Lecture Notes
Introduction to Differential Geometry
MATH 442

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Contents

Chapter 1

Manifolds

1.1 Submanifolds of Euclidean Space

- Idea: Manifold is a general space that looks locally like a Euclidean space of the same dimension. This allows to develop the differential and integral calculus.
- Let $n \in \mathbb{N}$ be a positive integer. The **Euclidean space** $\mathbb{R}^n$ is a set of points x described by ordered n-tuples $(x^1, \dots, x^n)$ or real numbers.
- The numbers $x^i \in \mathbb{R}$, $i = 1, \dots, n$, are called the **Cartesian coordinates** of the point x.
- The integer n is the **dimension** of the Euclidean space.
- The **distance** between two points of the Euclidean space is defined by

$$d(x,y) = \sqrt{\sum_{k=1}^{n} (x^k - y^k)^2}\,.$$

- The **open ball** of radius ε centered at x_0 is the set of points defined by

$$B_\varepsilon(x_0) = \{x \in \mathbb{R}^n \mid d(x, x_0) < \varepsilon\}\,.$$

- A **neighborhood** of a point x_0 is the set of points that contain an open ball around it.

- Let $x_0 \in \mathbb{R}^n$ be a fixed point with Cartesian coordinates x^i, $i = 1, \dots, n$, in the Euclidean space and $S \subset \mathbb{R}^n$ be a neighborhood of x_0. An injective (one-to-one) map
$$f : S \to \mathbb{R}^n$$
defined by
$$y^i = f^i(x^1, \dots, x^n), \qquad i = 1, \dots, n,$$
where $f^i(x)$ are smooth functions, is called a **coordinate system** in S.

- The map f is injective if for any point x in S
$$\det\left(\frac{\partial f^i}{\partial x^j}\right) \neq 0 .$$

1.1.1 Submanifolds of $\mathbb{R}^n$

- Let $n, r \in \mathbb{N}$ be positive integers and $m = n + r$. Let $M \subset \mathbb{R}^m$ be a subset of the Euclidean space $\mathbb{R}^m$.

- Then M is a **submanifold** of $\mathbb{R}^m$ of dimension n if for any point $x_0 \in M$ in M there exists a neighborhood with a coordinate system such that every point in this neighborhood has coordinates $(x^1, \dots, x^n, x^{n+1}, \dots, x^{n+r})$, where the last r coordinates $(x^{n+1}, \dots, x^{n+r})$ are given by smooth functions of the first n coordinates $(x^1, \dots, x^n)$:
$$x^\alpha = f^\alpha(x^1, \dots, x^n), \qquad \alpha = n + 1, \dots, n + r .$$

- The coordinates $(x^1, \dots, x^n)$ are called the **local coordinates** for M near x_0.

- More generally, let
$$F : \mathbb{R}^m \to \mathbb{R}^r$$
be a smooth map described by r equations
$$y^\alpha = F^\alpha(x^1, \dots, x^m), \qquad \alpha = 1, \dots, r .$$
Let $y_0 \in \mathbb{R}^r$ and
$$M = F^{-1}(y_0) = \{x \in \mathbb{R}^m \mid F(x) = y_0\}$$

be a subset of the Euclidean space $\mathbb{R}^m$ described by the locus of r equations

$$F^\alpha(x^1,\dots,x^m) = y_0^\alpha\,, \qquad \alpha = 1,\dots,r\,.$$

Suppose that M is non-empty and let $x_0 \in M$, that is

$$F(x_0) = y_0\,.$$

- Then the **Implicit Function Theorem** says that if

$$\det\left(\frac{\partial F^\alpha}{\partial x^\beta}(x_0)\right) \neq 0\,,$$

  where $\alpha = 1,\dots,r$ and $\beta = n+1,\dots,n+r$, then there is a neighborhood of x_0 such that the last r coordinates can be expressed as smooth functions of the first n coordinates:

$$x^\alpha = f^\alpha(x^1,\dots,x^n)\,, \qquad \alpha = n+1,\dots,n+r\,.$$

- If this is true for any point of M, then M is a n-dimensional submanifold of $\mathbb{R}^m$.

- The matrix

$$\left(\frac{\partial F^\alpha}{\partial x^j}\right),$$

  where $\alpha = 1,\dots,r$ and $j = 1,\dots,m$ is called the **Jacobian matrix**.

- The **General Implicit Function Theorem** says that if at a point x_0 the Jacobian matrix has the maximal rank equal to r,

$$\operatorname{rank}\left(\frac{\partial F^\alpha}{\partial x^j}(x_0)\right) = r\,,$$

  then there exists a coordinate system in a neighborhood of x_0 such that the last r coordinates can be expressed as smooth functions of the first n coordinates.

- If this is true for every point of M, then M is a n-dimensional submanifold of $\mathbb{R}^m$.

- The number r is called the **codimension** of M.

- If the codimension r is equal to 1, that is $n = m - 1$, then M is called a **hypersurface**.

1.1.2 Differential of a Map

- Let $\mathbb{R}^n$ be a Euclidean space and $x_0 \in \mathbb{R}^m$. Then the **tangent space** to $\mathbb{R}^m$ at x_0 is a vector space $\mathbb{R}^m_x$ of all vectors in $\mathbb{R}^m$ based at x.

- Let
$$F : \mathbb{R}^m \to \mathbb{R}^r$$
be a smooth map described by r smooth functions
$$y^\alpha = F^\alpha(x^1, \dots, x^n)\,, \qquad \alpha = 1, \dots, r\,.$$
Let $x_0 \in \mathbb{R}^m$ and $x = x(t)$, $t \in (-\varepsilon, \varepsilon)$, be a curve in $\mathbb{R}^m$ such that
$$x(0) = x_0 \text{ and } \frac{dx}{dt}(0) = \mathbf{v}\,,$$
where $\mathbf{v} \in \mathbb{R}^m_{x_0}$ is the tangent vector to the curve at x_0.

- Let $y_0 = F(x_0) \in \mathbb{R}^r$ and $y(t) = F(x(t))$. Then
$$y(0) = y_0 \text{ and } \frac{dy}{dt}(0) = \mathbf{w}\,,$$
where $\mathbf{w} \in \mathbb{R}^r_{x_0}$ is the tangent vector to the image of the curve at y_0.

- We compute
$$w^\alpha = \sum_{i=1}^m \left(\frac{\partial F^\alpha}{\partial x^i}\right)(x_0) v^i\,.$$

- Thus there is a linear transformation
$$F_* : \mathbb{R}^m_{x_0} \to \mathbb{R}^r_{y_0}\,,$$
so that
$$F_* \mathbf{v} = \mathbf{w}\,.$$
F_* is called the **differential** of the map F at x_0.

