \infty. \end{aligned}$$

- The **Lebesgue integral** of  $f$  is then defined by

$$\int f = \sum_{k=1}^{\infty} \int f_k$$

- Every Riemann integrable integral is Lebesgue integrable and both integrals are equal.

**Theorem 2.2.1** *If  $f$  is integrable and  $f \geq 0$  then*

$$\int f \geq 0.$$

- The set of all Lebesgue integrable functions on  $\mathbb{R}^n$  is denoted by  $L^1(\mathbb{R}^n)$ .

**Theorem 2.2.2** *The set  $L^1(\mathbb{R}^n)$  is a vector space and  $\int$  is a linear functional on it.*

- Let  $f \simeq \sum_{k=1}^{\infty} f_k$  and  $A$  be the set defined by

$$A = \{x \in \mathbb{R}^n \mid f(x) \neq \sum_{k=1}^{\infty} f_k\}.$$

Then

$$\int \chi_A = 0.$$

- A function  $f$  such that

$$\int |f| = 0$$

is called a **null** function.

- If  $f \simeq \sum_{k=1}^{\infty} f_k$ , then

the series  $\sum_{k=1}^{\infty} f_k(x)$  converges to  $f(x)$  for all  $x$  except a null set.

- Convergence for all  $x$  except a null set is called **convergence almost everywhere**.

**Theorem 2.2.3** *The absolute value of an integrable function is integrable and*

$$\left| \int f \right| \leq \int |f|.$$

**Theorem 2.2.4** *If  $f, g \in L^1(\mathbb{R}^n)$  and  $f \leq g$ , then*

$$\int f \leq \int g.$$

## 2.3 Series of Integrable Functions

- Let  $f \in L^1(\mathbb{R}^n)$  be an integrable function and  $(f_n)$  be a sequence of integrable functions in  $L^1(\mathbb{R}^n)$ . We say

$$f \simeq \sum_{k=1}^{\infty} f_k,$$

if two conditions are satisfied:

$$a) \sum_{k=1}^{\infty} \int |f_k| < \infty$$

$$b) f(x) = \sum_{k=1}^{\infty} f_k(x), \quad \forall x \in \mathbb{R}^n \text{ such that } \sum_{k=1}^{\infty} |f_k(x)| < \infty.$$

**Theorem 2.3.1** *If  $(f_k)$  is a sequence of integrable functions and*

$$f \simeq \sum_{k=1}^{\infty} f_k,$$

*then  $f$  is integrable and*

$$\int f = \sum_{k=1}^{\infty} \int f_k,$$

## 2.4 $L^1$ Norm

- The  $L^1$ -norm in  $L^1(\mathbb{R}^n)$  is defined by

$$\|f\| = \int |f|$$

- This functional is not a norm! To make  $L^1(\mathbb{R}^n)$  a normed space one has to consider instead of functions equivalence classes of equivalent functions.
- A function  $f$  is called a **null function** if it is integrable and  $\|f\| = 0$ .
- Two functions  $f$  and  $g$  are said to be **equivalent** if  $f - g$  is a null function.
- The **equivalence class** of  $f \in L^1(\mathbb{R}^n)$ , denoted by  $[f]$ , is the set of all functions equivalent to  $f$ .
- The space  $\mathcal{L}^1(\mathbb{R}^n)$  of equivalence classes of integrable functions is a normed space with the  $L^1$  norm.
- The space  $\mathcal{L}^1(\mathbb{R}^n)$  is complete.
- If  $F \in \mathcal{L}(\mathbb{R}^n)$ , then we cannot specify the value of  $F$  at a point  $x \in \mathbb{R}^n$  since  $F$  is not a function  $\mathbb{R}^n \rightarrow \mathbb{R}$  but an equivalence class of functions!
- A sequence of integrable functions  $(f_n)$  is said to **converge in norm** to an integrable function  $f$  if

$$\|f_n - f\| \rightarrow 0.$$

- A series  $\sum_{k=1}^{\infty} f_k$  is said to **converge in norm** to a function  $f$  if

$$\left\| \sum_{k=1}^n f_k - f \right\| \rightarrow 0.$$

- Notation. Convergence in norm is denoted by

$$f_n \rightarrow f \text{ i.n.}, \quad f_n \xrightarrow{\text{i.n.}} f \quad \sum_{k=1}^{\infty} f_k \stackrel{\text{i.n.}}{=} f$$

- Convergence in norm is the usual convergence in a normed space.

**Theorem 2.4.1** *If*

$$f \simeq \sum_{k=1}^{\infty} f_k,$$

*then*

$$f \stackrel{\text{i.n.}}{=} \sum_{k=1}^{\infty} f_k$$

## 2.5 Convergence Almost Everywhere

- A set  $X \subset \mathbb{R}^n$  is called a **null set** (or a **set of measure zero**) if its characteristic function is a null function.
- Every countable set is a null set.
- A countable union of null sets is a null set.
- Every subset of a null set is a null set.
- The closure of a null set is a null set.
- Two integrable functions,  $f, g \in L^1(\mathbb{R}^n)$ , are said to be **equal almost everywhere**,

$$f = g \text{ a.e.}, \quad \text{or} \quad f \stackrel{\text{a.e.}}{=} g$$

if the set of all  $x \in \mathbb{R}^n$  for which  $f(x) \neq g(x)$  is a null set.

- Let  $A = \text{supp}(f - g)$ . Obviously,  $f \stackrel{\text{a.e.}}{=} g$  if  $\int \chi_A = 0$ .

**Theorem 2.5.1** *Let  $f$  and  $g$  be integrable functions. Then*

$$f \stackrel{\text{a.e.}}{=} g \quad \text{if and only if} \quad \int |f - g| = 0$$

- A sequence of functions  $(f_n)$  is said to **converge almost everywhere** to a function  $f$  if

$$f_n(x) \rightarrow f(x) \text{ for every } x \in \mathbb{R}^n \text{ except a null set.}$$

- Notation. Convergence almost everywhere is denoted by

$$f_n \xrightarrow{\text{a.e.}} f \quad \text{or} \quad \lim f_n \stackrel{\text{a.e.}}{=} f$$

- A series  $\sum_{k=1}^{\infty} f_k$  converges to  $f$  almost everywhere if the sequence of partial sums converges to  $f$  almost everywhere.

- Notation.

$$\sum_{k=1}^{\infty} f_k \stackrel{\text{a.e.}}{=} f$$

- Convergence almost everywhere has properties similar to the converges in norm but is essentially different.
- Equivalence of functions and equality almost everywhere are the same thing.
- The value of a function at every point is not needed to find the value of the integral.
- Integral of a function is determined by the values of the function almost everywhere, that is everywhere except a null set.
- Uniform convergence does not imply convergence in norm.

**Theorem 2.5.2** *Let  $(f_n)$  be a sequence of integrable functions such that*

$$\sum_{k=1}^{\infty} \int |f_k| < \infty.$$

- *Then there exists a function  $f$  such that*

$$\sum_{k=1}^{\infty} f_k \stackrel{\text{a.e.}}{=} f$$

**Theorem 2.5.3** *If*

$$f \simeq \sum_{k=1}^{\infty} f_k,$$

- *then*

$$f \stackrel{\text{a.e.}}{=} \sum_{k=1}^{\infty} f_k.$$

**Theorem 2.5.4** Let  $(f_n)$  be a sequence of integrable functions such that

$$\sum_{k=1}^{\infty} \int |f_k| < \infty.$$

Then

$$f \stackrel{\text{a.e.}}{=} \sum_{k=1}^{\infty} f_k \quad \text{if and only if} \quad f \stackrel{\text{i.n.}}{=} \sum_{k=1}^{\infty} f_k.$$

## 2.6 Fundamental Theorems

**Theorem 2.6.1** The space  $L^1(\mathbb{R}^n)$  is complete.

- The space  $L^1(\mathbb{R}^n)$  is the completion of the space of step functions with respect to convergence in  $L^1$  norm.

**Theorem 2.6.2 (Riesz Theorem)** Let  $f$  be a function and  $(f_n)$  be a sequence of functions such that

$$f_n \xrightarrow{\text{i.n.}} f$$

Then there exists a subsequence  $(f_{p_n})$  such that

$$f_{p_n} \xrightarrow{\text{a.e.}} f.$$

- Convergence in norm implies convergence of integrals:

$$f_n \xrightarrow{\text{i.n.}} f \quad \text{implies} \quad \int f_n \rightarrow \int f$$

That is the order of the limit and the integral can be interchanged.

- Convergence almost everywhere does not have this property, that is

$$f_n \xrightarrow{\text{a.e.}} f \quad \text{does not imply} \quad \int f_n \rightarrow \int f$$

So, the order of the limit and the integral can not be interchanged.

- A sequence of functions is **monotone** if it is non-increasing or non-decreasing.

**Theorem 2.6.3 (Monotone Convergence Theorem)** *Let  $(f_n)$  be a monotone sequence of integrable functions such that the sequence of integrals  $(\int f_n)$  is bounded. Then there exists an integrable function  $f$  such that*

$$f_n \xrightarrow{\text{i.n.}} f \quad \text{and} \quad f_n \xrightarrow{\text{a.e.}} f$$

*In particular,*

$$\int \lim f_n = \lim \int f_n$$

*Moreover, if for some  $M$*

$$\left| \int f_n \right| \leq M, \quad \forall n \in \mathbb{N},$$

*then*

$$\left| \int f \right| \leq M.$$

**Theorem 2.6.4 (Dominated Convergence Theorem)** *Let  $f$  be a function,  $h$  be an integrable function and  $(f_n)$  be a sequence of integrable functions such that*

$$f_n \xrightarrow{\text{a.e.}} f$$

*and*

$$|f_n| \leq h, \quad \forall n \in \mathbb{N}.$$

*Then  $f$  is integrable and*

$$f_n \xrightarrow{\text{i.n.}} f.$$

*That is*

$$\int \lim f_n = \lim \int f_n$$

**Theorem 2.6.5 (Fatou Lemma)** *Let  $f$  be a function and  $(f_n)$  be a sequence of non-negative integrable functions such that*

$$f_n \xrightarrow{\text{a.e.}} f$$

and for some  $M$

$$\int f_n \leq M, \quad \forall n \in \mathbb{N}.$$

Then  $f$  is integrable and

$$\int f \leq M.$$

## 2.7 Locally Integrable Functions

- Let  $I$  be a bounded interval of  $\mathbb{R}^n$ . The integral of  $f$  over  $I$  is defined by

$$\int_I f = \int \chi_I f.$$

- If  $f$  is an integrable function then for every bounded interval  $I$  the integral  $\int_I f$  exists.
- The converse is not true.
- A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is called **locally integrable** if for every bounded interval  $I$  the function  $f\chi_I$  is integrable.
- Locally integrable functions form a vector space.
- The space  $L^1(\mathbb{R})$  is a subspace of the space of locally integrable functions.
- The absolute value of locally integrable function is locally integrable.
- The product of locally integrable functions may be locally nonintegrable!
- The product of a bounded function and a locally integrable function is locally integrable.

- If  $g$  is an integrable function and  $f$  is locally integrable function such that  $|f| \leq g$ , then  $f$  is integrable.
- The set of functions vanishing outside a bounded interval  $I$  is a vector space denoted by  $L^1(I)$ .
- The space  $L^1(I)$  is a normed space with the norm

$$\|f\| = \int_I |f|.$$

- The space  $L^1(I)$  is complete.

## 2.8 Lebesgue Measure

- A set  $\Omega$  is called **measurable** if the characteristic function  $\chi_\Omega$  of  $\Omega$  is a locally integrable function.
- The **measure**  $\mu(\Omega)$  of a measurable set  $\Omega$  is defined by

$$\mu(\Omega) = \int \chi_\Omega.$$

- The measure of any set is non-negative.
- The measure of the empty set is 0, that is  $\mu(\emptyset) = 0$ .
- The measure of a null set is 0.
- The **measure** of a nonmeasurable set  $\Omega$  is defined to be  $\infty$ .

**Theorem 2.8.1** *Let  $(\Omega_k)_{k=1}^\infty$  be a sequence of disjoint sets. If each set of the sequence is measurable, then the union of the sequence is also measurable and*

$$\mu\left(\bigcup_{k=1}^{\infty} \Omega_k\right) = \sum_{k=1}^{\infty} \mu(\Omega_k).$$

- The integral over any measurable set  $\Omega$  is defined by

$$\int_{\Omega} f = \int \chi_{\Omega} f$$

- For any measurable set  $\Omega$  the space  $L^1(\Omega)$  is the set of all integrable functions vanishing outside  $\Omega$ .
- The space  $L^1(\Omega)$  is a Banach space.
- A function  $f$  is called **measurable** if there exists a sequence of step functions  $(f_n)$  converging to  $f$  almost everywhere, that is

$$f_n \xrightarrow{\text{a.e.}} f.$$

- Every integrable function is measurable.
- Every locally integrable function is measurable.
- Not all measurable functions are locally integrable.

**Theorem 2.8.2** *The set of measurable functions form a vector space.*

- *The absolute value of a measurable function is measurable.*
- *The product of measurable functions is measurable.*

**Theorem 2.8.3** *Let  $f$  be a measurable function and  $g$  be a locally integrable function. If  $|f| \leq g$ , then  $f$  is locally integrable.*

## 2.9 Square Integrable Functions

- A locally integrable function  $f$  such that  $|f|^2$  is integrable is called **square integrable**.
- The space of square integrable functions is denoted by  $L^2(\mathbb{R}^n)$ .

**Theorem 2.9.1** *The space of square integrable functions is a vector space.*

**Corollary 2.9.1** *The product of two square integrable functions is an integrable function.*

**Theorem 2.9.2** *The space  $L^2(\mathbb{R}^n)$  is a normed space with the norm*

$$\|f\| = \left( \int |f|^2 \right)^{1/2}.$$



# Chapter 3

## Hilbert Spaces and Orthonormal Systems

### 3.1 Inner Product Spaces

**Definition 3.1.1 Inner Product Space.** *A complex vector space  $E$  is called an inner product space (or a pre-Hilbert space, or a unitary space) if there is a mapping  $(\cdot, \cdot) : E \times E \rightarrow \mathbb{C}$ , called an inner product, that satisfies the conditions:  $\forall x, y, z \in E, \forall \alpha \in \mathbb{C}$ :*

- 1.  $(x, x) \geq 0$
  2.  $(x, x) = 0$  if and only if  $x = 0$
  3.  $(x + y, z) = (x, z) + (y, z)$
  4.  $(\alpha x, y) = \alpha(x, y)$
  5.  $(x, y) = (y, x)^*$

- **Examples.**

### 3.1.1 Finite-dimensional spaces

- $\mathbb{C}^n$  is the space of  $n$ -tuples  $x = (x_1, \dots, x_N)$  of complex numbers. It is a Hilbert space with the inner product

$$(x, y) = \sum_{j=1}^n x_j y_j^*$$

### 3.1.2 Spaces of sequences

- $l^2$  is the space of sequences of complex numbers  $x = (x_n)_{n=1}^{\infty}$  such that

$$\sum_{n=1}^{\infty} |x_n|^2 < \infty.$$

It is an inner product space with the inner product

$$(x, y) = \sum_{j=1}^{\infty} x_j y_j^*$$

- $l_0$  is the space of sequences of complex numbers with zero tails with the inner product

$$(x, y) = \sum_{j=1}^{\infty} x_j y_j^*$$

### 3.1.3 Spaces of continuous functions

- $C([a, b])$  with the inner product

$$(f, g) = \int_a^b f g^*$$

- $C_0(\mathbb{R})$  (space of continuous functions with compact support) with the inner product

$$(f, g) = \int_{\mathbb{R}} f g^*$$

### 3.1.4 Spaces of square integrable functions

- $L^2([a, b])$  is the space of complex valued functions such that

$$\int_a^b |f|^2 < \infty.$$

It is an inner product space with the inner product

$$(f, g) = \int_{\mathbb{R}} fg^*$$

- $L^2([a, b], \mu)$  (space of square integrable functions with the measure  $\mu$ ) with the inner product

$$(f, g) = \int_a^b \mu fg^*,$$

where  $\mu > 0$  almost everywhere.

- Let  $\Omega$  be an open set in  $\mathbb{R}^n$  (in particular,  $\Omega$  can be the whole  $\mathbb{R}^n$ ). The space  $L^2(\Omega)$  is the set of complex valued functions such that

$$\int_{\Omega} |f|^2 < \infty,$$

where  $x = (x_1, \dots, x_n) \in \Omega$  and  $dx = dx_1 \cdots dx_n$ . It is an inner product space with the inner product

$$(f, g) = \int_{\Omega} fg^*$$

- $L^2(\mathbb{R}^n)$  with the inner product

$$(f, g) = \int_{\mathbb{R}^n} fg^*$$

- Let  $\Omega$  be an open set in  $\mathbb{R}^n$  (in particular,  $\Omega$  can be the whole  $\mathbb{R}^n$ ) and  $V$  be a finite-dimensional vector space. The space  $L^2(\Omega, V)$  is the set of vector valued functions  $f = (f_1, \dots, f_N)$  on  $\Omega$  such that

$$\sum_{i=1}^N \int_{\Omega} |f_i|^2 < \infty.$$

It is an inner product space with the inner product

$$(f, g) = \sum_{i=1}^N \int_{\Omega} f_i g_i^*$$

### 3.1.5 Real Inner Product Spaces

- **Remark.** The inner product in a *real* inner product space is symmetric.
- A finite dimensional real inner product space is called a **Euclidean space**.
- $\mathbb{R}^n$  is a Euclidean space.

### 3.1.6 Direct Sum of Inner Product Spaces

- Let  $E_1$  and  $E_2$  be inner product spaces. The **direct sum**  $E = E_1 \oplus E_2$  of  $E_1$  and  $E_2$  is an inner product space of ordered pairs  $z = (x, y)$  with  $x \in E_1$  and  $y \in E_2$  with the inner product defined by

$$(z_1, z_2)_E = (x_1, x_2)_{E_1} + (y_1, y_2)_{E_2}.$$

### 3.1.7 Tensor Products of Inner Product Spaces

- Let  $E_1$  and  $E_2$  be inner product spaces. For each  $\varphi_1 \in E_1$  and  $\varphi_2 \in E_2$  let  $\varphi_1 \otimes \varphi_2$  denote the conjugate bilinear form on  $E_1 \times E_2$  defined by

$$(\varphi_1 \otimes \varphi_2)(\psi_1, \psi_2) = (\psi_1, \varphi_1)_{E_1} (\psi_2, \varphi_2)_{E_2}$$

where  $\psi_1 \in E_1$  and  $\psi_2 \in E_2$ . Let  $E$  be the set of finite linear combinations of such bilinear forms. An inner product on  $E$  can be defined by

$$(\varphi \otimes \psi, \eta \otimes \mu)_E = (\varphi, \eta)_{E_1} (\psi, \mu)_{E_2}$$

(with  $\varphi, \eta \in E_1$  and  $\psi, \mu \in E_2$ ) and extending by linearity on  $E$ .

- Let  $E$  be an inner product space. The space

$$F(E) = \mathbb{C} \oplus_{n=1}^{\infty} \underbrace{E \otimes \cdots \otimes E}_n$$

is called the **Fock space** over  $E$ .

### 3.1.8 Homework

- Exercises: [4,5]

## 3.2 Norm in an Inner Product Space

**Definition 3.2.1 Norm in an Inner Product Space.** Let  $E$  be an inner product space. The norm in  $E$  is a functional  $\| \cdot \|: E \rightarrow \mathbb{R}$  defined by

$$\| x \| = \sqrt{(x, x)}.$$

**Theorem 3.2.1** Every inner product space is a normed space with the norm  $\| x \| = \sqrt{(x, x)}$  and a metric space with the metric  $d(x, y) = \sqrt{(x - y, x - y)}$ .

**Theorem 3.2.2 Schwarz's Inequality.** Let  $E$  be an inner product space. Then for any  $x, y \in E$  we have

$$|(x, y)| \leq \| x \| \| y \| .$$

The equality  $|(x, y)| = \| x \| \| y \|$  holds if and only if  $x$  and  $y$  are linearly dependent.

**Proof:**

1. ■

**Corollary 3.2.1 Triangle Inequality.** Let  $E$  be an inner product space. Then for any  $x, y \in E$  we have

$$\| x + y \| \leq \| x \| + \| y \| .$$

**Proof:**

1. ■

**Theorem 3.2.3 Parallelogram Law.** Let  $E$  be an inner product space. Then for any  $x, y \in E$  we have

$$\| x + y \|^2 + \| x - y \|^2 = 2 (\| x \|^2 + \| y \|^2)$$

**Proof:**

1. ■

- **Definition 3.2.2 Orthogonal Vectors.** *Let  $E$  be an inner product space. Two vectors  $x, y \in E$  are **orthogonal** if  $(x, y) = 0$ .*

- Notation. The orthogonality of the vectors  $x$  and  $y$  is denoted by

$$x \perp y$$

- The relation  $\perp$  is symmetric, that is, if  $x \perp y$  then  $y \perp x$ .