1.1.3 Main Theorem on Submanifolds of $\mathbb{R}^m$

- The matrix of the linear transformation F_* is exactly the Jacobian matrix.
- Therefore, the differential F_* at a point x_0 is a **surjective (onto)** map if and only if $m \geq r$ and the Jacobian at x_0 has the maximal rank equal to r.
- Recall that $F_* : \mathbb{R}^m_{x_0} \to \mathbb{R}^r_{y_0}$ is surjective if for any $\mathbf{w} \in \mathbb{R}^r_{y_0}$ there is $\mathbf{v} \in \mathbb{R}^m_{x_0}$ such that $F_*\mathbf{v} = \mathbf{w}$.
- Thus, we have the following theorem.
- **Theorem 1.1.1** *Let $F : \mathbb{R}^m \to \mathbb{R}^r$ with $m > r$, $y_0 \in \mathbb{R}^r$ and*

$$M = F^{-1}(y_0) = \{x \in \mathbb{R}^m \mid F(x) = y_0\} .$$

If M is non-empty and for any $x_0 \in M$ the differential $F_* : \mathbb{R}^m_{x_0} \to \mathbb{R}^r_{y_0}$ is surjective, then M is a $n = (m - r)$-dimensional submanifold of $\mathbb{R}^m$.

1.2 Manifolds

1.2.1 Basic Notions of Topology

- First we define the basic topological notions in the Euclidean space $\mathbb{R}^n$.

- Let $x_0 \in \mathbb{R}^n$ be a point in $\mathbb{R}^n$ and $\varepsilon > 0$ be a positive real number.

- The **open ball** in $\mathbb{R}^n$ of radius ε with the center at x_0 is the set

$$\bar{B}_\varepsilon(x_0) = \{x \in \mathbb{R}^n \mid d(x, x_0) < \varepsilon\}\,.$$

- The **closed ball** in $\mathbb{R}^n$ of radius ε with the center at x_0 is the set

$$\bar{B}_\varepsilon(x_0) = \{x \in \mathbb{R}^n \mid d(x, x_0) \le \varepsilon\}\,.$$

- Let $U \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$. A point $x \in U$ is an **interior point** of U if there is an open ball $B_\varepsilon(x)$ of some radius $\varepsilon > 0$ centered at x such that $B_\varepsilon(x) \subset U$.

- A point $x \in \mathbb{R}^n$ is a **boundary point** of U if every open ball $B_\varepsilon(x)$ of any radius $\varepsilon > 0$ centered at x contains at least one point from U and one point from its complement (that is not from U).

- The set U^o of all interior points of U is called the **interior** of U.

- The set ∂U of all boundary points of U is called the **boundary** of U.

- A set $U \subseteq \mathbb{R}^n$ is **open** if every point of U is an interior point of U, that is, $U = U^o$.

- A set $F \subseteq \mathbb{R}^n$ is **closed** if its complement $\mathbb{R}^n \setminus F$ is open, that is, $F = F^o \cup \partial F$.

- The sets $\emptyset$ and $\mathbb{R}^n$ are both open and closed.

- The union of *any* collection of open sets is open.

- The intersection of any *finite* number of open sets is open.

- **Definition 1.2.1** *A general* **topological space** *is a set M together with a collection of subsets of M, called* **open sets**, *that satisfy the following properties*

  1. *M and $\emptyset$ are open,*
  2. *the intersection of any finite number of open sets is open,*
  3. *the union of any collection of open sets is open.*

  Such a collection of open sets is called a **topology** *of M.*

- A subset of M is **closed** if its complement $M \setminus F$ is open.

- The **closure** $\bar{S}$ of a subset $S \subseteq\in M$ of a topological space M is the intersection of all closed sets that contain S; it is equal to $\bar{S} = S \cup \partial S$.

- A subset $S \subseteq M$ of a topological space is **dense** in M if $\bar{S} = M$, that is, every non-empty subset of M contains an element of S.

- A topological space is called **separable** if it contains a countable dense subset.

- A topology on M naturally induces a topology on any subset of M. Let $A \subseteq M$ be a subset of M. Then the **induced topology** on A is defined as follows. A subset $V \subseteq A$ is defined to be open subset of A if there is an open subset $U \subseteq M$ of M such that $V = U \cap A$.

- Let $x \in M$ be a point in a topological space M. An open set in M containing the point x is called a **neighborhood** of x.

- **Definition 1.2.2** *Let M and N be two topological spaces and $F : M \to N$ be a map from M into N. The map F is said to be* **continuous** *if the inverse image of any open set in N is an open set in M.*

  That is, if for any open set $V \subset N$ the set

$$F^{-1}(V) = \{x \in M \mid F(x) \in V\}$$

  is open in M.

- The direct images of open sets do not have to be open!

- Let the map F be bijective, that is, injective (one-to-one) and surjective (onto). Then there exists the inverse map $F^{-1} : N \to M$.

- **Definition 1.2.3** *A map $F : M \to N$ is called a* **homeomorphism** *if it is bijective and both F and F^{-1} are continuous.*

- Homeomorphisms preserve topology, that is, they take open sets to open sets and closed sets to closed sets.

- A topological space M is called **Hausdorff** if any two points of M have disjoint neighborhoods.

- A collection of subsets of a topological space M is called an **open cover** of M if the union of all subsets in the collection coincides with M.

- A subcollection of subsets that is itself a cover is called a **subcover**.

- **Definition 1.2.4** *A topological space M is called* **compact** *if every open cover of M has a finite subcover.*

- A subset $M \in \mathbb{R}^n$ of a Euclidean space if called **bounded** if there is an open ball $B_C(0)$ of some radius C centered at the origin such that $M \subseteq B_C(0)$.

- **Theorem 1.2.1** **Bolzano-Weierstrass Theorem.** *Let $M \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$ with the induced topology. Then M is compact if and only if M is closed and bounded in $\mathbb{R}^n$.*

- **Properties of Continuous Maps.**

- **Theorem 1.2.2** *Let M and N be two topological spaces and M be compact. Let $F : M \to N$ be a continuous map from M into N. Then the image $F(M)$ of M is compact in N.*

  That is, a continuous image of a compact topological space is compact.

  ***Proof*:**

  1. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of $F(M)$ in N.
  2. Then $\{F^{-1}(U_\alpha)\}_{\alpha\in A}$ is an open cover of M.
  3. Since M is compact it has a finite subcover $\{F^{-1}(U_i)\}_{i=1}^n$.
  4. Then $\{U_i\}_{i=1}^n$ is a finite subcover of $F(M)$.

5. Thus $F(M)$ is compact.

■

- **Theorem 1.2.3** *A continuous real-valued function $f : M \to \mathbb{R}$ on a compact topological space M is bounded.*

  Proof:

  1. $f(M)$ is compact in $\mathbb{R}^n$.
  2. Thus $f(M)$ is closed and bounded.

  ■

1.2.2 Idea of a Manifold

- A manifold M of dimension n is a topological space that is locally homeomorphic to $\mathbb{R}^n$.

- A manifold M is covered by a family of local coordinate systems $\{U_\alpha; x^1_\alpha, \ldots, x^n_\alpha\}_{\alpha \in A}$, called an **atlas**, consisting of open sets, called **patches** (or **charts**), U_α, and coordinates x_α.

- A point $p \in U_\alpha \cap U_\beta$ that lies in two coordinate patches has two sets of coordinates x_α and x_β related by smooth functions

$$x^i_\alpha = f^i_{\alpha\beta}(x^1_\beta, \ldots, x^n_\beta), \qquad i = 1, \ldots, n.$$

- The coordinates x_α and x_β are said to be **compatible**.

- If all the functions $f_{\alpha\beta}$ are smooth, then the manifold M is said to be **smooth**. If these functions are analytic, then the manifold is said to be **real analytic**.

- Each patch is homeomorphic to some open subset in $\mathbb{R}^n$.

- Thus, a manifold is **locally Euclidean**.

- The collection of all patches (charts) is called an **atlas**.

- The collection of all coordinate systems that are compatible with those used to define a manifold is called the **maximal atlas**.

- Let M be an n-dimensional manifold with local coordinate systems $\{U_\alpha; x^1_\alpha, \dots, x^n_\alpha\}_{\alpha\in A}$ and N be an m-dimensional manifold with local coordinate systems $\{V_\beta; y^1_\beta, \dots, y^m_\beta\}_{\beta\in B}$. The **product manifold** $L = M \times N$ is a manifold

$$L = M \times N = \{(p, q) \mid p \in M,\ q \in N\}$$

with local coordinate systems $\{W_{\alpha\beta}; z^1_{\alpha\beta}, \dots, z^{n+m}_{\alpha\beta}\}_{\alpha\in A, \beta\in B}$ where

$$W_{\alpha\beta} = U_\alpha \times V_\beta$$

and

$$(z^1_{\alpha\beta}, \dots, z^n_{\alpha\beta}) = (x^1_\alpha, \dots, x^n_\alpha)$$

and

$$(z^{n+1}_{\alpha\beta}, \dots, z^{n+m}_{\alpha\beta}) = (y^1_\beta, \dots, y^m_\beta)\,.$$

Examples

- **Unit Sphere** S^n.

$$S^n = \{x \in \mathbb{R}^{n+1} \mid d(x, 0) = 1\}$$

Stereographic projection:

$$\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^n$$

Let

$$R = \sqrt{1 - \sum_{j=1}^{n} (x^j)^2}$$

Then

$$x^{n+1} = \pm R\,, \qquad \Phi_{1,2}(x) = \left(\frac{x^1}{1 \pm R}, \cdots, \frac{x^n}{1 \pm R}\right)\,.$$

- **Torus** $T^n = S^1 \times \cdots \times S^1$.
  - T^n has local coordinates $(\theta^1, \cdots, \theta^n)$ (angles).
  - Topologically it is the cube $[0, 1]^n$ with the antipodal points identified.

- **Real Projective Space** $\mathbb{R}P^n$.
  - $\mathbb{R}P^n$ is the space of unoriented lines through the origin of $\mathbb{R}^{n+1}$.

– The set of oriented lines through the origin of $\mathbb{R}^{n+1}$ is S^n.
– Topologically $\mathbb{R}P^n$ is the sphere S^n in $\mathbb{R}^{n+1}$ with the antipodal points identified, which is the unit ball in $\mathbb{R}^n$ with the antipodal points on the boundary (which is a unit sphere S^{n-1}) identified.
– $\mathbb{R}P^n$ is covered by $(n+1)$ sets

$$U_j = \{L \in \mathbb{R}P^n \mid L \text{ with } x^j \neq 0\}, \qquad j = 1, \ldots, n+1\,.$$

– The local coordinates in U_j are

$$v^1 = \frac{x^1}{x^j}, \cdots, v^n = \frac{x^n}{x^j}\,.$$

– The $(n+1)$-tuple $(x^1, \ldots, x^{n+1})$ identified with $(\lambda x^1, \ldots, \lambda x^{n+1})$, $\lambda \neq 0$, are called **homogeneous coordinates** of a point in $\mathbb{R}P^n$.

1.2.3 Rigorous Definition of a Manifold

- Let M be a set (without topology) and $\{U_\alpha\}_{\alpha\in A}$ be a collection of subsets that is a cover of M, that is,

$$\bigcup_{\alpha\in A} U_\alpha = M\,.$$

- Let

$$\varphi_\alpha : U_\alpha \to \mathbb{R}^n, \qquad \alpha \in A\,,$$

  be injective maps such that $\varphi_\alpha(U_\alpha)$ are open subsets in $\mathbb{R}^n$.

- The set $\varphi_\alpha(U_\alpha \cap U_\beta)$ is an open subset in $\mathbb{R}^n$. The maps

$$f_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : \ \varphi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{R}$$

  are called the **transition functions** (or the **overlap functions**). We assume that the transition functions are smooth.

- The pair $(U_\alpha, \varphi_\alpha)$ is called a **coordinate patch** (or **chart**).

- A point $p \in U_\alpha$ is assigned coordinates of the point $\varphi_\alpha(p)$ in $\mathbb{R}^n$. Thus, φ_α is called a **coordinate map**.

- Now, we take the maximal atlas of such coordinate patches.
- The **topology** in M is defined as follows.
- Let $W \subset M$ be a subset of M. A point $p \in W$ in W is said to be an **interior** point if there is a coordinate chart $(U_\alpha, \varphi_\alpha)$ including p such that $U \subset W$.
- A subset W of M is declared to be **open** if all of its points are interior points.
- If the resulting topological space is Hausdorff and separable, then M is said to be an n-dimensional **smooth manifold**.
- The regularity of the transition functions determines the regularity of the manifold. If the transition functions are only differentiable once, then the manifold is called **differentiable**. If they are of class C^k, then the manifold is called a manifold **of class** C^k. If they are of class C^∞, then the manifold is called **smooth**. If the transition functions are analytic, then the manifold is called **analytic**.
- Let $F : M \to \mathbb{R}$ be a real-valued function on M. Let (U_α, x_α) be a local coordinate system. Then the function

  $$F_\alpha = F \circ \varphi_\alpha^{-1} : \ \varphi_\alpha(U_\alpha) \to \mathbb{R}$$

  is a function of n real variables $F_\alpha(x_\alpha^1, \dots, x_\alpha^n)$.
- The function F is said to be **smooth** if the function F_α is smooth in terms of local coordinates x_α.
- The process of replacing the map F by the function $F_\alpha = F \circ \varphi_\alpha^{-1}$ is usually omitted, and the functions F and F_α are identified, so that we think of the function F directly in terms of local coordinates.