- **Theorem 3.2.4 Pythagorean Theorem.** *Let  $E$  be an inner product space. If two vectors  $x, y \in E$  are **orthogonal** then*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 .$$

*Proof:*

1. ■

- **Examples.**

### 3.2.1 Homework

- Exercises: [9,10,11,12,13]

### 3.3 Hilbert Spaces

- **Definition 3.3.1 Hilbert Space.** *An inner product space is called a **Hilbert space** if it is complete as a normed space.*

- **Examples.**

#### 3.3.1 Finite dimensional Hilbert spaces

- $\mathbb{C}^n$  is complete.

#### 3.3.2 Spaces of sequences

- The space of square summable sequences ( $l^2$ ) is complete. Proved before.
- The space of sequences with vanishing tails ( $l_0$ ) is not complete. Counterexample.

#### 3.3.3 Spaces of continuous functions

- $C([a, b])$  is not complete. Counterexample.
- $C_0(\mathbb{R})$  (space of continuous functions with compact support) is not complete. Counterexample.

#### 3.3.4 Spaces of square integrable functions.

- **Theorem 3.3.1**  $L^2([a, b])$  is complete.

*Proof:*

1. ■

- **Theorem 3.3.2**  $L^2(\mathbb{R})$  is complete.

*Proof:*

1.

- **Theorem 3.3.3**  $L^2([a, b], \mu)$  is complete. ■

*Proof:*

1. ■

### 3.3.5 Sobolev Spaces

- Let  $\Omega$  be an open set in  $\mathbb{R}^n$  (in particular,  $\Omega$  can be the whole  $\mathbb{R}^n$ ) and  $V$  a finite-dimensional complex vector space. Let  $C^m(\Omega, V)$  be the space of complex vector valued functions that have continuous partial derivatives of all orders less or equal to  $m$ . Let

$$\alpha = (\alpha_1, \dots, \alpha_n),$$

$\alpha_j \in \mathbb{N}$ , be a multiindex of nonnegative integers,  $\alpha_i \geq 0$ , and let

$$|\alpha| = \alpha_1 + \dots + \alpha_n.$$

Define

$$D^\alpha f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} f.$$

Then  $f \in C^m(V, \Omega)$  iff  $\forall \alpha, |\alpha| \leq m, \forall i = 1, \dots, N, \forall x \in \Omega$  we have

$$|D^\alpha f_i(x)| < \infty.$$

The space  $\tilde{H}^m(\Omega, V)$  is the space of complex vector valued functions such that  $\forall \alpha, |\alpha| \leq m$ ,

$$D^\alpha f \in L^2(\Omega, V)$$

i.e. such that  $\forall \alpha, |\alpha| \leq m$ ,

$$\sum_{i=1}^N \int_{\Omega} |D^\alpha f_i(x)|^2 dx < \infty.$$

It is an inner product space with the inner product

$$(f, g) = \sum_{\alpha, |\alpha| \leq m} \sum_{i=1}^N \int_{\Omega} D^{\alpha} f_i (D^{\alpha} g_i)^*$$

The Sobolev space  $H^m(\Omega, V)$  is the completion of the space  $\tilde{H}^m(\Omega, V)$  defined above.

### 3.3.6 Homework

- Exercises: [15,16,17,18]

### 3.4 Strong and Weak Convergence

**Definition 3.4.1 Strong Convergence.** A sequence  $(x_n)$  of vectors in an inner product space  $E$  is **strongly convergent** to a vector  $x \in E$  if

$$\lim_{n \rightarrow \infty} \|x_n - x\| = 0$$

- Notation.  $x_n \rightarrow x$ .

**Definition 3.4.2 Weak Convergence.** A sequence  $(x_n)$  of vectors in an inner product space  $E$  is **weakly convergent** to a vector  $x \in E$  if for any  $y \in E$

$$\lim_{n \rightarrow \infty} (x_n - x, y) = 0$$

- Notation.  $x_n \xrightarrow{w} x$

**Theorem 3.4.1** A strongly convergent sequence is weakly convergent to the same limit.

**Proof:**

- 1.

■

- Converse is not true. Counterexample.

**Corollary 3.4.1** Let  $E$  be an inner product space. Then for every  $y \in E$  the linear functional  $\varphi_y : E \rightarrow \mathbb{C}$  defined by

$$\varphi_y(x) = (x, y) \quad \forall x \in E$$

is continuous.

**Theorem 3.4.2** Let  $(x_n)$  be a sequence in an inner product space  $E$ . If:

1.  $x_n \xrightarrow{w} x$  and
2.  $\|x_n\| \rightarrow \|x\|$ ,

then  $x_n \rightarrow x$ .

**Proof:**

1. ■

**Theorem 3.4.3** *Let  $S$  be a subset of an inner product space  $E$  and  $(x_n)$  be a sequence in  $E$ . If:*

- 1.  $\text{span } S$  is dense in  $E$ ,
  - 2.  $(x_n)$  is bounded, and
  - 3. for any  $y \in S$ ,  $\lim_{n \rightarrow \infty} (x_n - y, y) = 0$ ,
- then  $x_n \xrightarrow{w} x$ .

**Proof:**

1. ■

- **Theorem 3.4.4** *Weakly convergent sequences in a Hilbert space are bounded.*

**Proof:**

1. ■

### 3.4.1 Homework

- Exercises: [20,21,23,24]

## 3.5 Orthogonal and Orthonormal Systems

### Definition 3.5.1 Orthogonal and Orthonormal Systems.

Let  $E$  be an inner product space. A set  $S$  of vectors in  $E$  is called an **orthogonal system** if any pair of distinct vectors in  $S$  is orthogonal to each other.

An orthogonal system of unit vectors is an **orthonormal system**.

- Every orthogonal system can be made orthonormal.
- Let  $S$  be a set of vectors in  $E$ . If  $x \perp y$  for any  $y \in S$ , then  $x \perp \text{span } S$ .
- **Theorem 3.5.1** *Orthogonal systems are linearly independent.*

*Proof:*

1.

■

**Definition 3.5.2 Orthonormal Sequence.** A sequence of vectors which is an orthonormal system is an **orthonormal sequence**.

- **Examples.**
- Any orthogonal sequence can be always made orthonormal.
- **Gram-Schmidt orthonormalization process.** Any sequence of linearly independent vectors can be made orthonormal.
  1. Let  $(y_n)$  be a linearly independent sequence.
  2. Then the sequence  $(z_n)$  defined by

$$z_1 = y_1, \quad z_k = y_k - \sum_{n=1}^{k-1} \frac{(y_k, z_n)z_n}{\|z_n\|^2}$$

is orthogonal.

3. Then the sequence  $(e_n)$  defined by  $e_n = z_n / \|z_n\|$  is orthonormal.

### 3.5.1 Homework

- Exercises: [32,33,37]

### 3.6 Properties of Orthonormal Systems

**Theorem 3.6.1 Pythagorean Formula.** *Let  $E$  be an inner product space and  $\{e_n\}_{n=1}^N$  be an orthonormal set in  $E$ . Then*

$$\left\| \sum_{n=1}^N e_n \right\|^2 = \sum_{n=1}^N \|e_n\|^2$$

**Proof:**

1. ■

**Theorem 3.6.2 Bessel's Equality and Inequality.** *Let  $E$  be an inner product space and  $\{e_n\}_{n=1}^N$  be an orthonormal set in  $E$ . Then  $\forall x \in E$*

$$\|x\|^2 = \sum_{n=1}^N |x_n|^2 + \left\| x - \sum_{n=1}^N x_n e_n \right\|^2$$

and

$$\sum_{n=1}^N |x_n|^2 \leq \|x\|^2$$

where  $x_n = (x, e_n)$ .

**Proof:**

1. ■

- **Remarks.** Let  $(e_n)$  be an orthonormal sequence in an inner product space  $E$ .
- The complex numbers  $x_n = (x, e_n)$  are **generalized Fourier coefficients** of  $x$  with respect to the orthonormal sequence  $(e_n)$ .
- An orthonormal sequence  $(e_n)$  in  $E$  induces a mapping  $\varphi_e : E \rightarrow l^2$  defined by

$$\varphi(x) = \tilde{x} = (x_n)$$

- The sequence of Fourier coefficients  $\tilde{x} = (x_n)$  is square summable, that is  $\tilde{x} \in l^2$  since for any  $x \in E$

$$\sum_{n=1}^{\infty} |x_n|^2 \leq \|x\|^2$$

and, therefore, this series converges.

- The expansion

$$x \sim \sum_{n=1}^{\infty} x_n e_n$$

is a **generalized Fourier series** of  $x$ .

- The question is whether the mapping  $\varphi$  is bijective and whether the Fourier series converges.

**Theorem 3.6.3** *Let  $(e_n)$  be an orthonormal sequence in a Hilbert space  $H$  and  $(x_n)$  be a sequence of complex numbers. Then the series  $\sum_{n=1}^{\infty} x_n e_n$  converges if and only if  $(x_n) \in l^2$ , that is the series  $\sum_{n=1}^{\infty} |x_n|^2$  converges.*

*In this case*

$$\left\| \sum_{n=1}^{\infty} x_n e_n \right\|^2 = \sum_{n=1}^{\infty} |x_n|^2$$

**Proof:**

1. ■

- Fourier series of any  $x \in H$  in a Hilbert space  $H$  converges.
- Fourier series of  $x$  may converge to a vector different from  $x$ !
- **Example.**
- Let  $(e_n)$  be an orthonormal sequence in an inner product space  $E$ . The sequence of Fourier coefficients  $x_n = (x, e_n)$  is square summable, and, therefore,

$$\lim_{n \rightarrow \infty} (x, e_n) = 0 \quad \forall x \in E$$

- Thus, every orthonormal sequence weakly converges to zero.
- Orthonormal sequences are not strongly convergent since  $\|e_n\| = 1 \forall n \in \mathbb{N}$ .

**Definition 3.6.1 Complete Orthonormal Sequence.** Let  $E$  be an inner product space. An orthonormal sequence  $(e_n)$  in  $E$  is **complete** if  $\forall x \in E$  the Fourier series of  $x$  converges to  $x$ .

That is

$$x = \sum_{n=1}^{\infty} x_n e_n.$$

More explicitly,

$$\lim_{n \rightarrow \infty} \left\| x - \sum_{n=1}^{\infty} x_n e_n \right\| = 0$$

- **Example.**

**Definition 3.6.2 Orthonormal Basis.** Let  $E$  be an inner product space. An orthonormal system  $B$  in  $E$  is an **orthonormal basis** if for any  $x \in E$  there exists a unique orthonormal sequence  $(e_n)$  in  $B$  and a unique sequence  $(x_n)$  of nonzero complex numbers such that

$$x = \sum_{n=1}^{\infty} x_n e_n.$$

- **Remarks.**
- A complete orthonormal sequence in an inner product space is an orthonormal basis.
- Let  $E$  be an inner product space and  $(e_n)$  be a complete orthonormal sequence. Then the set

$$S = \text{span} \{e_n \mid n \in \mathbb{N}\}$$

is dense in  $E$ .

- **Theorem 3.6.4** *Let  $H$  be a Hilbert space. An orthonormal sequence in  $H$  is complete if and only if the only vector orthogonal to this sequence is the zero vector.*

**Proof:**

1. ■

- **Theorem 3.6.5 Parseval's Formula.** *Let  $H$  be a Hilbert space. An orthonormal sequence  $(e_n)$  in  $H$  is complete if and only if  $\forall x \in H$*

$$\|x\|^2 = \sum_{n=1}^{\infty} |x_n|^2$$

where  $x_n = (x, e_n)$ .

**Proof:**

1. ■

- **Theorem 3.6.6** *Let  $H_1$  and  $H_2$  be Hilbert spaces. If  $\{\varphi_k\}$  and  $\{\psi_l\}$  are orthonormal bases for  $H_1$  and  $H_2$  respectively, then  $\{\varphi_k \otimes \psi_l\}$  is an orthonormal basis for the tensor product  $H_1 \otimes H_2$ .*

**Proof:**

1. ■

- **Examples.**

### 3.6.1 Homework

- Exercises: [38,39,40,41]

### 3.7 Trigonometric Fourier Series

- Consider the Hilbert space  $L^2([-\pi, \pi])$ .
- The sequence

$$\varphi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}, \quad n \in \mathbb{Z}$$

is an orthonormal sequence in  $L^2([-\pi, \pi])$ .

- Consider the space  $L^1([-\pi, \pi])$ .
- Identify the elements of  $L^1([-\pi, \pi])$  with  $2\pi$  periodic functions on  $\mathbb{R}$ .
- Then for any  $f \in L^1([-\pi, \pi])$ ,

$$\int_{-\pi}^{\pi} dt f(t) = \int_{-\pi}^{\pi} dt f(t - x)$$

- Define the sequence

$$h_n = \sum_{k=-n}^n (f, \varphi_k) \varphi_k, \quad n \in \mathbb{N}.$$

More explicitly,

$$h_n(x) = \int_{-\pi}^{\pi} dt G_n(x - t) f(t)$$

where

$$G_n(x) = \frac{1}{2\pi} \sum_{k=-n}^n e^{ikx}$$

- Define the sequence

$$F_n = \frac{1}{n+1} \sum_{k=0}^n h_k = \frac{1}{n+1} \sum_{k=0}^n \sum_{j=-k}^k (f, \varphi_j) \varphi_j$$

More explicitly,

$$F_n(x) = \int_{-\pi}^{\pi} dt K_n(x - t) f(t),$$

where

$$K_n(x) = \frac{1}{2\pi(n+1)} \sum_{k=-n}^n (n+1-|k|) e^{ikx}$$

is the **Fejer's kernel**.

**Lemma 3.7.1** *We have*

$$K_n(x) = \frac{1}{2\pi(n+1)} \frac{\sin^2 \left[ (n+1) \frac{x}{2} \right]}{\sin^2 \left( \frac{x}{2} \right)}$$

**Proof:**

1. ■

**Definition 3.7.1** *A sequence  $\kappa_n$  of  $2\pi$ -periodic continuous functions is a **summability kernel** if it satisfies the conditions:*

1.  $\forall n \in \mathbb{N}$

$$\int_{-\pi}^{\pi} dt \kappa_n(t) = 1,$$

2. *There is  $M \in \mathbb{R}$  such that  $\forall n \in \mathbb{N}$*

$$\int_{-\pi}^{\pi} dt |\kappa_n(t)| \leq M,$$

3. *For any  $\delta \in (0, \pi)$*

$$\lim_{n \rightarrow \infty} \int_{\delta}^{2\pi-\delta} dt |\kappa_n(t)| = 0$$

**Lemma 3.7.2** *The Fejer's kernel is a summability kernel.*

**Proof:**

1. ■

**Theorem 3.7.1** Let  $(\kappa_n)$  be a summability kernel and  $f \in L^1([-\pi, \pi])$ . Let  $(F_n)$  be a sequence defined by

$$F_n = \int_{-\pi}^{\pi} dt \kappa_n(t) f(x - t).$$

Then  $F_n$  strongly converges to  $f$  in  $L^1$  norm.

**Proof:**

1. ■

**Theorem 3.7.2** Let  $f \in L^1([-\pi, \pi])$ . If all Fourier coefficients  $f_n = (f, \varphi_n)$  vanish, then  $f = 0$  almost everywhere.

**Proof:**

1. ■

**Theorem 3.7.3** The sequence

$$\varphi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}, \quad n \in \mathbb{Z}$$

is a complete orthonormal sequence, (an orthonormal basis), in  $L^2([-\pi, \pi])$ .

**Proof:**

1. ■

- Let  $f \in L^2([-\pi, \pi])$ . The series

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \varphi_n(x),$$

where

$$f_n = \int_{-\pi}^{\pi} dt f(t) \varphi_n(t),$$

is the **Fourier series**. The scalars  $f_n$  are the **Fourier coefficients**.

- Fourier series does not converge pointwise!
- Fourier series of a function  $f \in L^2([-\pi, \pi])$  converges almost everywhere.

### 3.7.1 Homework

- Exercises: [45]

### 3.8 Orthonormal Complements and Projection Theorem

- A subspace of a Hilbert space is an inner product space.
- A closed subspace of a Hilbert space is a Hilbert space.

**Definition 3.8.1 Orthogonal Complement.** Let  $H$  be a Hilbert space and  $S \subset H$  be a nonempty subset of  $H$ . We say that  $x \in H$  is **orthogonal** to  $S$ , denoted by  $x \perp S$ , if  $\forall y \in S$ ,  $(x, y) = 0$ .

The set

$$S^\perp = \{x \in H \mid x \perp S\}$$

of all vectors orthogonal to  $S$  is called the **orthogonal complement** of  $S$ .

Two subsets  $A$  and  $B$  of  $H$  are **orthogonal**, denoted by  $A \perp B$ , if every vector of  $A$  is orthogonal to every vector of  $B$ .

- If  $x \perp H$ , then  $x = 0$ , that is

$$H^\perp = \{0\}, \quad \{0\}^\perp = H.$$

- If  $A \perp B$ , then  $A \cap B = \{0\}$  or  $\emptyset$ .

**Theorem 3.8.1** The orthogonal complement of any subset of a Hilbert space is a Hilbert subspace.

**Proof:**

1. Let  $H$  be a Hilbert space and  $S \subset H$ .
2. Check directly that  $S^\perp$  is a vector subspace.
3. Claim:  $S^\perp$  is closed.
4. Let  $(x_n)$  be a sequence in  $S^\perp$  such that  $x_n \rightarrow x \in H$ .
5. Then  $\forall y \in S$

$$(x, y) = \lim_{n \rightarrow \infty} (x_n, y) = 0.$$

6. Thus  $x \in S^\perp$ .



- **Remarks.**

- $S$  does not have to be a vector subspace.

**Definition 3.8.2 Convex Sets.** A set  $U$  in a vector space  $E$  is called **convex** if  $\forall x, y \in U$  and  $\forall \alpha \in (0, 1)$ ,

$$\alpha x + (1 - \alpha)y \in U.$$

- A vector subspace is a convex set.

**Theorem 3.8.2 The Closest Point Property.** Let  $H$  be a Hilbert space and  $S$  be a closed convex subset of  $H$ . Then  $\forall x \in H$  there exists a unique  $y \in S$  such that

$$\|x - y\| = \inf_{z \in S} \|x - z\|.$$

**Proof:**

1. (I). Existence. Let  $d = \inf_{z \in S} \|x - z\|$ .
2. Let  $(y_n)$  be a sequence in  $S$  such that

$$\lim_{n \rightarrow \infty} \|x - y_n\| = d.$$

3. By convexity we have  $\forall n, m \in \mathbb{N}$

$$\|x - \frac{1}{2}(y_n + y_m)\| \geq d$$

4. By the parallelogram law, we have

$$\|y_n - y_m\|^2 = 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - 4\|x - \frac{1}{2}(y_m + y_n)\|^2$$

5. As  $n, m \rightarrow \infty$ , we have

$$\|y_n - y_m\|^2 \rightarrow 0.$$

6. So,  $(y_n)$  is Cauchy.

### 3.8. ORTHONORMAL COMPLEMENTS AND PROJECTION THEOREM 77

7. Since  $H$  is complete  $\exists \lim y_n = y \in H$ .

8. Since  $S$  is closed  $y \in S$ .

9. By continuity, we get

$$\|x - y\| = \lim \|x - y_n\| = d$$

10. (II). Uniqueness. Let  $y_1$  be another such point.

11. Then, by convexity

$$\|y - y_1\|^2 = 4d^2 - 4\|x - \frac{1}{2}(y + y_1)\|^2 \leq 0$$

12. Thus  $y = y_1$ . ■

**Theorem 3.8.3** *Let  $H$  be a real Hilbert space. Let  $S$  be a closed convex subset of  $H$ ,  $y \in S$  and  $x \in H$ . Then*

$$\|x - y\| = \inf_{z \in S} \|x - z\|$$

*if and only if*

$$\forall z \in S, \quad (x - y, z - y) \leq 0.$$

**Proof:**

1. (I). Let  $x \in H$ ,  $y, z \in S$  and  $\lambda \in (0, 1)$ .

2. Suppose  $\|x - y\| = \inf_{z \in S} \|x - z\|$ .

3. By convexity we have

$$\begin{aligned} \|x - y\|^2 &\leq \|(x - y) - \lambda(z - y)\|^2 \\ &\leq \|x - y\|^2 - 2\lambda(x - y, z - y) + \lambda^2 \|z - y\|^2 \end{aligned}$$

4. Therefore, as  $\lambda \rightarrow 0$ , we get

$$(x - y, z - y) \leq 0.$$

5. (II). Let  $x \in H$  and  $y \in S$ .

6. Suppose  $\forall z \in S, (x - y, z - y) \leq 0$ .

7. Then  $\forall z \in S$

$$\|x - y\|^2 - \|x - z\|^2 = 2(x - y, z - y) - \|z - y\|^2 \leq 0.$$

8. Thus

$$\|x - y\|^2 \leq \|x - z\|^2$$

■

**Theorem 3.8.4 Orthogonal Projection.** *Let  $H$  be a Hilbert space and  $S$  be a closed subspace of  $H$ . Then  $\forall x \in H$  there exist unique  $y \in S$  and  $z \in S^\perp$  such that*

$$x = y + z.$$

**Proof:**

1. (I). Existence. If  $x \in S$ , then let  $y = x \in S$  and  $z = 0 \in S^\perp$  so that  $x = y + z$ .
2. Let  $x \notin S$ .
3. Let  $y \in S$  be the unique point closest to  $x$ .
4. Let  $z = x - y$ .
5. Then  $\|z\| = \inf_{w \in S} \|x - w\|$ .
6. Claim:  $z \in S^\perp$ .
7. Let  $w \in S$  and  $\lambda \in \mathbb{C}$ .
8. Then

$$\|z\|^2 \leq \|z - \lambda w\|^2 = \|z\|^2 - 2\operatorname{Re} \lambda(w, z) + |\lambda|^2 \|w\|^2$$

9. Thus

$$-2\operatorname{Re} \lambda(w, z) + |\lambda|^2 \|w\|^2 \geq 0.$$

10. We obtain

$$\operatorname{Re}(w, z) \leq 0, \quad \operatorname{Im}(w, z) \leq 0.$$

11. Since this is true for any  $w \in S$ , we obtain

$$(w, z) = 0.$$

12. Thus,  $z \in S^\perp$ .

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13. (II). Uniqueness. Suppose  $x = y + z = y_1 + z_1$ .
14. Then  $y - y_1 = z_1 - z$ .
15. Since  $y - y_1 \in S$  and  $z - z_1 \in S^\perp$ , then  $y - y_1 = z - z_1 = 0$ .

■

- **Orthogonal decomposition of  $H$ .** If every element of  $H$  can be uniquely represented as the sum of an element of  $S$  and an element of  $S^\perp$ , then  $H$  is the **direct sum** of  $S$  and  $S^\perp$ , which is denoted by

$$H = S \oplus S^\perp$$

- The union of a basis of  $S$  and a basis of  $S^\perp$  gives a basis of  $H$ .
- **Orthogonal projection.** An orthogonal decomposition  $H = S \oplus S^\perp$  of  $H$  induces a map  $P : H \rightarrow S$  defined by  $P(y + z) = y$ , where  $y \in S$  and  $z \in S^\perp$ .
- **Examples.**

- **Theorem 3.8.5** *Let  $H$  be a Hilbert space and  $S$  be a closed subspace of  $H$ . Then*

$$(S^\perp)^\perp = S.$$

**Proof:**

1. Let  $x \in S$ .
2. Then  $x \perp S^\perp$ , or  $x \in S^{\perp\perp}$ .
3. So,  $S \subseteq S^{\perp\perp}$ .
4. Let  $x \in S^\perp$ .
5. Since  $S$  is closed, there exist  $y \in S$  and  $z \in S^\perp$  such that  $x = y + z$ .
6. Then  $y \in S^{\perp\perp}$ .
7. Since  $S^{\perp\perp}$  is a vector space,  $z = x - y \in S^{\perp\perp}$ .
8. Since  $z \in S^{\perp\perp}$  and  $z \in S^\perp$ .
9. Thus,  $z = 0$ , and  $x = y \in S$ .
10. Therefore,  $S^{\perp\perp} \subseteq S$ .

■

### 3.8.1 Homework

- Exercises: [51,52,53,55,56]

### 3.9 Linear Functionals and the Riesz Representation Theorem

- **Examples.**
- $L^2((a, b))$
- $\varphi(f) = (f, g)$  is a linear bounded functional.
- Let  $x_0 \in (a, b)$ . Then  $\varphi(f) = f(x_0)$  is a linear but unbounded functional.
- $\mathbb{C}^n$
- Let  $k \in \{1, \dots, n\}$ . Then  $\varphi(x) = x_k = (x, e_k)$  is a linear bounded functional.

**Lemma 3.9.1** *Let  $E$  be an inner product space and  $f : E \rightarrow \mathbb{C}$  be a bounded linear functional on  $E$ . Then*

$$\dim(N(f))^\perp \leq 1.$$

**Proof:**

1. If  $f = 0$ , then  $N(f) = E$ .
2. Therefore,  $(N(f))^\perp = \{0\}$  and  $\dim(N(f))^\perp = 0$ .
3. Suppose that  $f \neq 0$ .
4. Since  $f$  is bounded and linear it is continuous.
5. Therefore,  $N(f)$  is a closed subspace of  $E$ .
6. Thus,  $(N(f))^\perp$  is not empty.
7. Let  $x, y \in (N(f))^\perp$  be two nonzero vectors.
8. Then  $f(x) \neq 0$  and  $f(y) \neq 0$ .
9. Therefore, there exists  $\alpha \neq 0 \in \mathbb{C}$  such that  $f(x + \alpha y) = 0$ .
10. Hence,  $x + \alpha y \in N(f)$ .
11. Since  $x, y \in (N(f))^\perp$ , we also have  $x + \alpha y \in (N(f))^\perp$ .
12. Thus  $x + \alpha y = 0$ .

13. Therefore,  $x$  and  $y$  are linearly dependent, and, therefore,

$$\dim(N(f))^\perp = 1.$$

■

**Theorem 3.9.1 Riesz Representation Theorem.** *let  $H$  be a Hilbert space and  $f : H \rightarrow \mathbb{C}$  be a bounded linear functional on  $H$ . There exists a unique  $x_0 \in H$  such that*

$$f(x) = (x, x_0)$$

for all  $x \in H$ . Moreover,

$$\|f\| = \|x_0\|.$$

**Proof:**

1. (I). Existence.  
If  $f = 0$ , then  $x_0 = 0$ .
2. Suppose  $f \neq 0$ .
3. Then  $\dim(N(f))^\perp = 1$ .
4. Let  $u \in (N(f))^\perp$ .
5. Then  $\forall x \in H$ ,

$$x = y + z$$

where  $y = x - (x, u)u \in N(f)$  and  $z = (x, u)u \in (N(f))^\perp$ .

6. Therefore,  $f(y) = 0$ .
7. Further,

$$f(x) = f(z) = (x, u)f(u) = (x, x_0),$$

where

$$x_0 = (f(u))^*u.$$

8. (II). Uniqueness. Suppose there exists  $x_0$  and  $x_1$  such that  $\forall x \in H$

$$f(x) = (x, x_0) = (x, x_1).$$

9. Then  $\forall x \in H$

$$(x, x_0 - x_1) = 0.$$

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10. Thus,  $(x_0 - x_1) \in H^\perp = \{0\}$ .

11. So,  $x_0 = x_1$ .

12. Finally, we have

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{x \neq 0} \frac{|(x, x_0)|}{\|x\|} \leq \|x_0\| .$$

13. On another hand

$$\frac{|f(x_0)|}{\|x_0\|} = \frac{|(x_0, x_0)|}{\|x_0\|} = \|x_0\| .$$

14. Thus,  $\|f\| = \|x_0\|$ .

■

- **Remarks.** The set  $H'$  of all bounded linear functionals on a Hilbert space is a Banach space, called the **dual space**.
- The dual space  $H'$  of a Hilbert space  $H$  is isomorphic to  $H$ .

#### 3.9.1 Homework

- Exercises:

### 3.10 Separable Hilbert Spaces

• **Definition 3.10.1** **Separable Spaces.** *A Hilbert space is **separable** if it is finite-dimensional or contains a complete orthonormal sequence.*

• **Examples.**

•  $L^2([-\pi, \pi])$

•  $l^2$

• **Example (Non-separable Hilbert Space).** Let  $H$  be the space of all complex valued functions  $f : \mathbb{R} \rightarrow \mathbb{C}$  on  $\mathbb{R}$  such that they vanish everywhere except a countable number of points in  $\mathbb{R}$  and

$$\sum_{f(x) \neq 0} |f(x)|^2 < \infty.$$

Define the inner product by

$$(f, g) = \sum_{f(x)g(x) \neq 0} f(x)g(x)^*.$$

Then for any sequence  $f_n$  in  $H$ , there are non-zero functions  $f$  such that  $(f, f_n) = 0$  for all  $n \in \mathbb{N}$ . Therefore,  $H$  is not separable.

• **Theorem 3.10.1** *Let  $H$  be a separable Hilbert space. Then  $H$  contains a countable dense subset.*

**Proof:**

1. Let  $(e_n)$  be a complete orthonormal sequence in  $H$ .
2. Define the set

$$S = \left\{ \sum_{k=1}^n (\alpha_k + i\beta_k)e_k \mid \alpha_k, \beta_k \in \mathbb{Q}, n \in \mathbb{N} \right\}$$

3. Then  $S$  is countable.

4. Also,  $\forall x \in H$ ,

$$\lim_{n \rightarrow \infty} \left\| \sum_{k=1}^n (x, e_k) e_k - x \right\| = 0.$$

5. Therefore,  $S$  is dense in  $H$ . ■

• **Theorem 3.10.2** *Let  $H$  be a separable Hilbert space. Then every orthogonal set  $S$  in  $H$  is countable.*

**Proof:**

1. Let  $S$  be an orthogonal set in  $H$ .

2. Let

$$S_1 = \left\{ \frac{x}{\|x\|} \mid x \in S \right\}.$$

3. Then  $\forall x, y \in S_1, x \neq y$ ,

$$\|x - y\|^2 = 2.$$

4. Consider the collection of balls  $B_{2^{-1/2}}(x)$  for every  $x \in S_1$ .

5. Then, for any  $x, y \in S_1, x \neq y$ ,

$$B_{2^{-1/2}}(x) \cap B_{2^{-1/2}}(y) = \emptyset.$$

6. Since  $H$  is separable, it has a countable dense subset  $A$ .

7. Since  $A$  is dense in  $H$  it must have at least one point in every ball  $B_{2^{-1/2}}(x)$ .

8. Therefore,  $S_1$  must be countable.

9. Thus  $S$  is countable. ■

• **Definition 3.10.2 Unitary Linear Transformations.** *Let  $H_1$  and  $H_2$  be Hilbert spaces. A linear map  $T : H_1 \rightarrow H_2$  is **unitary** if  $\forall x, y \in H_1$*

$$(T(x), T(y))_{H_2} = (x, y)_{H_1}.$$

**Definition 3.10.3 Hilbert Space Isomorphism.** *let  $H_1$  and  $H_2$  be Hilbert spaces. Then  $H_1$  is isomorphic to  $H_2$  if there exists a linear unitary bijection  $T : H_1 \rightarrow H_2$  (called a **Hilbert space isomorphism**).*

- **Remark.** Every Hilbert space isomorphism has unit norm

$$\|T\| = 1.$$

**Theorem 3.10.3** *1. Every infinite-dimensional separable Hilbert space is isomorphic to  $l^2$ .*

*2. Every finite-dimensional separable Hilbert space  $H$  is isomorphic to  $\mathbb{C}^n$ , where  $n = \dim H$ .*

**Proof:**

1. (1). Let  $H$  be infinite-dimensional.
2. Let  $(e_n)$  be a complete orthonormal sequence in  $H$ .
3. Let  $x \in H$ .
4. Let  $x_n = (x, e_n)$ .
5. This defines a linear bijection  $T : H \rightarrow l^2$  by

$$T(x) = (x_n).$$

6. Let  $x, y \in H$ .
7. Then

$$(T(x), T(y))_{l^2} = \sum_{n=1}^{\infty} x_n y_n^*$$

8. On another hand

$$(x, y)_H = \left( \sum_{n=1}^{\infty} x_n e_n, y \right) = \sum_{n=1}^{\infty} x_n (e_n, y) = \sum_{n=1}^{\infty} x_n y_n^*$$

9. Therefore  $T$  is unitary, and is, therefore, an isomorphism from  $H$  onto  $l^2$ . ■

- **Remarks.**
- Isomorphism of Hilbert spaces is an equivalence relation.
- All separable infinite-dimensional Hilbert spaces are isomorphic.

### 3.10.1 Homework

- Exercises: 3.12[58,60,61,62]



# Chapter 4

## Linear Operators on Hilbert Spaces

### 4.1 Examples of Operators

- Let  $E$  be an inner product space. A linear operator is a linear map  $A : E \rightarrow E$ .
- Only linear operators will be considered.
- An operator  $A : E \rightarrow E$  is **bounded** if  $\exists K \in \mathbb{R}$  such that  $\forall x \in E$ ,

$$\| Ax \| \leq K \| x \| .$$

- The **norm** of an operator  $A$  is

$$\| A \| = \sup_{x \in E, x \neq 0} \frac{\| Ax \|}{\| x \|}$$

- A linear operator is bounded if and only if it is continuous.
- **Identity Operator**  $I : E \rightarrow E$  is defined by

$$\text{Id } x = x, \quad \forall x \in E$$

Obviously,

$$\| \text{Id} \| = 1 .$$

- **Null Operator**  $0 : E \rightarrow E$  is defined by

$$0x = 0, \quad \forall x \in E$$

Obviously,

$$\|0\| = 0.$$

- **Operators on finite-dimensional Hilbert spaces.**

Let  $e_k$ , ( $k = 1, \dots, n$ ), be the canonical orthonormal basis in  $\mathbb{C}^n$ . Let  $A : E \rightarrow E$  be an operator and define

$$(Ae_j, e_k) = A_{kj}.$$

Then

$$Ae_j = \sum_{k=1}^n e_k A_{kj}$$

and for any  $x \in E$

$$x = \sum_{j=1}^n e_j x_j, \quad x_j = (x, e_j)$$

we have

$$Ax = \sum_{j,k=1}^n e_k A_{kj} x_j$$

There is a one-to-one correspondence between the operators on  $\mathbb{C}^n$  and the  $n \times n$  complex matrices.

- The **trace** of the operator  $A$  is defined by

$$\text{tr } A = \sum_{k=1}^n (Ae_k, e_k) = \sum_{k=1}^n A_{kk}.$$

- The **adjoint** of the operator  $A$  is defined by

$$(A^\dagger e_j, e_k) = (e_j, Ae_k).$$

- The matrix  $(A^\dagger)_{jk}$  of the adjoint operator  $A^\dagger$  is Hermitian conjugate of the matrix  $A_{jk}$ , that is

$$(A^\dagger)_{jk} = (A_{kj})^*.$$

- The norm of the operator  $A$  is defined by

$$\|A\|^2 = \text{tr} AA^\dagger = \sum_{j,k=1}^n |A_{jk}|^2.$$

- Every operator on a finite-dimensional Hilbert space is bounded.
- **Differential Operator**  $D : C^\infty([a, b]) \rightarrow C^\infty([a, b])$  on the space of smooth functions is defined by

$$(Df)(x) = \frac{df}{dx}.$$

The differential operator is unbounded.

- **Integral Operator**  $K : L^2([a, b]) \rightarrow L^2([a, b])$  is defined by

$$(Kf)(x) = \int_a^b dy K(x, y)f(y).$$

The function  $K(x, y)$  is the **kernel** of the operator  $K$ .

The **trace** of the operator  $K$  is

$$\text{Tr} K = \int_a^b dx K(x, x)$$

when the integral exists.

The **adjoint**  $K^\dagger$  of the operator  $K$  is defined by

$$(K^\dagger f)(x) = \int_a^b dy K^*(y, x)f(y).$$

The **norm** of the operator  $K$  is defined by

$$\|K\|^2 = \text{Tr} KK^\dagger = \int_a^b \int_a^b dx dy |K(x, y)|^2.$$

when it exists.

- The operator  $K : L^2([a, b]) \rightarrow L^2([a, b])$  is bounded if its norm is finite.

*Proof:* By Schwarz inequality.

- **Multiplication Operator**  $\mu_f : L^2([a, b]) \rightarrow L^2([a, b])$ , where  $f \in C([a, b])$  is a continuous function called the **multiplier**, is defined by

$$(\mu_f g)(x) = f(x)g(x).$$

The operator  $\mu_f$  is bounded and

$$\|\mu_f\| = \max_{x \in [a, b]} |f(x)|.$$

- Two operators  $A$  and  $B$  on a vector space  $E$  are **equal** if  $A - B$  is a null operator.
- The set of all operators on a vector space  $E$  is a vector space with the addition and multiplication by scalars defined by

$$(A + B)(x) = A(x) + B(x), \quad (\alpha A)(x) = \alpha A(x).$$

- The product  $AB$  of the operators  $A$  and  $B$  is the composition of  $A$  and  $B$ .
- The integer powers of an operator are defined as the multiple composition of the operator with itself, i.e.

$$A^0 = \text{Id} \quad A^1 = A, \quad A^2 = AA, \dots$$

- In general,  $AB \neq BA$ .
- The operators  $A$  and  $B$  are **commuting operators** if  $AB = BA$ .
- Operators form an algebra.
- **Noncommuting Operators.** The differential operator  $D$  and the operator of multiplication by a nonconstant function  $\mu_f$  do not commute.

**Theorem 4.1.1** *The product of bounded operators is bounded and*

$$\|AB\| \leq \|A\| \|B\|$$

*Proof:*

1.

$$\| ABx \| \leq \| A \| \| Bx \| \leq \| A \| \| B \| \| x \| .$$

■

- **Theorem 4.1.2** *A bounded operator on a separable infinite-dimensional Hilbert space can be represented by an infinite matrix.*

**Proof:**1. Let  $A_{ij} = (Ae_j, e_i)$ .

2. Then

$$Ax = \sum_{k=1}^{\infty} (x, e_k) Ae_k$$

and

$$(Ax, e_k) = \sum_{j=1}^{\infty} A_{kj}(x, e_j).$$

■

### 4.1.1 Homework

- Exercises: 4.13[1,2,3]

## 4.2 Bilinear Functionals and Quadratic Forms

**Definition 4.2.1 Bilinear Functional.** Let  $E$  be a complex vector space. A **bilinear functional** on  $E$  is a mapping  $\phi : E \times E \rightarrow \mathbb{C}$  which is linear in the first argument and anti-linear in the second.

- **Examples.**

**Definition 4.2.2** Let  $E$  be a normed space and  $\phi$  be a **bilinear functional** on  $E$ . Then

1.  $\phi$  is **symmetric** if  $\forall x, y \in E$ ,

$$\phi(x, y) = \phi(y, x)^*.$$

2.  $\phi$  is **positive** if  $\forall x \in E$ ,

$$\phi(x, x) \geq 0.$$

3.  $\phi$  is **strictly positive** if  $\forall x \neq 0 \in E$ ,

$$\phi(x, x) > 0.$$

4.  $\phi$  is **bounded** if there exists a constant  $K > 0$  such that  $\forall x, y \in E$ ,

$$|\phi(x, y)| \leq K \|x\| \|y\|.$$

5. The **norm** of a bounded bilinear functional is

$$\|\phi\| = \sup_{x, y \neq 0} \frac{|\phi(x, y)|}{\|x\| \|y\|}.$$

so that  $\forall x, y \in E$

$$|\phi(x, y)| \leq \|\phi\| \|x\| \|y\|.$$

**Definition 4.2.3 Quadratic Form.** Let  $E$  be a normed space and  $\phi$  be a bilinear functional on  $E$ . The **quadratic form associated with the bilinear functional  $\phi$**  is the functional  $\Phi : E \rightarrow \mathbb{C}$  defined  $\forall x \in E$  by

$$\Phi(x) = \phi(x, x).$$

A quadratic form on  $E$  is **bounded** if there exists a constant  $K > 0$  such that  $\forall x \in E$

$$|\Phi(x)| \leq K \|x\|^2.$$

The **norm** of a bounded quadratic form is defined by

$$\|\Phi\| = \sup_{x \neq 0} \frac{|\Phi(x)|}{\|x\|^2}$$

so that  $\forall x \in E$

$$|\Phi(x)| \leq \|\Phi\| \|x\|^2.$$

**Theorem 4.2.1 Polarization Identity.** Let  $E$  be a complex vector space,  $\phi$  be a bilinear functional on  $E$  and  $\Phi$  be the associated quadratic form on  $E$ . Then  $\forall x, y \in E$

$$\phi(x, y) = \frac{1}{4} \{ \Phi(x + y) - \Phi(x - y) + i\Phi(x + iy) - i\Phi(x - iy) \}.$$

**Proof:**

1. Use

$$\Phi(\alpha x + \beta y) = |\alpha|^2 \Phi(x) + \alpha \beta^* \phi(x, y) + \alpha^* \beta \phi(y, x) + |\beta|^2 \Phi(y)$$

for  $\alpha, \beta = \pm 1, \pm i$ . ■

**Corollary 4.2.1** Let  $E$  be a vector space,  $\phi_1, \phi_2$  be bilinear functionals on  $E$  and  $\Phi_1, \Phi_2$  be the associated quadratic forms on  $E$ . Then, if  $\Phi_1 = \Phi_2$ , then  $\phi_1 = \phi_2$ .