1.2.4 Complex Manifolds

- Let M be a set and $\{U_\alpha\}_{\alpha \in A}$ be its cover. Let

  $$\varphi_\alpha : U_\alpha \to \mathbb{C}^n$$

  be injective maps from U_α to the complex n-space $\mathbb{C}^n$ so that $\varphi_\alpha(U_\alpha)$ are open subsets of $\mathbb{C}^n$.

- Let the transition functions

$$f_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : \ \varphi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{C}^n ,$$

defined by

$$z_\alpha^k = f_{\alpha\beta}^k(z_\beta^1, \dots, z_\beta^n) ,$$

where $z_\alpha^k = x_\alpha^k + i y_\alpha^k$ and $z_\beta^k = x_\beta^k + i y_\beta^k$, $k = 1, \dots, n$, be complex analytic. That is, they satisfy Cauchy-Riemann conditions

$$\frac{\partial x_\alpha^k}{\partial x_\beta^j} = \frac{\partial y_\alpha^k}{\partial y_\beta^j} , \qquad \frac{\partial x_\alpha^k}{\partial y_\beta^j} = -\frac{\partial y_\alpha^k}{\partial x_\beta^j} ,$$

or

$$\frac{\partial z_\alpha^k}{\partial \overline{z}_\beta^j} = 0 ,$$

with $j, k = 1, \dots, n$.

- Then M is called a n-dimensional **complex manifold**.

- The topological dimension of a n-dimensional complex manifold is $2n$.

1.3 Tangent Vectors and Mappings

- A tangent vector to a submanifold M of $\mathbb{R}^n$ at a point $p_0 \in M$ is the velocity vector $\dot{p}$ at p_0 to some parametrized curve $p = p(t)$ lying on M and passing through the point p_0.
- **Whitney theorem**: Every n-dimensional manifold can be realized as a submanifold of $\mathbb{R}^{2n}$, or as a smooth submanifold of $\mathbb{R}^{2n+1}$.
- So, every manifold is a submanifold of a Euclidean space.
- However, the intrinsic geometry of M does not depend on its embedding in a Euclidean space.

1.3.1 Tangent Vectors

- We fix a point $p_0 \in M$ on manifold M and consider a curve $p : (-\varepsilon, \varepsilon) \to M$ such that
$$p(0) = p_0\,.$$
- Let (U_α, x^j_α) be a local chart about p_0. The curve is described in local coordinates by
$$x^i_\alpha = x^i_\alpha(t)\,.$$
- The **velocity vector** $\dot{p}(0)$ is described in (U_α, x_α) by an n-tuple
$$\left(\dot{x}^1_\alpha(0), \dots, \dot{x}^n_\alpha(0)\right)\,.$$
- Let (U_β, x^j_β) be another local chart containing p_0. Then the **velocity vector** $\dot{p}(0)$ is described in (U_β, x_β) by an n-tuple
$$\left(\dot{x}^1_\beta(0), \dots, \dot{x}^n_\beta(0)\right)\,.$$
- These n-tuples are related by the chain rule
$$\dot{x}^i_\beta(0) = \sum_{j=1}^n \left(\frac{\partial x^i_\beta}{\partial x^j_\alpha}\right)(p_0)\dot{x}^j_\alpha(0)\,.$$

- **Definition 1.3.1** *A* **tangent vector** *at a point $p_0 \in M$ of a manifold M is a map that assigns to each coordinate chart (U_α, x_α) about p_0 an ordered n-tuple $(X^1_\alpha, \ldots, X^n_\alpha)$ such that*

$$X^i_\beta = \sum_{j=1}^n \left(\frac{\partial x^i_\beta}{\partial x^j_\alpha}\right)(p_0) X^j_\alpha \,.$$

1.3.2 Vectors as Differential Operators

- Let $f : M \to \mathbb{R}$ be a real-valued function on M.
- Let $p \in M$ be a point on M and X be a tangent vector at p.
- Let (U_α, x_α) be a coordinate chart about p.
- The (directional) **derivative of** f **with respect to** X at p (or along X, or in the direction of X) is defined by

$$X_p(f) = D_X(f) = \sum_{j=1}^n \left(\frac{\partial f}{\partial x^j_\alpha}\right)(p) X^j_\alpha \,.$$

- **Theorem 1.3.1** *$D_X(f)$ does not depend on the local coordinate system.*

Proof:

1. Chain rule.

■

- The **intrinsic** properties are **invariant** under a change of coordinate system. They should not depend on the choice of the local chart.
- There is a one-to-one correspondence between tangent vectors to M at p and first-order differential operators acting on real-valued functions in a local coordinate chart (U_α, x_α) by

$$X_p = \sum_{j=1}^n X^j_\alpha \frac{\partial}{\partial x^j_\alpha}\bigg|_p \,.$$

- Therefore, we can identify tangent vectors and the differential operators.

- Let us fix an index $j = 1, \ldots, n$. The curve

$$x^i(t) = x^i_0\,, \qquad i \neq j, \qquad x^j(t) = t\,,$$

is called the j-th **coordinate curve**.

- The velocity vector to this curve is given by

$$\dot{x}^i(0) = \delta^i_j\,,$$

where δ^i_j is the **Kronecker symbol** defined by

$$\delta^i_j = \begin{cases} 1, & \text{if } i = j\,, \\ 0, & \text{if } i \neq j\,. \end{cases}$$

- The differential operator corresponding to this velocity vector is

$$\frac{\partial}{\partial x^j_\alpha}\,.$$

1.3.3 Tangent Space

- Let M be a manifold and $p \in M$ be a point in M. The **tangent space** T_pM to M at p is the real vector space of all tangent vectors to M at p.

- Let (U, x) be a local chart about p. Then the vectors

$$\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p$$

form a basis in the tangent space called the **coordinate basis**, or the **coordinate frame**.

- A **vector field** X on an open set $U \subset M$ in a manifold M is the differentiable assignment of a tangent vector X_p to each point $p \in U$.

- In local coordinates

$$X = \sum_{j=1}^n X^j(x)\frac{\partial}{\partial x^j}\,.$$

- **Example.**

1.3.4 Mappings

- Let M be an n-dimensional manifold and N be an m-dimensional manifold and let $F : M \to N$ be a map from M to N. Let $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ be an atlas in M and $(V_\beta, \psi_\beta)_{\beta\in B}$ be an atlas in N. We define the maps

$$F_{\alpha\beta} = \psi_\beta \circ F \circ \varphi_\alpha^{-1} : \ \varphi_\alpha(U_\alpha) \to \psi_\beta(V_\beta)$$

  from open sets in $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by

$$y^a_\beta = F^a_{\alpha\beta}(x^1_\alpha, \dots, x^n_\alpha)\,,$$

  where $a = 1, \dots, m$.

- The map F is said to be **smooth** if $F^a_{\alpha\beta}$ are smooth functions of local coordinates x^i_α, $i = 1, \dots, n$.

- The process of replacing the map F by the functions $F_{\alpha\beta} = \psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is usually omitted, and the maps F and $F_{\alpha\beta}$ are identified, so that we think of the map F directly in terms of local coordinates.