In particular, if  $A, B$  are operators on  $E$  such that  $\forall x \in E$ ,  $(Ax, x) = (Bx, x)$ , then  $A = B$ . ■

**Proof:** Easy.

- **Theorem 4.2.2** *Let  $E$  be a vector space,  $\phi$  be a bilinear functional on  $E$  and  $\Phi$  be the associated quadratic form on  $E$ . Then,  $\phi$  is symmetric if and only if  $\Phi$  is real.*

**Proof:** Exercise. ■

- **Theorem 4.2.3** *Let  $E$  be a normed space,  $\phi$  be a bilinear functional on  $E$  and  $\Phi$  be the associated quadratic form on  $E$ . Then  $\phi$  is bounded if and only if  $\Phi$  is bounded. Moreover,*

$$\|\Phi\| \leq \|\phi\| \leq 2\|\Phi\|.$$

**Proof:**

1. By definition

$$\|\Phi\| \leq \|\phi\|.$$

2. Suppose  $\Phi$  is bounded.

3. By the polarization identity and the parallelogram law

$$|\phi(x, y)| \leq \|\Phi\| (\|x\|^2 + \|y\|^2).$$

- **Theorem 4.2.4** *Let  $E$  be a normed space,  $\phi$  be a bilinear functional on  $E$  and  $\Phi$  be the associated quadratic form on  $E$ . Let  $\phi$  be symmetric and bounded. Then  $\Phi$  is bounded and*

$$\|\Phi\| = \|\phi\|.$$

**Proof:**

1. We have

$$\|\Phi\| \leq \|\phi\|.$$

2. By the polarization identity

$$\operatorname{Re} \phi(x, y) = \frac{1}{4} [\Phi(x + y) - \Phi(x - y)]$$

3. By the parallelogram law

$$|\operatorname{Re} \phi(x, y)| \leq \frac{1}{2} \|\Phi\| [\|x\|^2 + \|y\|^2]$$

4. Let

$$\phi(x, y) = |\phi(x, y)|e^{i\alpha}.$$

5. Then  $\forall x, y \in E$  such that  $\|x\| = \|y\| = 1$ , we have

$$|\phi(x, y)| = \phi(e^{-i\alpha}x, y) = \operatorname{Re} \phi(e^{-i\alpha}x, y) \leq \|\Phi\|.$$

6. Thus

$$\|\phi\| \leq \|\Phi\|$$

■

**Theorem 4.2.5** *Let  $H$  be a Hilbert space and  $A$  be a bounded operator on  $H$ . Let  $\phi$  be a bilinear functional on  $H$  defined  $\forall x, y \in H$  by  $\phi(x, y) = (x, Ay)$ . Then  $\phi$  is bounded and*

$$\|\phi\| = \|A\|.$$

**Proof:**

1. By Schwartz inequality

$$|\phi(x, y)| \leq \|A\| \|x\| \|y\|$$

2. Thus,  $\|\phi\| \leq \|A\|$ .

3. For any  $Ax \neq 0$  we have

$$\|Ax\|^2 = |\phi(Ax, x)| \leq \|\phi\| \|Ax\| \|x\|$$

4. Thus,

$$\|Ax\| \leq \|\phi\| \|x\|$$

■

**Theorem 4.2.6** *Let  $H$  be a Hilbert space and  $\phi$  be a bounded bilinear functional on  $H$ . There exists a unique bounded operator  $A$  on  $H$  such that  $\forall x, y \in H$*

$$\phi(x, y) = (x, Ay).$$

**Proof:**

1. By Riesz theorem there exists a unique  $z = Ay \in H$  such that and  $\phi(x, y) = (x, Ay)$ .

2. Claim: the mapping  $y \mapsto z = Ay$  is bounded linear operator on  $E$ .

3. We also have

$$|(x, Ay)| = |\phi(x, y)| \leq \|\phi\| \|x\| \|y\|$$

4. Thus, for  $x = Ay \neq 0$  we have

$$\|Ay\|^2 \leq \|\phi\| \|Ay\| \|y\|$$

and

$$\|Ay\| \leq \|\phi\| \|y\|$$

5. Uniqueness is trivial. ■

**Definition 4.2.4 Coercive (Elliptic) Functional.** *Let  $E$  be a normed space,  $\phi$  be a bilinear functional on  $E$  and  $\Phi$  be the associated quadratic form on  $E$ . Then  $\phi$  is **elliptic** if there exists a constant  $K > 0$  such that  $\forall x \in E$*

$$\Phi(x) = \phi(x, x) \geq K \|x\|^2.$$

• **Example.**

**Theorem 4.2.7 Lax-Milgram Theorem.** *Let  $H$  be a Hilbert space and  $\phi$  be a bounded elliptic bilinear functional on  $H$  and  $A$  be the unique operator on  $H$  such that  $\forall x, y \in H$ ,  $\phi(x, y) = (x, Ay)$ . Let  $f$  be a bounded linear functional on  $H$ . Then there exists a unique vector  $x_f \in H$  such that  $\forall x \in H$*

$$f(x) = \phi(x, x_f) = (x, Ax_f).$$

**Proof:**

1. Let  $A$  be such that  $\forall x, y \in H$

$$\phi(x, y) = (x, Ay).$$

2. Claim:  $A$  is bijective.

3. Claim:  $A$  is injective.

4. We have

$$K \|x\|^2 \leq \phi(x, x) \leq \|Ax\| \|x\|$$

5. Thus  $\forall x \in H$

$$K \|x\| \leq \|Ax\|.$$

6. Therefore, if  $Ax = 0$ , then  $x = 0$ .

7. Also, if  $Ax_1 = Ax_2$ , then  $x_1 = x_2$ .

8. So,  $A$  is injective.

9. Let  $R(A)$  be the range of  $A$ .

10. Claim:  $R(A)$  is closed.

11. Let  $x_n$  be a sequence in  $H$  and  $y \in H$  such that

$$\lim \|Ax_n - y\| = 0.$$

12. Then  $(x_n)$  is Cauchy and, therefore, there exists  $x \in H$  such that

$$\lim \|x_n - x\| = 0$$

13. Then

$$\lim \|Ax_n - Ax\| = 0$$

14. Therefore,  $Ax = y$  and, hence,  $y \in R(A)$ .

15. Thus,  $R(A)$  is closed.

16. Claim:  $A$  is surjective, that is,  $R(A) = H$ .

17. By contradiction. Suppose  $\exists x \neq 0 \in H$  which is orthogonal to  $R(A)$ .

18. Then

$$0 = |(x, Ax)| = |\phi(x, x)| \geq K \|x\|^2,$$

which is a contradiction.

19. Thus,  $A$  is bijective.

20. Let  $f$  be a bounded linear functional on  $H$ .

21. Then there exists  $x_0 \in H$  such that  $\forall x \in H$ ,

$$f(x) = (x, x_0).$$

22. Then, there exists a unique  $x_f \in H$  such that  $x_0 = Ax_f$  and  $\forall x \in H$ ,

$$f(x) = (x, Ax_f) = \phi(x, x_f).$$

■

### 4.2.1 Homework

- Exercises: 4.13[4,5]

## 4.3 Adjoint and Self-Adjoint Operators

- Definition 4.3.1 Adjoint Operator.** *Let  $H$  be a Hilbert space and  $A$  a bounded operator on  $H$ . The **adjoint operator**  $A^*$  of  $A$  is defined by*

$$(Ax, y) = (x, A^*y), \quad \forall x, y \in H.$$

- The *adjoint operation*  $*$  :  $L(H) \rightarrow L(H)$  is an operator on the space of all bounded operators, which has the properties

$$(\alpha A + \beta B)^* = \alpha^* A^* + \beta^* B^*$$

$$(A^*)^* = A$$

$$(AB)^* = B^* A^*$$

$$I^* = I$$

where  $\alpha, \beta \in \mathbb{C}$  and  $I$  is the identity operator.

- Theorem 4.3.1** *Let  $H$  be a Hilbert space and  $A$  a bounded operator on  $H$ . Then the adjoint operator  $A^*$  is bounded and*

$$\|A\| = \|A^*\|, \quad \text{and} \quad \|AA^*\| = \|A\|^2.$$

**Proof:**

1. ■

- Definition 4.3.2 Self-adjoint (Hermitian) Operator.** *Let  $H$  be a Hilbert space and  $A$  a bounded operator on  $H$ . The operator  $A$  is **self-adjoint** if  $A = A^*$ .*

- Examples.**

- Theorem 4.3.2** *Let  $H$  be a Hilbert space and  $A$  a bounded operator on  $H$ . Then the operators  $AA^*$  and  $A + A^*$  are self-adjoint.*

**Proof:** Easy. ■

- **Theorem 4.3.3** Let  $H$  be a Hilbert space and  $A$  and  $B$  be bounded self-adjoint operators on  $H$ . The operator  $AB$  is self-adjoint if and only if  $AB = BA$ .

**Proof:** Easy. ■

- **Corollary 4.3.1** Let  $H$  be a Hilbert space,  $A$  be bounded operator on  $H$  and  $\alpha_0, \alpha_1, \dots, \alpha_n$  be real constants. Then the operator  $\alpha_0 I + \alpha_1 A + \dots + \alpha_n A^n$  is self-adjoint.

**Proof:** Exercise. ■

- **Definition 4.3.3** Let  $H$  be a Hilbert space and  $D(A)$  and  $D(B)$  be subspaces of  $H$ , and let  $A : D(A) \rightarrow H$  and  $B : D(B) \rightarrow H$  be operators. Then  $B$  is an **adjoint operator of  $A$**  if

$$(Ax, y) = (x, By), \quad \forall x \in D(A), y \in D(B).$$

- **Remarks.**

- In general, an adjoint is not unique!
- If  $D(A)$  is dense in  $H$  then the adjoint is unique.

- **Examples.**

- **Theorem 4.3.4** Let  $H$  be a Hilbert space and  $A$  be a bounded operator on  $H$ . Then there exist unique self-adjoint operators  $B$  and  $C$  on  $H$  such that  $A = B + iC$  and  $A^* = B - iC$ .

**Proof:**

1. ■

- **Theorem 4.3.5** Let  $H$  be a Hilbert space and  $A$  be a bounded self-adjoint operator on  $H$ . Then

$$\|A\| = \sup_{x \neq 0} \frac{|(Ax, x)|}{\|x\|^2}$$

**Proof:**

1.



- **Definition 4.3.4** **Anti-selfadjoint (Anti-Hermitian) Operator.** *Let  $H$  be a Hilbert space and  $A$  be a bounded operator on  $H$ . Then  $A$  is **anti-self-adjoint** if  $A = -A^*$ .*

- **Example.**

### 4.3.1 Homework

- Exercises: 4.13[]

## 4.4 Invertible, Normal, Isometric and Unitary Operators

**Definition 4.4.1 Inverse Operator.** Let  $E$  be a vector space  $E$  and  $A$  be an operator on  $E$  with a domain  $D(A)$  and a range  $R(A)$ . The operator  $A$  is **invertible** if there exists an operator  $A^{-1} : R(A) \rightarrow E$ , called the **inverse** of  $A$ , such that  $\forall x \in D(A)$  and  $y \in R(A)$ ,

$$A^{-1}Ax = x, \quad \text{and} \quad AA^{-1}y = y.$$

- The inverse of an invertible operator is unique.
- Domains and ranges

$$D(A^{-1}) = R(A), \quad R(A^{-1}) = D(A).$$

**Definition 4.4.2 Kernel of an Operator.** Let  $E$  be a vector space  $E$  and  $A$  be an operator on  $E$  with a domain  $D(A)$  and a range  $R(A)$ . The **kernel** of the operator  $A$  is the set of all vectors in  $E$  mapped to zero, that is

$$\text{Ker } A = \{x \in E \mid Ax = 0\}$$

**Theorem 4.4.1** Let  $E$  be a vector space  $E$  and  $A$  and  $B$  be linear operators on  $E$ . Then:

1.  $A^{-1}$  is a linear operator.
2.  $A$  is invertible if and only if  $Ax = 0$  implies  $x = 0$ , or  $\text{Ker } A = \{0\}$ .
3. If  $A$  is invertible and  $\{x_j\}_{j=1}^n$  is a collection of linearly independent vectors, then  $\{Ax_j\}_{j=1}^n$  is a collection of linearly independent vectors.
4. If  $A$  and  $B$  are invertible, then  $AB$  is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

**Proof:** Easy.



• **Corollary 4.4.1** *An invertible operator  $A : E \rightarrow E$  on a finite-dimensional vector space  $E$  is surjective, that is  $R(A) = E$ .*

• This is not true in infinite dimensions.

• **Example.**

• One-sided shift operator.

• Two-sided shift operator.

• Inverse of a bounded operator is not necessarily bounded.

• **Example.**  $l^2$

• The inverse of an invertible operator on a finite-dimensional vector space is bounded.

• **Theorem 4.4.2** *Let  $H$  be a Hilbert space and  $A$  be a bounded invertible operator on  $H$  such that  $R(A) = H$  and  $A^{-1}$  is bounded. Then  $A^*$  is invertible and*

$$(A^*)^{-1} = (A^{-1})^*.$$

**Proof:**

1. Show that  $\forall x \in H$

$$(A^{-1})^* A^* x = A^* (A^{-1})^* x = x$$



• **Corollary 4.4.2** *Let  $H$  be a Hilbert space and  $A$  be a bounded invertible self-adjoint operator on  $H$  such that  $R(A) = H$  and  $A^{-1}$  is bounded. Then  $A^{-1}$  is self-adjoint.*

**Proof:** Easy.



• **Definition 4.4.3 Normal Operator.** *Let  $H$  be a Hilbert space and  $A$  be a bounded operator on  $H$ . Then  $A$  is **normal** if*

$$AA^* = A^*A.$$

- Every self-adjoint operator is normal.

**Theorem 4.4.3** *Let  $H$  be a Hilbert space and  $A$  be a bounded operator on  $H$ . Then  $A$  is normal if and only if  $\forall x \in H$ ,*

$$\|Ax\| = \|A^*x\|.$$

**Proof:**

1. If  $A$  is normal, then

$$(A^*Ax, x) = \|A^*x\|^2$$

2. So,  $\|Ax\| = \|A^*x\|$ .

3. If  $\|Ax\| = \|A^*x\|$ , then

$$(A^*Ax, x) = (A^*Ax, x).$$

4. Therefore,

$$AA^* = A^*A$$

5. Thus  $A$  is normal. ■

- The condition  $\|Ax\| = \|A^*x\|$  is stronger than  $\|A\| = \|A^*\|$ .

- **Examples.**

**Theorem 4.4.4** *Let  $H$  be a Hilbert space,  $A$  be a bounded normal operator on  $H$  and  $\alpha \in \mathbb{C}$ . Then  $\alpha I - A$  is normal.*

**Proof:** Easy. ■

**Theorem 4.4.5** *Let  $H$  be a Hilbert space,  $A$  be a bounded operator on  $H$  and  $B$  and  $C$  be self-adjoint operators such that  $A = B + iC$ . Then  $A$  is normal if and only if  $A$  and  $B$  commute.*

**Proof:** Easy. ■

**Definition 4.4.4 Isometric Operator.** Let  $H$  be a Hilbert space and  $A$  be a bounded operator on  $H$ . Then  $A$  is **isometric** if

$\forall x \in H,$

$$\| Ax \| = \| x \| .$$

• **Examples.**

**Definition 4.4.5 Unitary Operator.** Let  $H$  be a Hilbert space and  $A$  be a bounded operator on  $H$  such that  $D(A) = R(A) = H$ .

Then  $A$  is **unitary** if

$$AA^* = A^*A = I \quad \text{on } H.$$

**Theorem 4.4.6** Let  $H$  be a Hilbert space and  $A$  be a bounded operator on  $H$ . Then  $A$  is isometric if and only if  $A$  is unitary.

**Proof:**

1. If  $\| Ax \|^2 = \| x \|^2$ , then  $(A^*Ax, x) = (AA^*x, x) = (x, x)$  for any  $x \in H$ .
2. So,  $A$  is unitary.
3. Similarly, if  $A$  is unitary, then  $A$  is isometric.

■

• Isometric operators preserve the inner product.

**Theorem 4.4.7** Let  $H$  be a Hilbert space and  $A$  be a bounded operator on  $H$ . Then  $A$  is unitary if and only if  $A$  is invertible and

$$A^{-1} = A^*.$$

**Proof:** Easy.

■

**Theorem 4.4.8** Let  $H$  be a Hilbert space and  $A$  be a bounded unitary operator on  $H$ . Then

1.  $A$  is isometric.
2.  $A$  is normal.
3.  $A^{-1}$  and  $A^*$  are unitary, and, therefore, isometric and normal.

**Proof:** Easy.

1.



- A normal operator is not necessarily unitary.
- **Examples.**
- $l^2$ . Let  $(x_n)_{n \in \mathbb{Z}}$  be a sequence. The operator  $Ax_n = x_{n-1}$  is unitary.
- $L^2([0, 1])$ . The operator  $Af(t) = f(1 - t)$  is unitary.

#### 4.4.1 Homework

- Exercises: 4.13[15,16,17,18,19,20,21,22,23,25,26,27,28,29]

## 4.5 Positive Operators

**Definition 4.5.1 Positive Operator.** *Let  $H$  be a Hilbert space. An operator  $A$  is **positive** if it is self-adjoint and  $\forall x \in H$*

$$(Ax, x) \geq 0.$$

**Definition 4.5.2 Strictly Positive Operator.** *Let  $H$  be a Hilbert space and  $A$  be a self-adjoint operator on  $H$ . Then  $A$  is **strictly positive** (or **positive definite**) if  $\forall x \in H, x \neq 0$ ,*

$$(Ax, x) > 0.$$

- **Examples.**  $L^2([0, 1])$

**Theorem 4.5.1** *Let  $H$  be a Hilbert space and  $A$  be a bounded operator on  $H$ . Then the operators  $AA^*$  and  $A^*A$  are positive.*

**Proof:** Easy. ■

**Theorem 4.5.2** *Let  $H$  be a Hilbert space and  $A$  be a invertible positive operator on  $H$ . Then the inverse operator  $A^{-1}$  is positive.*

**Proof:**

1. Let  $y \in H$ .
2. There is an  $x \in H$  such that  $Ax = y$ .
3. We have

$$(A^{-1}y, y) = (x, Ax) \geq 0. \quad \blacksquare$$

- **Remarks.**

- If  $A$  is positive, then we write

$$A \geq 0$$

- If  $A$  and  $B$  are two self-adjoint operators such that  $A - B$  is positive, then we write

$$A - B \geq 0, \text{ or } A \geq B.$$

**Proposition 4.5.1** *Let  $H$  be a Hilbert space,  $A, B, C, D$  be self-adjoint operators on  $H$  and  $\alpha \in \mathbb{R}, \alpha \geq 0$ . Then*

- 1. *If  $A \geq B$  and  $C \geq D$ , then  $A + C \geq B + D$ .*
- 2. *If  $A \geq 0$ , then  $\alpha A \geq 0$ .*
- 3. *If  $A \geq B$  and  $B \geq C$ , then  $A \geq C$ .*

**Proof:** Exercise.

**Theorem 4.5.3** *Let  $H$  be a Hilbert space and  $A$  be a bounded self-adjoint operator on  $H$  such that  $\|A\| \leq 1$ . Then  $A \leq I$ .*

**Proof:**

1. We have  $\forall x \in H$

$$((A - I)x, x) = (\|A\| - 1) \|x\|^2 \leq 0.$$

2. Thus  $A \leq I$ . ■

**Corollary 4.5.1** *Let  $H$  be a Hilbert space and  $A$  be a positive operator on  $H$ . Then there exists  $\alpha \in \mathbb{R}, \alpha > 0$ , such that  $\alpha A \leq I$ .*

**Proof:** Exercise. ■

- The product of positive operators is not necessarily positive.
- **Examples.**  $\mathbb{R}^2$

**Theorem 4.5.4** *Let  $H$  be a Hilbert space and  $A$  and  $B$  be commuting positive operators on  $H$ . Then the product  $AB$  is a positive operator on  $H$ , i.e.  $AB \geq 0$ .*

**Proof:**

1. Let  $A \neq 0$ .

2. Let  $\{P_n\}_{n=1}^\infty$  be a sequence of operators defined by

$$P_1 = \frac{A}{\|A\|}, \quad P_{n+1} = P_n - P_n^2 = P_n(I - P_n).$$

3. The operators  $P_n$  are polynomials in  $A$ . Therefore, they are self-adjoint and commute with  $A \forall n \in \mathbb{N}$ .