- If $n = m$ and the map $F : M \to N$ is bijective and both F and F^{-1} are differentiable, then F is called a **diffeomorphism**.

- That is, a diffeomorphism is a differentiable homeomorphism with differentiable inverse.

- If this is only true in a neighborhood of a point $p \in M$, then F is called a **local diffeomorphism**.

- **Example.**

- **Definition 1.3.2** *Let M and N be two manifolds and $F : M \to N$ be a map from M into N. Let $p_0 \in M$ be a point in M and $X \in T_{p_0}M$ be a tangent vector to M at p_0. Let $p = p(t)$, $t \in (-\varepsilon, \varepsilon)$, be a curve in M such that*
$$p(0) = p_0, \qquad \dot{p}(0) = X\,.$$
Then the **differential** *of F is the map*
$$F_* : T_{p_0}M \to T_{F(p_0)}N$$
defined by
$$F_*X = \frac{d}{dt}F(p(t))\Big|_{t=0}\,.$$

- F_* does not depend on the curve $p(t)$.

- The matrix of the linear transformation F_* in therms of the coordinate bases $\partial/\partial y^\alpha$ and $\partial/\partial x^i$ is the **Jacobian matrix**
$$(F_*)^\alpha{}_i = \frac{\partial y^\alpha}{\partial x^i}\,,$$
that is,
$$(F_*X)^\alpha{}_i = \sum_{i=1}^n \frac{\partial y^\alpha}{\partial x^i}X^i\,.$$

1.3.5 Submanifolds

- **Definition 1.3.3** *Let M be an n-dimensional manifold and $W \subset M$ be a subset of M. Then W is an r-dimensional* **embedded submanifold** *of M if W is locally described as the common locus of $(n-r)$ differentiable independent functions*
$$F^\alpha(x^1, \dots, x^n) = 0\,, \qquad \alpha = 1, \dots, n-r\,,$$
such that the Jacobian matrix has rank $(n-r)$ at each point of the locus, that is,
$$\operatorname{rank}\left(\frac{\partial F^\alpha}{\partial x^i}(x)\right) = n - r\,, \qquad \forall x \in W.$$

- Every embedded submanifold of a manifold is itself a manifold.

- **Theorem 1.3.2** *Let n and r be two positive integers such that $n > r$. Let M be an n-dimensional manifold and N be an r-dimensional manifold. Let $q \in N$ be a point in N such that the inverse image $W = F^{-1}(q) \neq \emptyset$ is nonempty. Suppose that for each point $p \in W$ the differential F_* of the map F is surjective, that is, has the maximal rank*

$$\operatorname{rank} F_*(p) = r\,.$$

Then W is an $(n-r)$ dimensional submanifold of M.

- **Example. Morse Map.** (Height function $F : M \to \mathbb{R}$ for trousers surface M in $\mathbb{R}^3$).

- Let $p_0 \in M$ and $\mathbf{v} \in T_{p_0}M$ be a tangent vector to M at p_0. Then $F_* : T_{p_0}M \to \mathbb{R} = T_{F(p_0)}M$ is the projection of $\mathbf{v}$ to z-axis defined by $F_*(v_1, v_2, v_3) = v_3$.

- Let $p_0 \in M$ and $z = F(p_0)$. $F^{-1}(z)$ is an embedded submanifold if F_* is onto, that is, $T_{p_0}M$ is not horizontal. If $T_{p_0}M$ is horizontal, then $F_* = 0$ (hence, not onto).

- **Definition 1.3.4** *Let M and N be two manifolds and $F : M \to N$ be a differentiable map from M into N.*

  A point $p \in M$ is a **regular point** *if the differential $F_* : T_pM \to T_{F(p)}N$ is surjective.*

  A point $p \in M$ is a **critical point** *if it is not regular.*

  A point $q \in N$ is a **regular value** *of F if the inverse image $F^{-1}(q)$ is either empty or consists only of regular points.*

  A point $q \in N$ is a **critical value** *of F if it is not regular.*

- Thus, we can reformulate the main theorem as follows.

  Theorem 1.3.3 *Let n and r be two positive integers such that $n > r$. Let M be an n-dimensional manifold and N be an r-dimensional manifold. Let $q \in N$ be a regular value of F. Then the inverse image $W = F^{-1}(q) \neq \emptyset$ is either empty or is an $(n-r)$ dimensional submanifold of M.*

- **Theorem 1.3.4 Sard's Theorem.** *Let M and N be two manifolds and $F : M \to N$ be a smooth map from M into N. Then the set of critical values of F is a set of measure zero in N, that is, almost all values of F are regular values.*

- The critical values of a map $F : M \to N$ cannot fill up any open set in N.

1.3.6 Change of Coordinates

- **Theorem 1.3.5 Inverse Function Theorem.** *Let M and N be two n-dimensional manifolds and $F : M \to N$ be a differentiable map from M into N. Let $p_0 \in M$ be such that the differential $F_* : T_{p_0}M \to T_{F(p_0)}N$ is bijective. Then F is a local diffeomorphism in a neighborhood U of p_0, that is, $F(U)$ is open in N and $F : U \to F(U)$ is a diffeomorphism.*

- **Theorem 1.3.6** *Let M be a manifold, $p_0 \in M$ be a point in M, and (U, x) be a local coordinate chart about p_0. Let $F : U \to \mathbb{R}^n$ be a differentiable map defined by $y^i = F^i(x)$, $i = 1, \dots, n$, such that*

  $$\det\left(\frac{\partial F^i}{\partial x^j}\right)(p_0) \neq 0\,.$$

  Then there is a neighborhood V of p_0 such that the n-tuple $(y^1, \dots, y^n)$ forms a compatible coordinate system in V.

- **Example.**

1.4 Vector Fields and Flows

1.4.1 Vector Fields in $\mathbb{R}^n$

- Let $x = (x^j) \in \mathbb{R}^n$ be a point in $\mathbb{R}^n$ and $\mathbf{v}(x) \in T_x\mathbb{R}^n$ be a vector at x given by

$$\mathbf{v} = \sum_{j=1}^{n} v^j(x)\partial_j ,$$

where $\partial_j = \partial/\partial x^j$ and the components $v^j(x)$ are smooth (or just differentiable) functions of x. Then $\mathbf{v}(x)$ is called a **vector field** in $\mathbb{R}^n$.

- Let $t \in (-\varepsilon, \varepsilon)$ and

$$\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$$

be a family of diffeomorphisms such that

$$\varphi_0 = \mathrm{Id}$$

and for any $t, t_1, t_2 \in (-\varepsilon, \varepsilon)$ such that $-t, t_1 + t_2 \in (-\varepsilon, \varepsilon)$ there holds

$$\varphi_{t_1} \circ \varphi_{t_2} = \varphi_{t_1+t_2}$$

and

$$\varphi_{-t} = \varphi_t^{-1} .$$

Such a **one-parameter group** of diffeomorphisms is called a **flow** on $\mathbb{R}^n$.