4. Claim:

$$A = \|A\| \sum_{n=1}^{\infty} P_n^2$$

5. First, we show (by induction)

$$0 \leq P_n \leq I.$$

6. Then we show that  $\forall x \in H$ ,

$$\sum_{n=1}^{\infty} \|P_n x\|^2 < \infty.$$

7. Therefore,

$$\|P_n x\| \rightarrow 0.$$

8. This leads to the needed representation of  $A$  as a series.

9. Now, we compute  $\forall x \in H$

$$(ABx, x) = \|A\| \sum_{n=1}^{\infty} (BP_n x, P_n x) \geq 0.$$

10. Therefore,  $AB \geq 0$ . ■

- **Remark.** This theorem can be proved much easier by using the square roots  $\sqrt{A}$  and  $\sqrt{B}$  of the operators  $A$  and  $B$  as follows:  $\forall x \in H$

$$(ABx, x) = (\sqrt{A}\sqrt{B}x, \sqrt{A}\sqrt{B}x) \geq 0.$$

- **Corollary 4.5.2** *Let  $H$  be a Hilbert space and  $A$  and  $B$  be self-adjoint operators on  $H$  such that  $A \leq B$ . Let  $C$  be a positive operator on  $H$  that commutes with both  $A$  and  $B$ . Then  $AC \leq BC$ .*

**Proof:** Exercise. ■

**Theorem 4.5.5** *Let  $H$  be a Hilbert space,  $\alpha, \beta \in \mathbb{R}$  be positive real numbers such that  $0 < \alpha < \beta$ , and  $A$  be a positive operator on  $H$  such that*

$$\alpha I \leq A \leq \beta I.$$

*Then*

- 1.  $A$  is invertible.
  2.  $A$  is surjective, that is,  $R(A) = H$ , and, therefore, bijective.
  - 3.

$$\frac{1}{\beta} I \leq A^{-1} \leq \frac{1}{\alpha} I.$$

**Proof:**

1. (1). We have  $\alpha \|x\|^2 \leq (Ax, x) \leq \beta \|x\|^2$ .
2. Thus, if  $Ax = 0$ , then  $x = 0$ .
3. Thus,  $\text{Ker } A = \{0\}$ , and  $A$  is invertible.
4. (2). Claim:  $R(A)$  is closed.
5. Let  $(y_n)$  be as sequence in  $R(A)$  that converges to some  $y \in H$ .
6. Then there is a sequence  $(x_n)$  in  $H$  such that  $y_n = Ax_n$ .
7. We have

$$\alpha \|x_n - x_m\| \leq \|y_n - y_m\|.$$

8. Therefore,  $(x_n)$  is Cauchy and converges to some  $x \in H$ .
9. By continuity of  $A$ , we obtain  $y = Ax$ .
10. Thus,  $y \in R(A)$ , and, hence,  $R(A)$  is closed.
11. Claim:  $(R(A))^\perp = \{0\}$ .
12. Let  $y \in R(A)$ .

13. Then  $\forall x \in H, (Ax, y) = 0$ .
14. Therefore,  $(Ay, y) = 0$ .
15. This means that  $y = 0$ .
16. Therefore,  $(R(A))^\perp = \{0\}$ .
17. Then  $D(A^{-1}) = R(A) = \{0\}^\perp = H$ .
18. Finally, we also have

$$\alpha A^{-1} \leq I \leq \beta A^{-1},$$

which leads to the last assertion of the theorem. ■

**Theorem 4.5.6** *Let  $H$  be a Hilbert space,  $B$  be a self-adjoint operator on  $H$ , and  $\{A_i\}_{i=1}^\infty$  be a sequence of self-adjoint operators on  $H$  such that: i) all operators  $A_n, n \in \mathbb{N}$ , commute with each other as well as with operator  $B$ , and ii)*

$$A_1 \leq A_2 \leq \cdots \leq A_n \leq A_{n+1} \leq \cdots \leq B,$$

*Then there exists a self-adjoint operator  $A$  on  $H$  such that*

$$\lim_{n \rightarrow \infty} A_n x = Ax, \quad \forall x \in H$$

*and*

$$A_n \leq A \leq B, \quad \forall n \in \mathbb{N}.$$

**Proof:**

1. Let  $C_n = B - A_n$ .
2. Then
 
$$C_1 \geq C_2 \geq \cdots \geq 0.$$
3. Then
 
$$C_{n+1}^2 \leq C_n C_{n+1} \leq C_n^2.$$
4. Let  $x \in H$  and  $a_n = (C_n^2 x, x) = \|C_n x\|^2$ .
5. Then  $(a_n)$  is a nonincreasing sequence of nonnegative real numbers.

6. Therefore, the sequence  $(a_n)$  converges to some  $\alpha \geq 0$ .

7. Hence, as  $m, n \rightarrow \infty$  we also have

$$\lim_{m, n \rightarrow \infty} (C_m x, C_n x) = \alpha.$$

8. Therefore, as  $m, n \rightarrow \infty$ , we have

$$\|C_m x - C_n x\|^2 = \|C_m x\|^2 + \|C_n x\|^2 - 2(C_m x, C_n x) \rightarrow 0.$$

9. Therefore,  $C_n x$  is Cauchy, and, hence, converges.

10. Thus,  $A_n x$  also converges for any  $x \in H$ .

11. Finally, we define the operator  $A$  by

$$Ax = \lim_{n \rightarrow \infty} A_n x.$$

12. Then  $A$  is self-adjoint, and  $\forall n \in \mathbb{N}$ ,

$$A_n \leq A \leq B.$$

■

- **Definition 4.5.3 Square Root.** *Let  $H$  be a Hilbert space and  $A$  be a positive operator on  $H$ . A **square root** of  $A$  is a self-adjoint operator  $B$  on  $H$  satisfying  $B^2 = A$ .*

- **Theorem 4.5.7** *Let  $H$  be a Hilbert space and  $A$  be a positive operator on  $H$ . Then  $A$  has a unique positive square root (denoted by  $\sqrt{A}$ ).*

*The square root  $\sqrt{A}$  commutes with every operator commuting with  $A$ .*

**Proof:**

1. Let  $D = \frac{A}{\|A\|}$ .
2. Then  $D \leq I$ .
3. Define the sequence  $(T_n)_{n \in \mathbb{N}}$  by

$$T_0 = 0, \quad T_{n+1} = T_n + \frac{1}{2}(D - T_n^2).$$

4. Claim:

$$0 \leq T_1 \leq \cdots \leq T_n \leq \cdots \leq I.$$

5. Let  $C_n = I - T_n$ .

6. Then, we have,

$$C_{n+1} = \frac{1}{2}C_n^2 + \frac{1}{2}(I - D) \geq 0.$$

7. Also

$$C_{n+1} - C_n = \frac{1}{2}(C_n + C_{n-1})(C_n - C_{n-1}).$$

8. We have  $C_0 = I$  and  $C_1 = I - \frac{1}{2}D$ . Therefore,

$$C_1 \leq C_0.$$

9. Therefore, by induction

$$C_{n+1} \leq C_n.$$

10. The sequence  $C_n$  is a decreasing sequence of self-adjoint operators squeezed between 0 and  $I$ , and, therefore, converges.

11. Therefore, the sequence  $T_n$  converges to a positive operator  $T$ .

12. As  $n \rightarrow \infty$  we obtain

$$T^2 = D.$$

13. Let  $B = \sqrt{\|A\|}T$ . Then

$$B^2 = A.$$

14. The operator  $B$  is positive and commutes with every operator that commutes with  $A$ .

15. Uniqueness. Suppose there are two positive operators  $B_1$  and  $B_2$  such that  $B_1^2 = B_2^2 = A$ .

16. Let  $x \in H$  and  $y = (B_1 - B_2)x$ .

17. Then

$$(B_1y, y) + (B_2y, y) = ((B_1 + B_2)y, y) = ((B_1^2 - B_2^2)x, y) = 0.$$

18. Since both  $B_1$  and  $B_2$  are positive, then

$$(B_1y, y) = (B_2y, y) = 0.$$

19. Now let  $C_1$  be a square root of  $B_1$  and  $C_2$  be a square root of  $B_2$ .

20. Then

$$0 = (B_1y, y) = (C_1y, C_1y), \quad \text{and} \quad 0 = (B_2y, y) = (C_2y, C_2y).$$

21. This means that

$$C_1y = B_1y = C_2y = B_2y = 0.$$

22. Finally, for any  $x \in H$ ,

$$\| (B_1 - B_2)x \|^2 = ((B_1 - B_2)^2x, x) = ((B_1 - B_2)y, x) = 0.$$

23. Thus  $B_1 = B_2$ . ■

### 4.5.1 Homework

- Exercises: 4.13[31,33,34,37,38,39]

## 4.6 Projection Operators

**Definition 4.6.1 Orthogonal Projection Operator.** *Let  $H$  be a Hilbert space and  $S$  be a closed subspace of  $H$ . Then  $H = S \oplus S^\perp$  and for any  $x \in H$  we have*

$$x = y + z,$$

where  $y \in S$  and  $z \in S^\perp$ .

- The vector  $y$  is called the **projection** of  $x$  onto  $S$ .

The **orthogonal projection operator** onto  $S$  is an operator  $P$  on  $H$  defined by

$$Px = y.$$

That is

$$P|_S = I \quad \text{and} \quad P|_{S^\perp} = 0.$$

- **Remarks.**

- Projection is a linear operator.
- Projection is bounded and

$$\|P\| \leq 1.$$

- Zero operator is the projection onto the zero subspace  $\{0\}$  and  $\|0\| = 0$ .
- The identity operator is the projection onto the whole space  $H$ .
- A nonzero projection operator has the unit norm

$$\|P\| = 1.$$

- 

$$P|_{S^\perp} = 0, \quad P|_S = I.$$

**Definition 4.6.2 Orthogonality of Projection Operators.**

*Let  $H$  be a Hilbert space and  $P$  and  $Q$  be two projections operators.*

- Then  $P$  and  $Q$  are **orthogonal** if

$$PQ = QP = 0.$$

- For any two projection operators  $P$  and  $Q$ ,  $PQ = 0$  if and only if  $QP = 0$ .
- The operator  $P^\perp = I - P$  is the projection onto  $S^\perp$ . It is called the **complementary projection**.
- For orthogonal projections we have

$$P^* = P, \quad (P^\perp)^* = P^\perp, \quad P^\perp + P = I, \quad PP^\perp = P^\perp P = 0.$$

- **Examples.**  $l^2$ ,  $L^2([-\pi, \pi])$ .

<p>• <b>Definition 4.6.3 Idempotent Operator.</b> An operator <math>A</math> is <b>idempotent</b> if</p> $A^2 = A.$
---

- Projection operators are idempotent.
- Not every idempotent operator is a projection.

- **Example.**

<p>• <b>Theorem 4.6.1</b> Let <math>H</math> be a Hilbert space and <math>P</math> be a bounded operator on <math>H</math>. Then <math>P</math> is a projection if and only if <math>P</math> is idempotent and self-adjoint.</p>
---

**Proof:**

1. (I). Let  $P$  be a projection onto a closed subspace  $S$ .
2. Then  $P$  is idempotent.
3. Let  $x, y \in H$ . Then

$$(Px, y) = (Px, Py) = (x, Py)$$

4. Thus  $P$  is self-adjoint.
5. (II). Let  $P$  be a self-adjoint idempotent operator.
6. Let  $S$  be a subspace of  $H$  defined by

$$S = \{x \in H \mid Px = x\}.$$

7. Then  $S$  is closed (since  $P$  is bounded).

8. The idempotency leads then to  $P|_S = I$ .
9. Similarly,  $P|_{S^\perp} = 0$ .
10. Thus  $P$  is the projection onto  $S$ .

■

**Corollary 4.6.1** *Let  $H$  be a Hilbert space,  $S$  be a closed subspace of  $H$  and  $P$  be the projection onto  $S$ . Then  $\forall x \in H$*

$$(Px, x) = \|Px\|^2 .$$

**Proof:** Easy.

■

- The sum of two projections is not necessarily a projection.
- **Example.**

**Theorem 4.6.2** *Let  $H$  be a Hilbert space,  $R$  and  $S$  be closed subspaces of  $H$  and  $P_R$  and  $P_S$  be the projections onto  $R$  and  $S$  respectively. Then  $P_R$  and  $P_S$  are orthogonal if and only if  $R \perp S$ .*

**Proof:**

1. (I). Let  $P_R P_S = 0$ .
2. Then  $R \perp S$  since for any  $x \in R$  and  $y \in S$

$$(x, y) = (P_R x, P_S y) = 0 .$$

3. (II). If  $R \perp S$ , then  $P_R P_S = 0$ .

■

**Theorem 4.6.3** *Let  $H$  be a Hilbert space,  $R$  and  $S$  be closed subspaces of  $H$  and  $P_R$  and  $P_S$  be the projections onto  $R$  and  $S$  respectively. Then the sum  $P = P_R + P_S$  is a projection operator if and only if  $P_R$  and  $P_S$  are orthogonal. The sum of the orthogonal projections  $P_R$  and  $P_S$  is the projection onto the direct sum  $R \oplus S$ ,*

$$P_R + P_S = P_{R \oplus S} .$$

**Proof:**

1. (I). Let  $P$  be a projection.
2. Then  $P^2 = P$  and
 
$$P_R P_S = 0.$$
3. (II). Let  $P_R P_S = 0$ .
4. Then  $P$  is idempotent and self-adjoint.
5. Thus  $P$  is a projection.
6. Finally,  $P|_{R \oplus S} = I$  and  $P|_{(R \oplus S)^\perp} = 0$ .
7. Thus  $P$  is the projection onto  $R \oplus S$ .

■

**Theorem 4.6.4** *Let  $H$  be a Hilbert space,  $R$  and  $S$  be closed subspaces of  $H$  and  $P_R$  and  $P_S$  be the projections onto  $R$  and  $S$  respectively. Let  $P = P_R P_S$ . Then  $P$  is a projection operator if and only if  $P_R$  and  $P_S$  commute. In this case  $P$  is the projection onto  $R \cap S$ ,*

$$P_R P_S = P_{R \cap S}.$$

**Proof:**

1. (I). Suppose that  $P$  is a projection.
2. Then  $P^* = P$  and  $P_R P_S = P_S P_R$ .
3. (II). Let  $P_R P_S = P_S P_R$ .
4. Then  $P$  is self-adjoint and idempotent.
5. Thus  $P$  is a projection.
6. Moreover,

$$P|_{R \cap S} = I \quad \text{and} \quad P|_{(R \cap S)^\perp} = 0.$$

7. Thus  $P$  is the projection onto  $R \cap S$ .

■

**Theorem 4.6.5** *Let  $H$  be a Hilbert space,  $R$  and  $S$  be closed subspaces of  $H$  and  $P_R$  and  $P_S$  be the projections onto  $R$  and  $S$  respectively. Then the following conditions are equivalent:*

1.  $R \subset S$ .
2.  $P_S P_R = P_R$ .
3.  $P_R P_S = P_R$ .
4.  $\| P_R x \| \leq \| P_S x \| \quad \forall x \in H$ .

**Proof:**

1. (I). Suppose  $R \subset S$ .
2. Then  $P_S P_R = P_R$ .
3. (II). If  $P_S P_R = P_R$ , then  $P_R P_S = P_R$ .
4. (III). If  $P_R P_S = P_R$ , then for all  $x \in H$

$$\| P_R x \| = \| P_R P_S x \| \leq \| P_S x \| .$$

5. (IV). Finally, suppose that for all  $x \in H$ ,  $\| P_R x \| \leq \| P_S x \|$ .
6. Suppose that  $R$  is not a subset of  $S$ .
7. Then there is  $x \in R$  such that  $x \notin S$ .
8. Then

$$\| P_R x \|^2 = \| x \|^2 = \| P_S x \|^2 + \| P_S^\perp x \|^2 .$$

9. Thus, if  $x \notin S$ , then  $P_S^\perp x \neq 0$ , and, therefore,

$$\| P_R x \| > \| P_S x \| ,$$

which contradicts the assumptions. ■

### 4.6.1 Homework

- Exercises: 4.13[40,43,44,45,46]

## 4.7 Compact Operators

**Definition 4.7.1 Compact Operator.** *An operator on a Hilbert space is **compact** (or **completely continuous**) if the image of every bounded sequence in  $H$  contains a convergent subsequence.*

- **Remark.** Every operator on a finite-dimensional space is compact.

- **Example.**

• **Theorem 4.7.1** *Every compact operator is bounded.*

**Proof:**

1. If  $A : H \rightarrow H$  is not bounded, then there is a sequence  $(x_n)$  in  $H$  such that  $\|x_n\| = 1$ ,  $n \in \mathbb{N}$ , and  $\|Ax_n\| \rightarrow \infty$  as  $n \rightarrow \infty$ .
2. Then  $(Ax_n)$  does not contain a convergent subsequence. ■

- **Remark.** Not every bounded operator is compact.

- The identity operator on an infinite-dimensional Hilbert space is not compact.

- Projection operator on finite-dimensional subspaces are compact.

- **Examples.**

• **Theorem 4.7.2** *Integral operators in  $L^2([a, b])$  with continuous kernels are compact.*

**Proof:** No proof. Read the textbook.

• **Theorem 4.7.3** *The set of all compact operators on a Hilbert space is a vector space.*

**Proof:** Exercise. ■

• **Theorem 4.7.4** *A product of a compact operator and a bounded operator on a Hilbert space is a compact operator.*

**Proof:**

1. Let  $A$  be a compact operator and  $B$  be a bounded operator on a Hilbert space  $H$ .
2. (I). Let  $(x_n)$  be a bounded sequence in  $H$ .
3. Then  $(Bx_n)$  is bounded and  $(ABx_n)$  contains a convergent subsequence.
4. Thus,  $AB$  is compact.
5. (II). We have  $(Ax_n)$  contains a convergent subsequence  $(Ax_{n_k})$ .
6. Therefore, the sequence  $BAx_{n_k}$  converges.
7. Thus  $BA$  is compact.

■

• **Definition 4.7.2 Finite Dimensional Operator.** *An operator is **finite-dimensional** if it has a finite-dimensional range.*

• **Theorem 4.7.5** *Finite-dimensional bounded operators are compact.*

**Proof:**

1. Let  $A$  be a finite-dimensional bounded operator and  $R(A)$  be its range.
2. Let  $P$  be projection onto  $R(A)$ .
3. Then  $A = PA$ .
4. Since  $A$  is bounded and  $P$  compact, the product  $A = PA$  is compact as well.

■

• **Theorem 4.7.6** *The limit of a convergent sequence of compact operators is compact.*

**Proof:**

1. Let  $(T_n)$  be a sequence of compact operators and  $T$  be an operator such that  $\|T_n - T\| \rightarrow 0$  as  $n \rightarrow \infty$ .
2. Let  $(x_n)$  be a bounded sequence in  $H$  such that  $\|x_n\| \leq M$  for all  $n \in \mathbb{N}$ .

3. Then there exists a subsequence  $(x_{1,n})$  of  $(x_n)$  such that  $(T_1x_{1,n})$  converges.
4. Similarly, there exists a subsequence  $(x_{2,n})$  of  $(x_{1,n})$  such that  $(T_2x_{2,n})$  converges.
5. By induction, we get a sequence of sequences  $(x_{k,n})$  such that  $(x_{k,n})$  is a subsequence of  $(x_{k-1,n})$  and  $T_kx_{k,n}$  converges.
6. Let  $(y_n)$  be a sequence defined by  $y_n = x_{n,n}$ .
7. All of these sequences, in particular, the sequence  $(y_n)$  are subsequences of the sequence  $(x_n)$ . Therefore,  $y_n = x_{p_n}$ , where  $(p_n)$  is an increasing sequence in  $\mathbb{N}$ .
8. Then, the sequence  $T_kx_{p_n}$  converges for any  $k \in \mathbb{N}$ .
9. Claim: the sequence  $(Tx_{p_n})$  converges.
10. Let  $\varepsilon > 0$ .
11. Let  $\varepsilon_1 > 0$ . Then there is  $K \in \mathbb{N}$  such that

$$\|T_K - T\| < \varepsilon_1.$$

12. Next, there exists  $L \in \mathbb{N}$  such that for any  $n, m \geq L$ ,

$$\|T_Kx_{p_n} - T_Kx_{p_m}\| < \varepsilon_1.$$

13. Then for  $n, m \geq \max\{K, L\}$ ,

$$\begin{aligned} \|Tx_{p_n} - Tx_{p_m}\| &\leq \|Tx_{p_n} - T_Kx_{p_n}\| + \|T_Kx_{p_n} - T_Kx_{p_m}\| \\ &\quad + \|T_Kx_{p_m} - Tx_{p_m}\| \\ &< \varepsilon_1M + \varepsilon_1 + \varepsilon_1M = (2 + M)\varepsilon_1 = \varepsilon \end{aligned}$$

by choosing  $\varepsilon_1 = \varepsilon/(2 + M)$ .

14. So,  $(Tx_{p_n})$  is Cauchy in  $H$  and, therefore, converges.
15. Thus  $T$  is compact. ■

• **Corollary 4.7.1** *The limit of a convergent sequence of finite-dimensional operators on a Hilbert space is a compact operator.*

**Proof:** Obvious. ■

- **Theorem 4.7.7** *The adjoint of a compact operator on a Hilbert space is compact.*

**Proof:**

1. Let  $A$  be a compact operator on a Hilbert space  $H$ .
2. Let  $(x_n)$  be a bounded sequence in  $H$  such that for any  $n \in \mathbb{N}$ ,  $\|x_n\| \leq M$ .
3. Let  $y_n = A^*x_n$ ,  $n \in \mathbb{N}$ .
4. Then  $(y_n)$  is bounded.
5. Since  $A$  is compact, there exists a subsequence  $(y_{p_n})$  such that  $Ay_{p_n}$  converges in  $H$ .
6. Claim:  $\forall n, m \in \mathbb{N}$ ,

$$\|y_{p_m} - y_{p_n}\|^2 \leq 2M \|Ay_{p_n} - Ay_{p_m}\| \rightarrow 0.$$

7. Thus,  $(y_{p_n})$  is Cauchy in  $H$  and, hence, converges.
8. Thus,  $T^*$  is compact. ■

- **Theorem 4.7.8** *An operator on a Hilbert space is compact if and only if it maps every weakly convergent sequence into a strongly convergent sequence.*

**Proof:**

1. Let  $T$  be a compact operator on a Hilbert space  $H$ .
  2. (I). Let  $(x_n)$  be a sequence in  $H$  that converges weakly to  $x \in H$ .
  3. Assume that  $(Tx_n)$  does not converge (strongly) to  $Tx$ .
  4. Then there is  $\varepsilon > 0$  and a subsequence  $(x_{p_n})$  such that for all  $n \in \mathbb{N}$ ,
- $$\|Tx_{p_n} - Tx\| > \varepsilon.$$
5. The sequence  $(x_{p_n})$  is bounded.
  6. There is a strongly convergent subsequence  $(Tx_{q_n})$  of the sequence  $(Tx_{p_n})$ .

7. We have, for any  $y \in H$ ,

$$(Tx_n, y) \rightarrow (Tx, y).$$

8. Thus

$$Tx_n \xrightarrow{w} Tx$$

9. Then (weakly)

$$Tx_{q_n} \xrightarrow{w} Tx.$$

10. Moreover, (strongly)

$$Tx_{q_n} \rightarrow Tx,$$

which contradicts the assumption.

11. (II). Assume that  $T$  maps every weakly convergent sequence to a strongly convergent sequence.

12. Let  $(x_n)$  be a bounded sequence in  $H$  such that for all  $n \in \mathbb{N}$ ,  $\|x_n\| \leq M$ .

13. Claim:  $(Tx_n)$  has a convergent subsequence.

14. Let  $(e_n)$  be a complete orthonormal sequence in  $H$ .

(separability?)

15. There a subsequence  $(x_{1,n})$  of  $(x_n)$  such that  $(x_{1,n}, e_1)$  converges.

16. By induction, there is a subsequence  $(x_{k+1,n})$  of  $(x_k, n)$  for all  $k \in \mathbb{N}$  such that  $(x_{k,n}, e_k)$  converges as  $n \rightarrow \infty$ .

17. Let  $y_n = x_{n,n}$ ,  $n \in \mathbb{N}$ .

18. Then  $(y_n)$  is a subsequence of  $(x_n)$ .

19. Also, for any  $k \in \mathbb{N}$  the sequence  $(y_n, e_k)$  converges as  $n \rightarrow \infty$ .  
Let

$$\alpha_k = \lim_{n \rightarrow \infty} (y_n, e_k).$$

20. Claim: the sequence  $(y_n)$  is weakly convergent.

21. We have,  $\forall l, n \in \mathbb{N}$ ,

$$\sum_{k=1}^l |(y_n, e_k)|^2 \leq \sum_{k=1}^{\infty} |(y_n, e_k)|^2 = \|y_n\|^2 \leq M^2.$$

22. As  $n, l \rightarrow \infty$  we get

$$\sum_{k=1}^{\infty} |\alpha_k|^2 \leq M^2.$$

23. Let

$$y = \sum_{k=1}^{\infty} \alpha_k e_k.$$

24. Then for all  $k \in \mathbb{N}$  as  $n \rightarrow \infty$  we obtain

$$((y_n - y), e_k) = (y_n, e_k) - \alpha_k \rightarrow 0.$$

25. Thus, (weakly)

$$y_n \xrightarrow{w} y.$$

26. Therefore, (strongly)

$$Ty_n \rightarrow Ty.$$

27. Thus  $T$  is compact. ■

- Corollary 4.7.2** *Let  $A$  be a compact operator on a Hilbert space  $H$  and  $(e_n)$  be an orthonormal sequence in  $H$ . Then  $Te_n \rightarrow 0$  as  $n \rightarrow \infty$ .*
- *Proof:* Obvious. ■

- **Remarks.**

- The inverse of a compact invertible operator on an infinite-dimensional Hilbert space is unbounded.
- Compactness of operators is a stronger version of continuity.
- That is why compact operators are also called completely continuous operators.

### 4.7.1 Homework

- Exercises: 4.13[49,50]

## 4.8 Eigenvalues and Eigenvectors

**Definition 4.8.1 Eigenvalue.** *Let  $A$  be an operator on a complex vector space  $E$ . A complex number  $\lambda$  is called an **eigenvalue** of the operator  $A$  if there is a non-zero vector  $u \in E$  such that*

$$Au = \lambda u.$$

*The vector  $u$  is called the **eigenvector** corresponding to the eigenvalue  $\lambda$ .*

- **Example.** Projection operator.
- **Remarks.**
- There are infinitely many eigenvectors corresponding to an eigenvalue.

**Theorem 4.8.1** *Let  $A$  be an operator on a complex vector space  $E$  and  $\lambda$  be an eigenvalue of the operator  $A$ . The collection of all eigenvectors corresponding to the eigenvalue  $\lambda$  is a vector space.*

**Proof:** Exercise. ■

**Definition 4.8.2 Eigenvalue Space.** *Let  $A$  be an operator on a complex vector space  $E$  and  $\lambda$  be an eigenvalue of the operator  $A$ . The set of all eigenvectors corresponding to the eigenvalue  $\lambda$  is called the **eigenvalue space** (or **eigenspace**) of  $\lambda$ .*

*The dimension of the eigenspace of the eigenvalue  $\lambda$  is called the **multiplicity** of  $\lambda$ .*

- *An eigenvalue of multiplicity one is called **simple** (or **non-degenerate**)*

*An eigenvalue of multiplicity greater than one is called **multiple** (or **degenerate**). The multiplicity is then called the **degree of degeneracy**.*

*The problem of finding the eigenvalues and the eigenvectors is the **eigenvalue problem**.*

- **Example.** Let  $H = L^2([0, 2\pi])$ ,  $f(x) = \sin x$ ,  $g(x) = \cos x$  and  $A : H \rightarrow H$  be defined by

$$Au = (u, f)g + (u, g)f.$$

Show that  $A$  has exactly one non-zero eigenvalue  $\lambda = \pi$  of multiplicity 2 and the eigenvalue  $\lambda = 0$  of infinite multiplicity. Find the eigenvectors.

check this!!!

- **Remarks.**
- The operator  $A - \lambda I$  is invertible if and only if  $\lambda$  is not an eigenvalue of the operator  $A$ .
- If the vector space  $E$  is finite-dimensional and  $\lambda$  is not an eigenvalue of the operator  $A$ , then the operator  $A - \lambda I$  is bounded.

**Definition 4.8.3 Resolvent and Spectrum.** Let  $A$  be an operator on a normed vector space  $E$  and  $\lambda \in \mathbb{C}$  be a complex number. The operator

$$R_\lambda = (A - \lambda I)^{-1} : E \rightarrow E$$

- is called the **resolvent** of the operator  $A$ .

The values of  $\lambda \in \mathbb{C}$  for which the resolvent  $R_\lambda$  is well defined and bounded are called **regular points** of  $A$ .

The set of values of  $\lambda \in \mathbb{C}$  which are not regular is called the **spectrum** of the operator  $A$ .

- **Remarks.**
- Every eigenvalue belongs to the spectrum.
- Not all points in the spectrum are eigenvalues.
- **Example.** Let  $E = C([a, b])$ ,  $u \in E$  and  $A$  be an operator on  $E$  defined by

$$(Af)(t) = u(t)f(t).$$

Show that:

- 1) the spectrum of  $A$  is exactly the range of  $u$ ,

- 2) if  $u(t) = c$  is constant, then  $\lambda = c$  is an eigenvalue of  $A$ ,
- 3) if  $u$  is strictly increasing, then  $A$  has no eigenvalues.

- **Remarks.**

**Theorem 4.8.2** *Let  $A$  be an operator on a Hilbert space  $H$ .*

1. *If  $T$  is an invertible operator on  $E$ , then the eigenvalues of the operators  $TAT^{-1}$  and  $A$  are the same.*
2. *If  $A$  is self-adjoint, then all eigenvalues of  $A$  are real.*
3. *If  $A$  is positive, then all eigenvalues of  $A$  are non-negative.*
4. *If  $A$  is strictly positive, then all eigenvalues of  $A$  are positive.*
5. *If  $A$  is unitary, then all eigenvalues of  $A$  have modulus equal to 1.*
- 6. *If  $A$  is a projection, then it can only have two eigenvalues: 1 and 0.*
7. *If  $A$  is bounded, then for every eigenvalue  $\lambda$  of  $A$  we have*

$$|\lambda| \leq \|A\| .$$

8. *If  $A$  is bounded and self-adjoint, then for every eigenvalue  $\lambda$  of  $A$  we have*

$$|\lambda| \leq \sup_{x \in H} \frac{|(Ax, x)|}{\|x\|^2} .$$

9. *If  $A$  is self-adjoint or unitary, then the eigenvectors corresponding to distinct eigenvalues are orthogonal.*

**Proof:** Easy. ■

- **Remark.**

- All eigenvalues of a bounded operator lie inside the circle of radius  $\|A\|$  in the complex plane.

- **Theorem 4.8.3** *Let  $A$  be a non-zero, compact, self-adjoint operator on a Hilbert space  $H$ . Then  $A$  has an eigenvalue  $\lambda$  equal to either  $\|A\|$  or  $-\|A\|$ .*

**Proof:**

1. Let

$$B = \frac{A}{\|A\|},$$

so that  $\|B\| = 1$ .

2. Let  $(x_n)$  be a sequence of unit vectors in  $H$  and  $(\alpha_n)$  be a sequence of real numbers defined by

$$\alpha_n = \|Bx_n\|^2$$

We can always choose a sequence  $(x_n)$  so that

$$\alpha_n \rightarrow 1.$$

3. Then one can show that

$$\|(B^2 - \alpha_n)x_n\|^2 \leq \alpha_n(1 - \alpha_n).$$

4. Therefore,

$$\|(B^2 - \alpha_n)x_n\|^2 \rightarrow 0.$$

5. The operator  $B^2$  is compact.

6. Therefore there exists a subsequence  $(x_{p_n})$  such that  $B^2x_{p_n}$  converges to some vector  $x$

$$B^2x_{p_n} \rightarrow x.$$

7. Then we have

$$\|x - x_{p_n}\| \leq \|x - B^2x_{p_n}\| + \|B^2x_{p_n} - \alpha_{p_n}x_{p_n}\| + \|\alpha_{p_n}x_{p_n} - x_{p_n}\|.$$

8. Therefore,

$$\|x - x_{p_n}\| \rightarrow 0$$

9. Thus

$$x_{p_n} \rightarrow x$$

and

$$B^2x = x.$$

10. This means

$$(B - I)(B + I)x = 0.$$

11. We conclude that

$$\text{either } (B - I)x = 0 \text{ or } (B + I)x = 0,$$

which means that

$$\text{either } +1 \text{ or } -1 \text{ is an eigenvalue of } B,$$

and  $\|A\|$  or  $-\|A\|$  is an eigenvalue of  $A$ . ■

**Corollary 4.8.1** *Let  $A$  be a non-zero compact self-adjoint operator on a Hilbert space  $H$ . Then there is a unit vector  $u \in H$  such that  $\|u\| = 1$  and*

$$|(Au, u)| = \sup_{\|x\| \leq 1} |(Ax, x)|.$$

**Proof:**

1. Let  $u, \|u\| = 1$ , be the eigenvector corresponding to the eigenvalue  $\|A\|$  or  $-\|A\|$ . ■

**Theorem 4.8.4** *Let  $A$  be a self-adjoint compact operator on a Hilbert space  $H$ . Then the set of distinct non-zero eigenvalues of  $A$  is either finite or forms a sequence  $(\lambda_n)$  that converges to 0*

$$\lim_{n \rightarrow \infty} \lambda_n = 0.$$

**Proof:**

1. Suppose  $A$  has infinitely (countably) many distinct eigenvalues  $(\lambda_n)$  with the corresponding unit eigenvectors  $(u_n)$ .
2. Then  $(u_n)$  is an orthonormal sequence.

3. Hence  $(u_n)$  weakly converges to 0.
4. Thus, since  $A$  is compact

$$Au_n \rightarrow 0$$

and

$$\lim_{n \rightarrow \infty} \lambda_n^2 = \lim_{n \rightarrow \infty} \|Au_n\|^2 = 0.$$

■

- **Example.** Let  $H = L^2([0, 2\pi])$ ,  $k$  is a locally square integrable periodic function with period  $2\pi$ ,  $k_t$  be a function defined by  $k_t(x) = k(t - x)$  and  $A$  be an operator on  $H$  defined by

$$(Af)(t) = (f, k_t).$$

Show that:

- 1)  $A$  is self-adjoint if  $k(-x) = k(x)$ , and
- 2) the eigenvalues and the eigenfunctions of  $A$  are

$$\lambda_n = (u_n, k) \quad u_n(x) = e^{inx}, \quad n \in \mathbb{Z}.$$

**Theorem 4.8.5** *Let  $H$  be a Hilbert space and  $(P_n)$  be a sequence of pairwise orthogonal projections on  $H$ . Let  $(\lambda_n)$  be a sequence of complex numbers converging to 0. Then:*

1. *The series*

$$A = \sum_{n=1}^{\infty} \lambda_n P_n$$

*converges in  $B(H, H)$  and defines a bounded operator on  $H$ .*

2. *Each  $\lambda_n$  is an eigenvalue of the operator  $A$ . The only other possible eigenvalue of  $A$  is 0.*
3. *If all  $\lambda_n$  are real, then  $A$  is self-adjoint.*
4. *If all  $P_n$  are finite-dimensional, then  $A$  is compact.*

**Proof:**

1. (I). Since  $B(H, H)$  is complete we only need to show that  $s_N = \sum_{n=1}^N \lambda_n P_n$  is a Cauchy sequence.

2. Let  $\varepsilon > 0$ . Then there is  $n_0 \in \mathbb{N}$  such that

$$|\lambda_n| < \varepsilon \text{ for any } n > n_0.$$

3. For any  $x \in H$  and any  $k, m \in \mathbb{N}$  such that  $m > k > n_0$  we obtain

$$\left\| \sum_{n=k}^m \lambda_n P_n x \right\|^2 \leq \varepsilon^2 \left\| \sum_{n=k}^m P_n \right\|^2 \|x\|^2 \leq \varepsilon^2 \|x\|^2,$$

4. So,

$$\|s_m - s_k\| = \left\| \sum_{n=k}^m \lambda_n P_n \right\| \leq \varepsilon,$$

and, hence,  $s_n$  is a Cauchy sequence.

5. (II). Let  $k \in H$  and  $u \in P_k(H)$ .

6. Then

$$Au = \lambda_k u$$

and, hence,  $\lambda_k$  is the eigenvalue of  $A$ .

7. Suppose  $\lambda$  is a complex number such that  $\lambda \neq 0, \lambda_n$  for any  $n \in \mathbb{N}$  and  $u$  is a vector such that

$$Au = \lambda u.$$

8. Let  $R(A)$  be the range of  $A$ ,  $P$  be the projection on  $R(A)$  and  $P^\perp$  be the projection on the orthogonal complement  $(R(A))^\perp$ .

9. Then

$$\sum_{n=1}^{\infty} P_n + P^\perp = I.$$

10. Therefore,

$$(A - \lambda I)u = \sum_{n=1}^{\infty} (\lambda_n - \lambda) P_n u - \lambda P^\perp u = 0.$$

11. Since  $\lambda \neq \lambda_n$  for any  $n$  and  $\lambda \neq 0$ , then  $P_n u = 0$  for all  $n \in \mathbb{N}$  and  $P^\perp u = 0$ .

12. Thus,  $u = 0$  and  $\lambda$  is not an eigenvalue.

13. (III). Suppose  $\lambda_n$  are real.

14. Then for any  $x, y \in H$

$$(Ax, y) = (x, Ay).$$

15. (IV). If all  $P_n$  are finite-dimensional, then  $A$  is compact since the limit of a convergent sequence of finite-dimensional operators is a compact operator. ■

**Definition 4.8.4 Approximate Eigenvalue.** *Let  $T$  be an operator on a Hilbert space  $H$ . A complex number  $\lambda$  is called an **approximate eigenvalue** of  $T$  if there exists a sequence of unit vectors  $(x_n)$  such that  $\|x_n\| = 1, \forall n \in \mathbb{N}$  and*

$$\lim_{n \rightarrow \infty} \|(T - \lambda I)x_n\| = 0.$$

- Every eigenvalue is an approximate eigenvalue.
- **Example.** Let  $H$  be a Hilbert space and  $(e_n)$  be a complete orthonormal sequence in  $H$ . Let  $\lambda$  be a real number and  $(\lambda_n)$  be a sequence of real numbers such that  $\lambda_n \neq \lambda$  and  $\lim_{n \rightarrow \infty} \lambda_n = \lambda$ . Let  $T$  be an operator on  $H$  defined by

$$Tu = \sum_{n=1}^{\infty} \lambda_n (u, e_n) e_n.$$

Show that  $\lambda_n$  is an eigenvalue of  $T$  and  $\lambda$  is an approximate eigenvalue of  $T$  but not an eigenvalue.

### 4.8.1 Homework

- Exercises: 4.13[51,52,53,54,55,58,59]

## 4.9 Spectral Decomposition

**Theorem 4.9.1 Hilbert-Schmidt Theorem.** *Let  $H$  be an infinite-dimensional Hilbert space and  $A$  be a self-adjoint compact operator on  $H$ . Then there exists an orthonormal system of eigenvectors  $(u_n)$  corresponding to non-zero eigenvalues  $(\lambda_n)$  such that every vector  $x \in H$  has a unique representation*

$$x = \sum_{n=1}^{\infty} \alpha_n u_n + v,$$

where  $(\alpha_n)$  is a sequence in  $\mathbb{C}$ , and  $v$  is a vector such that

$$Av = 0.$$

If  $A$  has infinitely many distinct eigenvalues  $(\lambda_n)$ , then

$$\lambda_n \rightarrow 0.$$

That is, there is a direct sum decomposition of the Hilbert space

$$H = \bigoplus_{n=1}^{\infty} E_n \oplus E_0,$$

where  $E_n$  are the eigenspaces corresponding to the eigenvalues  $\lambda_n$  and  $E_0 = \text{Ker } A$  is the kernel of the operator  $A$ .