- A flow φ_t defines a vector field $\mathbf{v}$ by

$$\mathbf{v}(x) = \frac{d}{dt}\varphi_t(x)\Big|_{t=0}$$

with the components

$$v^j(x) = \frac{dx^j}{dt} .$$

- The corresponding differential operator

$$\mathbf{v}(f)(x) = \frac{d}{dt} f(\varphi_t(x))\Big|_{t=0} = \sum_{j=1}^{n} v^j(x)\frac{\partial f}{\partial x^j}$$

is the derivative along the streamline of the flow through the point p.

- Conversely, every vector field **v** determines a flow, which is determined by requiring **v** to be the velocity field of the flow.

- Such a flow is defined as the solution of the system of ordinary first-order differential equations

$$\frac{dx^j}{dt} = v^j(x^1(t), \dots, x^n(t)), \qquad j = 1, \dots, n,$$

with the initial conditions

$$x^j(0) = x_0^j .$$

The solution of this system defines the **integral curves** of the vector field **v**.

- **Theorem 1.4.1** *Fundamental Theorem on Vector Fields in $\mathbb{R}^n$ Let $\subset \mathbb{R}^n$ be an open set in $\mathbb{R}^n$ and $\mathbf{v} : U \to \mathbb{R}^n$ be a smooth vector field on U. Then for any point $x_0 \in U$ there is $\varepsilon > 0$ and a neighborhood V of x_0 such that:*

  1. *there is a unique curve (called the* **integral curve** *of* **v***) $x(t) = \phi_t(x_0)$, $t \in (-\varepsilon, \varepsilon)$ such that for any $t \in (-\varepsilon, \varepsilon)$*

  $$\dot{x}(t) \equiv \frac{dx(t)}{dt} = v(x(t))$$

  and $x(0) = x_0$;

  2. *the map*

  $$V \times (-\varepsilon, \varepsilon) \to \mathbb{R}^n$$

  defined by $(x, t) \mapsto \varphi_t(x)$ is smooth and for any $t_1, t_2 \in (-\varepsilon, \varepsilon)$ such that $t_1 + t_2 \in (-\varepsilon, \varepsilon)$

  $$\varphi_{t_1} \circ \varphi_{t_2} = \varphi_{t_1 + t_2}$$

  holds in V. The family of maps φ_t is called a **local one-parameter group of diffeomorphisms** *or a* **local flow**.

- **Remarks.**

- The local flow is only defined in a small neighborhood of the point x_0.

- The one-parameter group is not a group in the strict sense.

- The integral curves exist only for a small time.
- If the vector field is not differentiable, then the integral curve is not unique.

1.4.2 Vector Fields on Manifolds

- Let W be an open subset of a mnifold M and $\mathbf{v}$ be a smooth vector field on W.
- Let (U_α, x_α) be a local chart in W.
- If $W \subset U_\alpha$, then one can proceed as in $\mathbb{R}^n$.
- If W is not contained in a single chart, then we choose a cover of W and proceed as follows. Let $p \in W$ and (U_α, x_α) and (U_β, x_β) be two charts covering p.
- Then the integral curves in both local coordinate systems have the same meaning and define a unique integral curve in M. This defines a local flow on W in M. We just need to check that if the flow equations are satisfied in one coordinate system, then they are satisfied in another coordinate system.

1.4.3 Straightening Flows

- Let U be an open set in a manifold M and $\varphi_t : U \to M$ be a local flow on a M such that $\varphi_0(p) = p$. Then $\varphi_t(p)$ depends smoothly on both the time t and the point p.
- Note that if a vector field does not vanish at a point p, i.e. $\mathbf{v}_p \neq 0$, then it does not vanish in a neighborhood of p.
- Let p_0 be a fixed point in M and W be a sufficiently small hypersurface passing through p_0 such that $\mathbf{v}_p$ is **transversal** to W at every point of $p \in W$. This just means that $\mathbf{v}$ is nowhere tangent to W.
- If W is small enough it can be covered by a single coordinate chart. Let $(u^1, \dots, u^{n-1})$ be local coordinates for W such that the the point p_0 has coordinates $(0, \dots, 0)$.
- Then for a small neighborhood of a point p and sufficiently small t the n-tuple $(u^1, \dots, u^{n-1}, t)$ gives a local coordinate system near p in M.

- Proof: (follows from Inverse Function Theorem).
- Consider a map $F : W \times (-\varepsilon, \varepsilon) \to M$ given by $F(p, t) = \varphi_t(p)$.
- The differential of this map at p_0 (that is, at $u = 0$), is

  $$F_* \frac{\partial}{\partial u^j} = \frac{\partial}{\partial u^j}\,, \qquad j = 1, \ldots, n-1\,.$$

  Also,

  $$F_* \mathbf{v} = \frac{d}{dt}\varphi_t(p_0) = \mathbf{v}\,.$$

  Thus

  $$F_* = \mathrm{Id}\,,$$

  and, hence, $(u^1, \ldots, u^{n-1}, t)$ can be used as a coordinate system near p_0.
- In these coordinates the flow is

  $$\varphi_s(u, t) = (u, t + s)$$

  and

  $$\mathbf{v} = \frac{\partial}{\partial t}\,.$$

- Thus, near a non-singular point of a vector field $\mathbf{v}$ one can introduce local coordinates $(u^1, \ldots, u^{n-1}, u^n)$ such that

  $$\frac{du^j}{dt} = \delta^j_n$$

  that is,

  $$\frac{du^j}{dt} = 0, \qquad \text{if } j = 1, \ldots, n-1, \qquad \text{and} \qquad \frac{du^n}{dt} = 1\,.$$

- Near a non-singular point of a vector field all flows are qualitatively the same.
- Let $\mathbf{v}$ be a vector field on a manifold M. A point $p \in M$ is called a **singular point** of a vector field $\mathbf{v}$ if $\mathbf{v}_p = 0$ and a **non-singular** if $\mathbf{v}_p \neq 0$.

Chapter 2

Tensors

2.1 Covectors and Riemannian Metric

2.1.1 Linear Functionals and Dual Space

- Let E be a real n-dimensional vector space.

- Let $B = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis in E.

- Then for any $\mathbf{v} \in E$

$$\mathbf{v} = \sum_{j=1}^{n} v^j \mathbf{e}_j$$

- The real numbers $(v^1, \dots, v^n)$ are called the **components** of $\mathbf{v}$ with respect to the basis B.

- **Remark.** Every real vector space of dimension n is isomorphic to $\mathbb{R}^n$.