**Proof:**

1. Since  $A$  is self-adjoint and compact there is an eigenvalue  $\lambda_1$  of  $A$  such that

$$|\lambda_1| = \sup_{x \in H, \|x\| \leq 1} |(Ax, x)|.$$

2. Let  $E_1 \subset H$  be the eigenspace corresponding to  $\lambda_1$ .
3. Then  $E_1^\perp$  is a closed invariant subspace of  $H$ .
4. Thus, there exists an eigenvalue  $\lambda_2$  such that

$$|\lambda_2| = \sup_{x \in E_1^\perp, \|x\| \leq 1} |(Ax, x)|.$$

5. By induction, at the  $n$ -th step, we get the eigenvalues  $(\lambda_1, \dots, \lambda_n)$  and their eigenspaces  $(E_1, \dots, E_n)$  and we choose an eigenvalue  $\lambda_{n+1}$  such that

$$|\lambda_{n+1}| = \sup_{x \in E_n^\perp, \|x\| \leq 1} |(Ax, x)|.$$

6. We have

$$|\lambda_n| \geq |\lambda_{n+1}|$$

and

$$E_n \perp E_k, \quad \text{for } n \neq k.$$

7. If at the step  $k$  we get an eigenspace  $E_k$  such that  $(Ax, x) = 0$  for any  $x \in E_k^\perp$ , then this process terminates. The space  $E_0 = E_k^\perp$  is the eigenspace of the zero eigenvalue.
8. In this case we have

$$H = E_1 \oplus \dots \oplus E_k \oplus E_0,$$

and every vector  $x \in H$  has a unique representation

$$x = \sum_{j=1}^k \alpha_j u_j + v,$$

where  $v \in E_0$  is a zero eigenvector.

9. Suppose that the process does not terminate. Then we get a sequence of eigenvalues  $(\lambda_n)$  with nonincreasing moduli and the corresponding eigenspaces  $(E_n)$ .
10. Then a sequence  $(u_n)$  of unit eigenvectors from  $E_n$  converges weakly to 0.
11. Since  $A$  is compact, the sequence  $(Au_n)$  converges strongly to 0.
12. Thus

$$|\lambda_n| = \|Au_n\| \rightarrow 0.$$

13. Now, let

$$S = \bigoplus_{n=1}^{\infty} E_n = \text{Span}\{u_n \mid u_n \in E_n, n \in \mathbb{N}\}.$$

14. Then

$$H = S \oplus E_0,$$

and for any  $x \in H$  there is a unique decomposition

$$x = \sum_{j=1}^{\infty} \alpha_j u_j + v,$$

where  $v \in E_0$ .

15. Claim:  $E_0 = S^\perp$  is the zero eigenspace.

16. Let  $w \in E_0$  be a unit vector.

17. Then for any  $n \in \mathbb{N}$

$$|(Aw, w)| \leq \sup_{x \in E_n^\perp, \|x\| \leq 1} |(Ax, x)| = |\lambda_{n+1}|.$$

18. Since  $\lambda_n \rightarrow 0$ , we have  $(Aw, w) = 0$ , and therefore,  $Aw = 0$ . ■

**Theorem 4.9.2 Spectral Theorem for Self-Adjoint Compact Operators.** *Let  $H$  be an infinite-dimensional Hilbert space and  $A$  be a self-adjoint compact operator on  $H$ . Then there exists a complete orthonormal system  $(v_n)$  (an orthonormal basis) in  $H$  consisting of eigenvectors of  $A$  corresponding to the eigenvalues  $(\lambda_n)$ .*

*Then for every  $x \in H$*

$$Ax = \sum_{n=1}^{\infty} \lambda_n (x, v_n) v_n.$$

**Proof:**

1. The sequence  $(v_n)$  we add to the eigenvectors  $(u_n)$  corresponding to non-zero eigenvalues an orthonormal basis in the zero eigenspace  $E_0$ . ■

**Theorem 4.9.3** *Let  $H$  be a Hilbert space and  $A$  and  $B$  be two commuting self-adjoint compact operators on  $H$ . Then there exists an orthonormal basis in  $H$  consisting of the common eigenvectors of the operators  $A$  and  $B$ .*

**Proof:**

1. Let  $\lambda$  be an eigenvalue of  $A$  and  $E_\lambda$  be the corresponding eigenspace.
2. Then  $E_\lambda$  is an invariant subspace of  $B$  and has an orthonormal basis consisting of the eigenvectors of  $B$ . These vectors are also eigenvectors of  $A$ .

■

**Theorem 4.9.4** *Let  $H$  be a Hilbert space and  $A$  be a self-adjoint compact operator on  $H$ . Let  $(\lambda_n)$  be the eigenvalues of  $A$  and  $(v_n)$  be the corresponding orthonormal system of eigenvectors. Let  $(P_n)$  be the projection operators onto the one-dimensional spaces spanned by  $(v_n)$ . Then, for all  $x \in H$*

$$x = \sum_{n=1}^{\infty} P_n x$$

and

$$A = \sum_{n=1}^{\infty} \lambda_n P_n.$$

More generally, if  $p$  is a polynomial, such that  $p(0) = 0$ , then

$$p(A) = \sum_{n=1}^{\infty} p(\lambda_n) P_n.$$

**Proof:**

1. The projections  $P_n$  are defined by

$$P_k x = (x, v_k) v_k,$$

so that for any  $x \in H$  we have

$$x = \sum_{n=1}^{\infty} P_n x.$$

or

$$\sum_{n=1}^{\infty} P_n = I.$$

2. Therefore,

$$Ax = \sum_{n=1}^{\infty} \lambda_n P_n x.$$

3. We immediately obtain

$$A^k x = \sum_{n=1}^{\infty} \lambda_n^k P_n x,$$

which proves the theorem. ■

**Definition 4.9.1 Function of an Operator.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a real-valued function on  $\mathbb{R}$  such that

$$f(0) = \lim_{\lambda \rightarrow 0} f(\lambda) = 0.$$

Let  $A$  be a self-adjoint compact operator on a Hilbert space  $H$  given by its spectral decomposition

$$A = \sum_{n=1}^{\infty} \lambda_n P_n.$$

Then the operator  $f(A)$  on  $H$  defined by

$$f(A) = \sum_{n=1}^{\infty} f(\lambda_n) P_n$$

is self-adjoint and compact.

- **Example.** If all eigenvalues of a self-adjoint compact operator  $A$  are non-negative, that is  $\lambda_n \geq 0$ , then for any  $\alpha > 0$  we define

$$A^\alpha = \sum_{n=1}^{\infty} \lambda_n^\alpha P_n.$$

- If the function  $f$  is not zero at zero but is just bounded at zero, then we can still define

$$f(A) = \sum_{n=1}^{\infty} f(\lambda_n) P_n.$$

In this case the operator  $f(A)$  is not compact.

- Theorem 4.9.5** *Let  $H$  be a Hilbert space and  $T$  be a self-adjoint operator on  $H$  such that all eigenvalues  $(\lambda_n)$  of the operator  $T$  are non-negative (or positive) and the eigenvectors  $(u_n)$  of  $T$  form an orthonormal basis on  $H$ . Then the operator  $T$  is positive (or strictly positive).*
- 

**Proof:**

1. Let  $x$  be a non-zero in  $H$ .
2. Then

$$(Tx, x) = \sum_{n=1}^{\infty} \lambda_n |(x, u_n)|^2 \geq 0.$$

■

### 4.9.1 Homework

- Exercises: 4.13[60,61,64,65,66]

## 4.10 The Fourier Transform

### 4.10.1 Fourier Transform in $L^1(\mathbb{R})$

- The Fourier transform on  $L^2(\mathbb{R})$  is defined as an extension of the Fourier transform in  $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ .

**Definition 4.10.1** **Fourier Transform in  $L^1(\mathbb{R})$ .** Let  $L^1(\mathbb{R})$  be the space of integrable functions on  $\mathbb{R}$ . The **Fourier transform**

$$\mathcal{F} : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$$

- 

is a linear operator on  $L^1(\mathbb{R})$  defined by

$$(\mathcal{F}f)(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} dx e^{-i\omega x} f(x).$$

- **Example.**

**Theorem 4.10.1** *The Fourier transform of an integrable function is a continuous function. That is,*

- 

$$\mathcal{F}(L^1(\mathbb{R})) \subset C(\mathbb{R}).$$

**Proof:**

1. Let  $f \in L^1(\mathbb{R})$  and  $h \in \mathbb{R}$ .
2. We get an estimate

$$|\hat{f}(\omega + h) - \hat{f}(\omega)| \leq (2\pi)^{-1/2} \int_{-\infty}^{\infty} dx |e^{-ihx} - 1| |f(x)|.$$

Since

$$\lim_{h \rightarrow 0} \int_{-\infty}^{\infty} dx |e^{-ihx} - 1| |f(x)| = 0$$

$\hat{f}$  is continuous. ■

- $L^1(\mathbb{R})$  is the normed space with the norm

$$\|f\|_1 = \int_{-\infty}^{\infty} dx |f(x)|.$$

- Let  $C_0(\mathbb{R})$  be the space of all continuous functions on  $\mathbb{R}$  vanishing at infinity, that is

$$\lim_{|x| \rightarrow \infty} |f(x)| = 0.$$

It is a normed space with the uniform convergence norm (or the sup norm)

$$\|f\|_{\infty} = \sup_{x \in \mathbb{R}} |f(x)|.$$

**Theorem 4.10.2** *Let  $(f_n)$  be a sequence in  $L^1(\mathbb{R})$  such that*

$$\|f_n - f\|_1 \rightarrow 0.$$

*Then*

$$\|\mathcal{F}f_n - \mathcal{F}f\|_{\infty} \rightarrow 0,$$

*that is, the sequence  $(\mathcal{F}f_n)$  converges to  $\mathcal{F}f$  uniformly on  $\mathbb{R}$ .*

**Proof:**

1. We notice that  $\forall \omega \in \mathbb{R}$

$$|(\mathcal{F}f)(\omega)| \leq (2\pi)^{-1/2} \|f\|_1.$$

2. Thus

$$\|\mathcal{F}(f_n - f)\|_{\infty} \leq (2\pi)^{-1/2} \|f_n - f\|_1.$$

■

**Theorem 4.10.3 Riemann-Lebesgue Lemma.** *The Fourier transform of an integrable function is a continuous function vanishing at infinity. That is,*

$$\mathcal{F}(L^1(\mathbb{R})) \subset C_0(\mathbb{R}).$$

**Proof:**

1. We estimate

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{2} (2\pi)^{-1/2} \int_{-\infty}^{\infty} dx e^{-i\omega x} \left| f(x) - f\left(x - \frac{\pi}{\omega}\right) \right| \\ &= \frac{1}{2} (2\pi)^{-1/2} \int_{-\infty}^{\infty} dx \left| f(x) - f\left(x - \frac{\pi}{\omega}\right) \right|. \end{aligned}$$

- Thus, the Fourier transform is a continuous linear map

$$\mathcal{F} : L^1(\mathbb{R}) \rightarrow C_0(\mathbb{R}).$$

**Theorem 4.10.4** *Let  $f \in L^1(\mathbb{R})$ . Then*

$$(\mathcal{F}\{e^{i\alpha x} f(x)\})(\omega) = (\mathcal{F}(f))(\omega - \alpha),$$

$$(\mathcal{F}\{f(x - x_0)\})(\omega) = (\mathcal{F}(f))(\omega) e^{-i\omega x_0},$$

$$(\mathcal{F}\{f(\alpha x)\})(\omega) = \frac{1}{\alpha} [\mathcal{F}(f)]\left(\frac{\omega}{\alpha}\right), \quad \alpha > 0,$$

$$\mathcal{F}\{\bar{f}(x)\} = \overline{\mathcal{F}\{f(-x)\}}.$$

**Proof:** Easy.

- **Example (Modulated Gaussian Function).**

$$\mathcal{F}\left(\exp\left(i\omega_0 x - \frac{x^2}{2}\right)\right)(\omega) = \exp\left(-\frac{(\omega - \omega_0)^2}{2}\right).$$

**Theorem 4.10.5** *Let  $f \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$  such that  $f$  is piecewise differentiable and  $f' \in L^1(\mathbb{R})$ . Then*

$$(\mathcal{F}f')(\omega) = i\omega(\mathcal{F}f)(\omega).$$

**Proof:**

1. Integration by parts.

**Corollary 4.10.1** *Let  $n \in \mathbb{N}$ . Let  $f \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$  be a continuous integrable function such that: 1)  $f$  is  $n$  times piecewise differentiable and 2)  $f^{(k)} \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$  for  $k = 0, 1, \dots, n$  and  $f^{(n)} \in L^1(\mathbb{R})$ . Then*

$$(\mathcal{F}f^{(k)})(\omega) = (i\omega)^k (\mathcal{F}f)(\omega).$$

**Proof:**

1. Induction. ■

- The **convolution** of two functions  $f, g \in L^1(\mathbb{R})$  is defined by

$$(f * g)(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} dt f(x-t)g(t).$$

- **Theorem 4.10.6 Convolution Theorem.** *Let  $f, g \in L^1(\mathbb{R})$ . Then*

$$\mathcal{F}(f * g) = (\mathcal{F}f)(\mathcal{F}g).$$

**Proof:**

1. Computation. ■

### 4.10.2 Fourier Transform in $L^2(\mathbb{R})$

- $L^2(\mathbb{R})$  is the normed space of all square integrable functions with the norm

$$\|f\|_2^2 = \int_{-\infty}^{\infty} dx |f(x)|^2.$$

- Let  $C_c(\mathbb{R})$  be the space of all continuous functions on  $\mathbb{R}$  with compact support, that is vanishing outside a bounded interval.
- The space  $C_c(\mathbb{R})$  is dense in  $L^2(\mathbb{R})$ .
- Let  $C$  be the operator of complex conjugation and Id be the identity operator on  $L^2(\mathbb{R})$ .

**Theorem 4.10.7** *Fourier transform is a continuous linear mapping*

$$\mathcal{F} : C_c(\mathbb{R}) \rightarrow L^2(\mathbb{R})$$

*preserving the  $L^2$ -norm.*

•

*That is*

$$\mathcal{F}(C_c(\mathbb{R})) \subset L^2(\mathbb{R})$$

*and for any  $f \in C_c(\mathbb{R})$*

$$\| \mathcal{F}f \|_2 = \| f \|_2 .$$

**Proof:**

1. Suppose  $f$  has a compact support within the interval  $[-\pi, \pi]$ .
2. Then, we compute

$$\| f \|_2^2 = \| \mathcal{F}f \|_2^2 .$$

3. If  $f$  has a compact support but it is not within the interval  $[-\pi, \pi]$ , then there is a  $\lambda > 0$  such that

$$g(x) = f(\lambda x)$$

has a compact support within  $[-\pi, \pi]$ .

4. Then

$$(\mathcal{F}g)(x) = \frac{1}{\lambda} (\mathcal{F}f) \left( \frac{x}{\lambda} \right) .$$

5. Therefore,

$$\| f \|_2^2 = \lambda \| g \|_2^2 = \lambda \| \mathcal{F}g \|_2^2 = \| \mathcal{F}f \|_2^2 .$$

■

- The Fourier transform is a densely defined operator in  $L^2(\mathbb{R})$  and has a unique extension to the whole space  $L^2(\mathbb{R})$ .

**Definition 4.10.2** **Fourier Transform in  $L^2(\mathbb{R})$ .** Let  $f \in L^2(\mathbb{R})$  and  $(\varphi_n)$  be a sequence in  $C_c(\mathbb{R})$  converging to  $f$  in  $L^2(\mathbb{R})$ , that is such that

$$\|\varphi_n - f\|_2 \rightarrow 0.$$

- The Fourier transform of  $f$  is defined by

$$\mathcal{F}f = \lim_{n \rightarrow \infty} \mathcal{F}\varphi_n,$$

where the limit is with respect to the  $L^2$ -norm.

**Theorem 4.10.8 Parseval's Relation.** For any  $f \in L^2(\mathbb{R})$

$$\|\mathcal{F}f\|_2 = \|f\|_2.$$

- That is, the Fourier transform is an isometry on  $L^2(\mathbb{R})$ .

More generally, for any  $f, g \in L^2(\mathbb{R})$

$$(\mathcal{F}f, \mathcal{F}g) = (f, g).$$

**Proof:** Easy. Every isometry preserves inner product (by polarization identity). ■

- We define a sequence of operators  $\mathcal{F}_n$  in  $L^2(\mathbb{R})$  by

$$(\mathcal{F}_n f)(\omega) = (2\pi)^{-1/2} \int_{-n}^n dx e^{-i\omega x} f(x).$$

**Theorem 4.10.9** Let  $f \in L^2(\mathbb{R})$ . Then

$$\|(\mathcal{F}_n - \mathcal{F})f\|_2 \rightarrow 0.$$

- That is

$$\mathcal{F}_n f \rightarrow \mathcal{F}f$$

with respect to the  $L^2$ -norm.

**Proof:**

1. Let  $\chi_n(x)$  be the characteristic function of the interval  $[-n, n]$ . Define for  $n \in \mathbb{N}$

$$f_n(x) = \chi_n(x)f(x).$$

2. Then  $\|f - f_n\|_2 \rightarrow 0$  and  $\|\mathcal{F}f - \mathcal{F}f_n\|_2 \rightarrow 0$ . ■

**Theorem 4.10.10** *Let  $f, g \in L^2(\mathbb{R})$ . Then*

$$(f, CFCg) = (\mathcal{F}f, g).$$

**Proof:**

1. Let

$$f_n = \chi_n f \quad \text{and} \quad g_n = \chi_n g.$$

2. Then we compute

$$(\mathcal{F}f_n, g_n) = (f_n, \overline{\mathcal{F}g_n}).$$

3. As  $n, m \rightarrow \infty$ , we get

$$(\mathcal{F}f, g) = (f, \overline{\mathcal{F}g}).$$
■

**Lemma 4.10.1** *In  $L^2(\mathbb{R})$*

$$CFCF = \text{Id}.$$

**Proof:**

1. By computation show that

$$\|f - \mathcal{F}(\overline{\mathcal{F}f})\|_2^2 = 0.$$
■

We define a sequence of operators  $\mathcal{F}_n^{-1}$  in  $L^2(\mathbb{R})$  by

$$\begin{aligned} (\mathcal{F}_n^{-1}f)(\omega) &= (\mathcal{F}_n f)(-\omega) \\ &= (2\pi)^{-1/2} \int_{-n}^n dx e^{i\omega x} f(x). \end{aligned}$$

**Theorem 4.10.11 Inversion of the Fourier Transform in  $L^2(\mathbb{R})$ .** Let  $f \in L^2(\mathbb{R})$ . Then

$$\mathcal{F}_n^{-1} \mathcal{F} f \rightarrow f,$$

where the convergence is with respect to the  $L^2$ -norm.

**Proof:**

1. By computation. ■

• The inverse Fourier transform  $\mathcal{F}^{-1}$  is defined by

$$(\mathcal{F}^{-1} f)(x) = (\mathcal{F} f)(-x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\omega e^{i\omega x} f(\omega).$$

• **Corollary 4.10.2** Let  $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ . Then almost everywhere in  $\mathbb{R}$  it holds

$$f = \mathcal{F}^{-1}(\mathcal{F} f).$$

**Proof:** Follows from previous theorem. ■

• **Corollary 4.10.3 Duality.** Let  $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ . Then almost everywhere in  $\mathbb{R}$  it holds

$$(\mathcal{F}^2 f)(x) = f(-x).$$

**Proof:** Easy. Obvious. ■

**Theorem 4.10.12 Plancherel's Theorem.** *There is a linear bounded operator*

$$\mathcal{F} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$$

such that for any  $f, g \in L^2(\mathbb{R})$ :

1. if  $f \in L^1(\mathbb{R})$ , then

$$(\mathcal{F}f)(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} dx e^{-i\omega x} f(x),$$

- 2.

$$\|(\mathcal{F} - \mathcal{F}_n)f\|_2 \xrightarrow{n \rightarrow \infty} 0,$$

- 3.

$$\|(\text{Id} - \mathcal{F}_n^{-1}\mathcal{F})f\|_2 \xrightarrow{n \rightarrow \infty} 0,$$

- 4.

$$\|\mathcal{F}f\|_2 = \|f\|_2,$$

- 5.

$$\mathcal{F}^* = C\mathcal{F}C,$$

- 6.

$$C\mathcal{F}C\mathcal{F} = \mathcal{F}C\mathcal{F}C = \text{Id},$$

- 7.

$$\mathcal{F}^4 = \text{Id},$$

8.  $\mathcal{F}$  is a unitary operator on  $L^2(\mathbb{R})$

$$\mathcal{F}^*\mathcal{F} = \text{Id},$$

9.  $\mathcal{F}$  is a Hilbert space isomorphism of  $L^2(\mathbb{R})$  onto itself.

**Proof:**

1. The surjectivity of  $\mathcal{F}$  follows from the fact that for any  $f \in L^2(\mathbb{R})$ ,

$$f = \mathcal{F}C\mathcal{F}Cf = \mathcal{F}g,$$

where  $g = C\mathcal{F}Cf$ .



- **Examples.**

### 4.10.3 Homework

- Exercises: 4.13[69,70,71]

## 4.11 Unbounded Operators

- Let  $H$  be a Hilbert space and  $A$  be an operator in  $H$ .
- We say that  $A$  is defined in a Hilbert space if the range of  $A$  is a proper subset of  $H$ , that is  $A$  is not surjective.
- $A$  is **unbounded** if it is not bounded.
- To show that  $A$  is unbounded, one has to find a bounded sequence  $(x_n)$  in  $H$  such that

$$\|Ax_n\| \rightarrow \infty.$$

- Unboundedness is equivalent to discontinuity at every point.
- To show that  $A$  is unbounded, one has to find a sequence  $(x_n)$  converging to 0 such that the sequence  $(Ax_n)$  does not converge to 0.
- **Convention.** If the domain  $D(A)$  of the operator  $A$  is the whole space  $H$  then we say that

$A$  is an operator **on**  $H$ .

If the domain  $D(A)$  of the operator  $A$  is a proper subset of  $H$  then we say that

$A$  is an operator **in**  $H$ .

- If  $A$  is a bounded operator in a Hilbert space  $H$ , then  $A$  has a unique extension to a bounded operator defined on the closure of  $D(A)$ .
- There exists a bounded operator  $B$  defined on the closure  $\overline{D(A)}$  such that

$$Ax = Bx \text{ for every } x \in D(A).$$

The operator  $B$  is defined by continuity. That is, for any  $x \in \overline{D(A)}$  let  $(x_n)$  be a sequence in  $D(A)$  such that  $x_n \rightarrow x$ . Then

$$Bx = \lim_{n \rightarrow \infty} Ax_n.$$

In this case,

$$\|A\| = \|B\|.$$

- Then the operator  $B$  can be extended to a bounded operator  $C$  defined on the whole space  $H$  by

$$C = BP_{D(B)}.$$

Then  $\|C\| = \|B\|$ .

- Thus, a bounded operator can always be extended to the whole space  $H$ , so one can always assume that the domain of a bounded operator is the whole space  $H$ .
- An unbounded operator does not have a natural unique extension onto the closure of its domain.

**Definition 4.11.1 Extension of Operators.** *Let  $E$  be a vector space and  $A$  be an operator in  $E$ . An operator  $B$  in  $E$  is called an **extension** of the operator  $A$  (denoted by  $A \subset B$ ) if*

- $$D(A) \subset D(B),$$

and

$$Ax = Bx \quad \text{for all } x \in D(A).$$

- We have

$$D(A + B) = D(A) \cap D(B) \text{ and } D(AB) = \{x \in D(B) \mid Bx \in D(A)\}$$

- We have

$$[(A + B)C] = (AC + BC)$$

but only

$$(AB + AC) \subset [A(B + C)].$$

**Definition 4.11.2 Densely Defined Operator.** *Let  $E$  be a vector space and  $A$  be an operator in  $E$ . We say that  $A$  is **densely defined** if  $D(A)$  is dense in  $E$ , that is  $\overline{D(A)} = E$ .*

-

**Definition 4.11.3 Adjoint of a Densely Defined Operator.**

Let  $H$  be a Hilbert space and  $A$  be a densely defined operator in  $H$ . Let  $y \in H$  and  $\varphi_y : D(A) \rightarrow \mathbb{C}$  be a linear functional on  $D(A)$  defined for  $x \in D(A)$  by

$$\varphi_y(x) = (Ax, y).$$

•

The **adjoint**  $A^*$  of  $A$  is an operator with the domain

$$D(A^*) = \{y \in H \mid \varphi_y \text{ is continuous on } D(A)\}$$

and such that

$$(Ax, y) = (x, A^*y) \text{ for all } x \in D(A) \text{ and } y \in D(A^*).$$

- If  $A$  is not densely defined, then  $A^*$  is not uniquely defined.

**Theorem 4.11.1** Let  $H$  be a Hilbert space and  $A$  and  $B$  be densely defined operators in  $H$ .

1. If  $A \subset B$ , then  $B^* \subset A^*$ .

(The adjoint of a densely defined operator  $A$  is the extension of the adjoint of the extension of  $A$ .)

•

2. If  $D(B^*)$  is dense in  $H$ , then  $B \subset B^{**}$ .

(If the adjoint of a densely defined operator  $B$  is densely defined, then the adjoint of the adjoint is the extension of the operator  $B$ .)

**Proof:**

1. (I). Let  $A \subset B$ . Then for all  $x \in D(A)$  and  $y \in D(B^*)$

$$(Ax, y) = (Bx, y) = (x, B^*y).$$

2. We also have for all  $x \in D(A)$  and  $y \in D(A^*)$

$$(Ax, y) = (x, A^*y).$$

3. Thus  $D(B^*) \subset D(A^*)$  and for all  $y \in D(B^*)$

$$A^*y = B^*y.$$

4. Therefore,

$$B^* \subset A^* .$$

5. (II). Let  $D(B^*)$  be dense in  $H$ . Then since for all  $x \in D(B)$  and  $y \in D(B^*)$

$$(B^*y, x) = (y, Bx) ,$$

$B^{**}$  exists and for all  $x \in D(B^{**})$  and  $y \in D(B^*)$

$$(B^*y, x) = (y, B^{**}x) .$$

6. Therefore,  $D(B) \subset D(B^{**})$  and for any  $x \in D(B)$

$$Bx = B^{**}x .$$

■

**Theorem 4.11.2** *Let  $H$  be a Hilbert space and  $A$  be a densely defined injective operator in  $H$  such that the inverse  $A^{-1}$  is densely defined. Then the adjoint  $A^*$  is also injective and*

$$(A^*)^{-1} = (A^{-1})^* .$$

**Proof:**

1. Let  $y \in D(A^*)$ . Then for any  $x \in D(A^{-1})$

$$(A^{-1}x, A^*y) = (AA^{-1}x, y) = (x, y) .$$

2. Thus  $A^*y \in D((A^{-1})^*)$  and

$$(A^{-1})^*A^*y = y .$$

3. Then for any  $y \in D((A^{-1})^*)$  and  $x \in D(A)$

$$(Ax, (A^{-1})^*y) = (x, y) .$$

4. Thus  $(A^{-1})^* \in D(A^*)$  and

$$A^*(A^{-1})^*y = y .$$

■

**Theorem 4.11.3** *Let  $H$  be a Hilbert space and  $A, B$  and  $AB$  be densely defined operators in  $H$ . Then*

$$B^*A^* \subset (AB)^*.$$

**Proof:**

1. Let  $x \in D(AB)$  and  $y \in D(B^*A^*)$ .

2. Then

$$(ABx, y) = (Bx, A^*y) = (x, B^*A^*y).$$

3. Therefore  $y \in D((AB)^*)$  and

$$B^*A^*y = (AB)^*y.$$

■

**Definition 4.11.4 Self-Adjoint Operator.** *A densely defined operator  $A$  in a Hilbert space  $H$  is **self-adjoint** if*

$$A^* = A,$$

*in particular,  $D(A^*) = D(A)$  and*

$$(Ax, y) = (x, Ay) \text{ for all } x, y \in D(A).$$

- If  $A$  is a bounded densely defined operator in  $H$ , then  $A$  has a unique extension to a bounded operator on  $H$ . Then

$$D(A) = D(A^*) = H.$$

- For unbounded operators, it is possible that

$$Ax = A^*x \text{ for any } x \in D(A) \cap D(A^*), \text{ but } D(A) \neq D(A^*).$$

Then  $A$  is not self-adjoint.

**Definition 4.11.5 Symmetric Operator.** *A densely defined operator  $A$  in a Hilbert space  $H$  is **symmetric** if*

$$(Ax, y) = (x, Ay) \text{ for all } x, y \in D(A).$$

- Every self-adjoint operator is symmetric.
- **Example.** Let  $H = l^2$  and  $A$  be a self-adjoint injective operator on  $H$  defined by

$$A(x_n) = \left( \frac{x_n}{n} \right).$$

The domain of the inverse operator  $D(A^{-1}) = R(A)$  is dense in  $H$ . The inverse operator is defined by

$$A^{-1}(x_n) = (nx_n).$$

Then  $A^{-1}$  is an unbounded operator. We also have

$$(A^{-1})^* = (A^*)^{-1} = A^{-1}.$$

Thus,  $A^{-1}$  is self-adjoint unbounded operator.

- **Example.** Let  $H = L^2([0, 1])$  and

$$A = i \frac{d}{dt}$$

be the differential operator in  $H$  with the domain

$$D(A) = \{f \in H \mid f' \text{ is continuous and } f(0) = f(1) = 0\}.$$

Then for any  $f, g \in D(A)$

$$(Af, g) = (f, Ag).$$

Thus  $A$  is symmetric.

Let  $g \in H$  and  $\varphi_g$  be a functional on  $D(A)$  defined by

$$\varphi_g(f) = (Af, g).$$

Then  $\varphi_g$  is continuous on  $D(A)$  for any continuously differentiable function  $g \in H$ , not necessarily satisfying  $g(0) = g(1) = 0$ . Therefore,

$$D(A) \subset D(A^*)$$

and  $A$  is not self-adjoint.

- **Theorem 4.11.4** *Let  $H$  be a Hilbert space and  $A$  be a densely defined operator in  $H$ . Then  $A$  is symmetric if and only if  $A \subset A^*$ .*

**Proof:**

1. (I). Let  $A \subset A^*$ . Then for all  $x, y \in D(A)$

$$(Ax, y) = (x, Ay).$$

2. Thus  $A$  is symmetric.

3. (II). Suppose that  $A$  is symmetric. Then for all  $x, y \in D(A)$

$$(Ax, y) = (x, Ay)$$

and for all  $x \in D(A), y \in D(A^*)$

$$(Ax, y) = (x, A^*y).$$

4. Thus  $A \subset A^*$ . ■

- **Definition 4.11.6** *Let  $E_1$  and  $E_2$  be vector spaces,  $D(A) \subset E_1$  and  $R(A) \subset E_2$ . The **graph**  $G(A)$  of an operator  $A : D(A) \rightarrow R(A)$  is a subset of  $E_1 \times E_2$  defined by*

$$G(A) = \{(x, Ax) \mid x \in D(A)\}.$$

- If  $A \subset B$  ( $B$  is an extension of  $A$ ), then  $G(A) \subset G(B)$ .

- **Definition 4.11.7 Closed Operator.** *Let  $E_1$  and  $E_2$  be normed spaces and  $A : E_1 \rightarrow E_2$  be an operator from  $E_1$  into  $E_2$ . The operator  $A$  is **closed** if its graph  $G(A)$  is a closed subspace of  $E_1 \times E_2$ , that is*

*if  $x_n \in D(A)$ ,  $x_n \rightarrow x$ , and  $Ax_n \rightarrow y$ , then  $x \in D(A)$  and  $Ax = y$ .*

- The domain  $D(A)$  of a closed operator does not have to be closed.

- **Theorem 4.11.5 Closed Graph Theorem.** *Every closed operator from a Banach space into a Banach space is bounded.*

- The domain of an unbounded operator in a Hilbert space cannot be closed. It cannot be the whole space.

- **Theorem 4.11.6** *The inverse of a closed operator is closed.*

**Proof:** Obvious. ■

- **Theorem 4.11.7** *Let  $H$  be a Hilbert space and  $A$  be a densely defined operator in  $H$ . Then the adjoint  $A^*$  is a closed operator.*

**Proof:**

1. Let  $y_n \in D(A^*)$  be a sequence such that

$$y_n \rightarrow y \quad \text{and} \quad A^*y_n \rightarrow z.$$

2. Then, for any  $x \in D(A)$ ,

$$(Ax, y) = \lim_{n \rightarrow \infty} (Ax, y_n) = \lim_{n \rightarrow \infty} (x, A^*y_n) = (x, z).$$

3. Therefore,  $y \in D(A^*)$  and  $A^*y = z$ . ■

- A symmetric operator can be always extended to a closed operator.

- **Theorem 4.11.8** *Let  $H$  be a Hilbert space and  $A$  be a densely defined symmetric operator in  $H$ . Then there exist a closed symmetric operator  $B$  in  $H$  such that  $A \subset B$ .*

*A densely defined symmetric operator has a closed symmetric extension.*

**Proof:**

1. Let  $D(B)$  be the set of all  $x \in H$  for which there is a sequence  $(x_n)$  in  $D(A)$  and  $y \in H$  such that

$$x_n \rightarrow x \quad \text{and} \quad Ax_n \rightarrow y.$$

2. Then  $D(B)$  is a vector space and  $D(A) \subset D(B)$ .

3. Let  $x \in D(B)$  and  $x_n$  be such that  $x_n \rightarrow x$  and the limit  $y = \lim_{n \rightarrow \infty} Ax_n$  exists. Then  $B$  is defined by

$$Bx = \lim_{n \rightarrow \infty} Ax_n.$$

4. Claim: The value  $Bx$  does not depend on the representing sequence  $x_n$ .
5. Let  $x_n \rightarrow x$  and  $z_n \rightarrow x$ . Let  $Ax_n \rightarrow y$  and  $Az_n \rightarrow w$ .
6. Then for any  $u \in D(A)$

$$(u, Ax_n - Az_n) = (Au, x_n - z_n).$$

7. As  $n \rightarrow \infty$

$$(u, (y - w)) = 0.$$

8. Thus,  $y - w \in D(A)^\perp$ .
9. Since  $D(A)$  is dense in  $H$ ,  $y - w = 0$ .
10. So,  $B$  is well defined.
11. Claim:  $B$  is an extension of  $A$ .
12. For any  $x \in D(A)$  we can just take a constant sequence  $x_n = x$ . Thus  $Bx = Ax$ .
13. Claim:  $B$  is symmetric.
14. Let  $x, y \in D(B)$ . Then there are sequences  $x_n$  and  $y_n$  in  $D(A)$  such that  $x_n \rightarrow x$ ,  $y_n \rightarrow y$  and  $Ax_n \rightarrow Bx$  and  $Ay_n = By$ .
15. Then

$$(Ax_n, y_n) = (x_n, Ay_n),$$

and as  $n \rightarrow \infty$

$$(Bx, y) = (x, By).$$

16. Claim:  $B$  is closed.
17. Let  $x \in D(B)$  and  $x_n$  be a sequence in  $D(B)$  such that  $x_n \rightarrow x$  and  $Bx_n \rightarrow y$ .
18. Claim:  $x \in D(B)$  and  $Bx = y$ .
19. For any  $m \in \mathbb{N}$  there is  $y_m \in D(B)$  such that

$$\|x_n - y_m\| < \frac{1}{m}, \quad \text{and} \quad \|Bx_m - Ay_m\| < \frac{1}{m}.$$

20. Therefore,

$$y_m \rightarrow x, \quad \text{and} \quad Ay_m \rightarrow y.$$

21. This means  $x \in D(B)$  and

$$Bx = y.$$

■

**Theorem 4.11.9** *Let  $H$  be a Hilbert space and  $A$  be a densely defined closed operator in  $H$ .*

1. *For any  $u, v \in H$  there exists unique  $x \in D(A)$  and  $y \in D(A^*)$  such that*

$$Ax + y = u \quad \text{and} \quad x - A^*y = v.$$

2. *For any  $v \in H$ , there exists a unique  $x \in D(A^*)$  such that*

$$A^*Ax + x = v.$$

**Proof:**

1. (I). Let  $H_1 = H \times H$ .
2. Then  $G(A)$  is a closed subspace of  $H_1$ .
3. Thus,

$$H_1 = G(A) \oplus G(A)^\perp.$$

4. We have  $(z, y) \in G(A)^\perp$  if and only if for all  $x \in D(A)$

$$(x, z) + (Ax, y) = 0, \quad \text{or} \quad (Ax, y) = (x, -z);$$

equivalently  $y \in D(A^*)$  and  $z = -A^*y$ .

5. Thus, if  $(v, u) \in H \times H$ , then there is unique  $x \in D(A)$  and  $y \in D(A^*)$  such that

$$(v, u) = (x, Ax) + (-A^*y, y).$$

6. (II). If  $u = 0$  above, then there are unique  $x \in D(A)$  and  $y \in D(A^*)$  such that

$$Ax + y = 0, \quad \text{and} \quad x - A^*y = v.$$

7. Thus,

$$x + A^*Ax = v.$$



- **Remark.** Let  $H$  be a Hilbert space and  $A$  be a closed operator in  $H$ . It is possible to redefine the inner product on  $D(A)$  by

$$(x, y)_1 = (x, y) + (Ax, Ay).$$

Then  $D(A)$  is complete with respect to the norm

$$\|x\|_1^2 = \|x\|^2 + \|Ax\|^2.$$

$D(A)$  is a Hilbert space with the inner product  $(\cdot, \cdot)_1$ . The operator  $A$  is a bounded operator on  $D(A)$  in this norm.

#### 4.11.1 Homework

- Exercises: 4.13[72,73,75,76,77]

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# Notation

## Notation

### Logic

$A \implies B$	$A$ implies $B$
$A \impliedby B$	$A$ is implied by $B$
iff	if and only if
$A \iff B$	$A$ implies $B$ and is implied by $B$
$\forall x \in X$	for all $x$ in $X$
$\exists x \in X$	there exists an $x$ in $X$ such that

### Sets and Functions (Mappings)

$x \in X$	$x$ is an element of the set $X$
$x \notin X$	$x$ is not in $X$
$\{x \in X \mid P(x)\}$	the set of elements $x$ of the set $X$ obeying the property $P(x)$
$A \subset X$	$A$ is a subset of $X$
$X \setminus A$	complement of $A$ in $X$
$\overline{A}$	closure of set $A$
$X \times Y$	Cartesian product of $X$ and $Y$
$f : X \rightarrow Y$	mapping (function) from $X$ to $Y$
$f(X)$	range of $f$
$\chi_A$	characteristic function of the set $A$
$\emptyset$	empty set
$\mathbb{N}$	set of natural numbers (positive integers)
$\mathbb{Z}$	set of integer numbers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers

$\mathbb{R}_+$	set of positive real numbers
$\mathbb{C}$	set of complex numbers

### Vector Spaces

$H \oplus G$	direct sum of $H$ and $G$
$H^*$	dual space
$\mathbb{R}^n$	vector space of $n$ -tuples of real numbers
$\mathbb{C}^n$	vector space of $n$ -tuples of complex numbers
$l^2$	space of square summable sequences
$l^p$	space of sequences summable with $p$ -th power

### Normed Linear Spaces

$\ x\ $	norm of $x$
$x_n \longrightarrow x$	(strong) convergence
$x_n \xrightarrow{w} x$	weak convergence

### Function Spaces

$\text{supp } f$	support of $f$
$H \otimes G$	tensor product of $H$ and $G$
$C_0(\mathbb{R}^n)$	space of continuous functions with bounded support in $\mathbb{R}^n$
$C(\Omega)$	space of continuous functions on $\Omega$
$C^k(\Omega)$	space of $k$ -times differentiable functions on $\Omega$
$C^\infty(\Omega)$	space of smooth (infinitely differentiable) functions on $\Omega$
$\mathcal{D}(\mathbb{R}^n)$	space of test functions (Schwartz class)
$L^1(\Omega)$	space of integrable functions on $\Omega$
$L^2(\Omega)$	space of square integrable functions on $\Omega$
$L^p(\Omega)$	space of functions integrable with $p$ -th power on $\Omega$
$H^m(\Omega)$	Sobolev spaces
$C_0(V, \mathbb{R}^n)$	space of continuous vector valued functions with bounded support in $\mathbb{R}^n$
$C^k(V, \Omega)$	space of $k$ -times differentiable vector valued functions on $\Omega$

$C^\infty(V, \Omega)$	space of smooth vector valued functions on $\Omega$
$\mathcal{D}(V, \mathbb{R}^n)$	space of vector valued test functions (Schwartz class)
$L^1(V, \Omega)$	space of integrable vector valued functions on $\Omega$
$L^2(V, \Omega)$	space of square integrable vector valued functions on $\Omega$
$L^p(V, \Omega)$	space of vector valued functions integrable with $p$ -th power on $\Omega$
$H^m(V, \Omega)$	Sobolev spaces of vector valued functions

### Linear Operators

$D^\alpha$	differential operator
$L(H, G)$	space of bounded linear transformations from $H$ to $G$
$H^* = L(H, \mathbb{C})$	space of bounded linear functionals (dual space)