- **Definition 2.1.1** *Let E be a real vector space. A* **linear functional** *on E is a linear real-valued function on E. That is, it is a map*

$$\alpha : E \to \mathbb{R}$$

satisfying the linearity conditions: $\forall a, b, \in \mathbb{R}$ and $\forall \mathbf{v}, \mathbf{w} \in E$

$$\alpha(a\mathbf{v} + b\mathbf{w}) = a\alpha(\mathbf{v}) + b\alpha(\mathbf{w}) \,.$$

- Given a basis $\{\mathbf{e}_j\}$, $j = 1, \ldots, n$, we define

$$a_j = \alpha(\mathbf{e}_j)\,.$$

Then for any $\mathbf{v} \in E$ we have

$$\alpha(\mathbf{v}) = \sum_{j=1}^{n} a_j v^j\,.$$

- **Definition 2.1.2** *Let E be a real vector space. The set of all linear functionals on E is called the* **dual space** *and denoted by E^*.*

- The dual space is a real vector space under the natural operations of addition and multiplication by scalar defined by: $\forall \alpha, \beta \in E^*$, $c_1, c_2 \in \mathbb{R}$, $\mathbf{v} \in E$,

$$(c_1\alpha + c_2\beta)(\mathbf{v}) = c_1\alpha(\mathbf{v}) + c_2\beta(\mathbf{v})\,.$$

- Let $\{\mathbf{e}_j\}$ be a basis in E and $\{\sigma^j\}$, $j = 1, \ldots, n$, be linear functionals on E such that

$$\sigma^j(\mathbf{e}_i) = \delta^j_i\,.$$

- Then

$$\sigma^j(\mathbf{v}) = v^j\,.$$

- **Theorem 2.1.1** *Let E be a real vector space and $\{\mathbf{e}_j\}$, $j = 1, \ldots, n$, be a basis in E. Let and $\{\sigma^j\}$, $j = 1, \ldots, n$, be linear functionals on E such that*

$$\sigma^j(\mathbf{e}_i) = \delta^j_i\,.$$

Then the linear functionals $\{\sigma^j\}$ for a basis in the dual space E^*, called the **dual basis** *to the basis $\{\mathbf{e}_j\}$, so that for any $\alpha \in E^*$*

$$\alpha = \sum_{j=1}^{n} a_j \sigma^j\,.$$

The real numbers

$$a_j = \alpha(\mathbf{e}_j)$$

are called the **components** *of the linear functional α with respect to the basis $\{\sigma^j\}$.*

Proof:

1.

■

2.1.2 Differential of a Function

- **Definition 2.1.3** *Let M be a manifold and $p \in M$ be a point in M. The space T^*_pM dual to the tangent space T_pM at p is called the* **cotangent space**.

- **Definition 2.1.4** *Let M be a manifold and $f : M \to \mathbb{R}$ be a real valued smooth function on M. Let $p \in M$ be a point in M. The* **differential** *$df \in T^*_pM$ of f at p is the linear functional*

  $$df : T_p \to \mathbb{R}$$

  defined by

  $$df(\mathbf{v}) = \mathbf{v}_p(f)\,.$$

- In local coordinates x^j the differential is defined by

  $$df(\mathbf{v}) = \sum_{j=1}^{n} v^j(x)\frac{\partial f}{\partial x^j}$$

  In particular,

  $$df\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j}\,.$$

  Thus

  $$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j\,.$$

  and

  $$dx^i(\mathbf{v}) = v^i\,.$$

- The differentials $\{dx^i\}$ form a basis for the cotangent space T^*_pM called the **coordinate basis**.

- Therefore, every linear functional has the form

  $$\alpha = \sum_{j=1}^{n} a_j dx^j\,.$$

- That is why, the linear functionals are also called **differential forms**, or 1**-forms**, or **covectors**, or covariant vectors.

- **Definition 2.1.5** *A* **covector field** α *is a differentiable assignment of a covector* $\alpha_p \in T^*_pM$ *to each point* p *of a manifold. This means that the components of the covector field are differentiable functions of local coordinates.*

- Therefore, a covector field has the form
$$\alpha = \sum_{j=1}^{n} a_j(x) dx^j$$

- Under a change of local coordinates $x^j_\alpha = x^j_\alpha(x_\beta)$ the differentials transform according to
$$dx^j_\alpha = \sum_{i=1}^{n} \frac{\partial x^j_\alpha}{\partial x^i_\beta} dx^i_\beta .$$

- Therefore, the components of a covector transform as
$$a^\alpha_i = \sum_{j=1}^{n} \frac{\partial x^j_\alpha}{\partial x^i_\beta} a^\beta_j .$$

2.1.3 Inner Product

- Let E be a n-dimensional real vector space.

- The **inner product** (or **scalar product**) on E is a symmetric bilinear non-degenerate functional on $E \times E$, that is, it is a map $\langle\ ,\ \rangle : E \times E \to \mathbb{R}$ such that

    1. $\forall \mathbf{v}, \mathbf{w} \in E$,
    $$\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$$
    2. $\forall \mathbf{v}, \mathbf{w}, \mathbf{u} \in E$, $\forall a, b \in \mathbb{R}$
    $$\langle a\mathbf{v} + b\mathbf{u}, \mathbf{w} \rangle = a\langle \mathbf{v}, \mathbf{w} \rangle + b\langle \mathbf{u}, \mathbf{w} \rangle$$
    3. $\forall \mathbf{v} \in E$,
    $$\text{if} \qquad \langle \mathbf{v}, \mathbf{w} \rangle = 0, \quad \forall \mathbf{w} \in E, \qquad \text{then } \mathbf{v} = \mathbf{0} .$$

4. If, in addition, $\forall \mathbf{v} \in E$,
$$\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$$
and
$$\langle \mathbf{v}, \mathbf{v} \rangle = 0 \qquad \text{if and only if } \mathbf{v} = \mathbf{0}\,,$$
then the inner product is called **positive definite**.

- For a positive-definite inner product the **norm** of a vector $\mathbf{v}$ is defined by
$$||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

- Let $\{\mathbf{e}_j\}$ be a basis in E.

- Then the matrix g_{ij} defined by
$$g_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle$$
is a **metric tensor**, more precisely it gives the components of the **metric tensor** in that basis.

- The matrix g_{ij} is symmetric and nondegenerate, that is,
$$g_{ij} = g_{ji}, \qquad \det g_{ij} \neq 0\,.$$
For a positive definite inner product, this matrix is positive-definite, that is, it has only positive real eigenvalues. One says, that the metric has the **signature** $(+\cdots+)$. In special relativity one considers metrics wich are not positive definite but have the signature $(-+\cdots+)$.

- Two vectors $\mathbf{v}, \mathbf{w} \in E$ are **orthogonal** if
$$\langle \mathbf{v}, \mathbf{w} \rangle = 0\,.$$

- A vector $\mathbf{u} \in E$ is called **unit vector** if
$$||\mathbf{u}|| = 1\,.$$

- The basis is called **orthonormal** if
$$g_{ij} = \delta_{ij}\,.$$

- The inner product is given then by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i,j=1}^{n} g_{ij} v^i w^j \,.$$

- Let $\mathbf{v} \in E$. Then we can define a linear functional $v \in E^*$ by

$$v(\mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle \,.$$

- Therefore, each vector $\mathbf{v} \in E$ defines a covector $v \in E^*$ called the **covariant version of the vector v**.

- Given a basis $\{\mathbf{e}_j\}$ in E and the dual basis σ^j in E^* we find

$$v_i = \sum_{j=1}^{n} g_{ij} v^j \,.$$

- This operation is called **lowering the index** of a vector.

- Therefore, we can denote the components of the covector v corresponding to a vector $\mathbf{v}$ by the same symbol and call them the **covariant components** of the vector.

- In an orthonormal basis, of course,

$$v_i = v^i \,.$$

- Similarly, given a covector $v \in E^*$ we can define a vector $\mathbf{v} \in E$ such that $\forall \mathbf{w} \in E^*$

$$v(\mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle \,.$$

- Let g^{ij} represent the matrix inverse to the matrix g_{ij}.

- Then the contravariant components can be computed from the covariant components by

$$v^i = \sum_{j=1}^{n} g^{ij} v_j \,.$$

  This operation is called **raising the index of a covector**.

- Thus, the vector spaces E and E^* are isomorphic. The isomorphism is provided by the inner product (the metric).

- The space E can also be considered as the space of linear functionals on E^*. A vector $\mathbf{v} \in E$ defines a linear functional $\mathbf{v} : E^* \to \mathbb{R}$ by, for any $\alpha \in E^*$

$$\mathbf{v}(\alpha) = \alpha(\mathbf{v}).$$

2.1.4 Riemannian Manifolds

- **Definition 2.1.6** *Let M be a manifold. A* **Riemannian metric** *on M is a differentiable assignment of a positive definite inner product in each tangent space T_pM to the manifold at each point $p \in M$.*

  If the inner product is non-degenerate but not positive definite, then it is a **pseudo-Riemannian metric**.

  A **Riemannian manifold** *is a pair (M, g) of a manifold with a Riemannian metric on it.*

- Let $p \in M$ be a point in M and (U, x_α) be a local coordinate system about p. Let ∂_i be the coordinate basis in T_pM and $g^\alpha_{ij} = \langle \partial^\alpha_i, \partial^\alpha_j \rangle$ be the components of the metric tensor in the coordinate system x_α. Let (U_β, x_β) be another coordinate system containing p. Then the components of the metric tensor transform as

$$g^\alpha_{ij} = \sum_{k,l=1}^{n} \frac{\partial x^k_\beta}{x^i_\alpha} \frac{x^l_\beta}{x^j_\alpha} g^\beta_{kl} .$$

- **Definition 2.1.7** *Let (M, g) be a Riemannian manifold and $f : M \to \mathbb{R}$ be a smooth function. Then the* **gradient of** f *is a vector field* $\mathbf{grad}\, f$ *associated with the covector field df. That is, for any $p \in M$ and any $\mathbf{v} \in T_pM$*

$$\langle \mathbf{grad}\, f, \mathbf{v} \rangle = df(\mathbf{v}) .$$

- In local coordinates,

$$(\mathbf{grad}\, f)^i = \sum_{j=1}^{n} g^{ij} \frac{\partial f}{\partial x^j} .$$

2.1.5 Curves of Steepest Ascent

- Let (M, g) be a Riemannian manifold.
- Let $p \in M$ be a point in M. For positive-definite inner product there is the Schwartz inequality: for any $\mathbf{v}, \mathbf{w} \in T_pM$,
$$|\langle \mathbf{v}, \mathbf{w} \rangle| \le ||\mathbf{v}||\, ||\mathbf{w}||\, .$$
- Let $\mathbf{u} \in T_pM$ be a unit vector. Then for any $f \in C^\infty(M)$
$$\mathbf{u}(f) = \langle\, \mathbf{grad}\, f, \mathbf{u} \rangle\, .$$
- Therefore,
$$|\mathbf{u}(f)| = |\langle\, \mathbf{grad}\, f, \mathbf{u} \rangle| \le ||\, \mathbf{grad}\, f||\, .$$
- Thus, f has a maximum rate of change in the direction of the gradient.
- **Definition 2.1.8** *Let $a \in \mathbb{R}$ be a real number. Then the* **level set** *of $f : M \to \mathbb{R}$ is the subset of M defined by*
$$S = f^{-1}(a) = \{p \in M \mid f(p) = a\}\, .$$
- Let $f : M \to \mathbb{R}$ be a smooth function, $a \in \mathbb{R}$ and $S = f^{-1}(a)$ be the level set of f. Let $p_0 \in S$ be a point in the level set S and let $p = p(t)$, $t \in (-\varepsilon, \varepsilon)$, be a curve in S so that $p(0) = p_0$, $f(p(t)) = a$ and $\dot{p}(0) \in T_{p_0}S$. Then
$$\frac{d}{dt} f(p(t)) = \langle\, \mathbf{grad}\, f, \dot{p} \rangle = df(\dot{p}) = 0\, .$$
- Thus, the gradient of f is orthogonal to the level set of f.
- The flow
$$\frac{dp}{dt} = \frac{\mathbf{grad}\, f}{||\, \mathbf{grad}\, f||}$$
defined by the gradient of a smooth function f is called the **Morse deformation**. It has the property that
$$\frac{df}{dt} = 1$$
and, therefore, in time t it maps the level set $f^{-1}(a)$ into the level set $f^{-1}(a + t)$.

2.2 Tangent Bundle

2.2.1 Fiber Bundles

- Let M and E be smooth manifolds and $\pi : E \to M$ be a smooth map. Then the triple (E, π, M) is called a **bundle**.

- The manifold M is called the **base manifold** of the bundle and the manifold E is called the **bundle space** of the bundle (or the **bundle space manifold**). The map π is called the **projection**.

- The inverse image $\pi^{-1}(p)$ of a point $p \in M$ is called the **fiber** over p.

- The projection map is supposed to be surjective, that is, the differential π_* has the maximal rank equal to dim M.

- Let $\{U_\alpha\}_{\alpha \in A}$ be an atlas of local charts covering the base manifold M and let $U_{\alpha\beta} = U_\alpha \cap U_\beta$, $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$ etc.

- A **fiber bundle** is a bundle all fibers of which, $\pi^{-1}(p)$, $\forall p \in M$, are diffeomorphic to a common manifold F called the **typical fiber** of the bundle (or just the **fiber**).

- For a fiber bundle, the inverse images $\pi^{-1}(U_\alpha)$ are diffeomorphic to $U_\alpha \times F$. That is, there are diffeomorhisms

  $$h_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha)\,,$$

  such that for any $p \in U_\alpha \subset M$, $\sigma \in F$

  $$\pi(h_\alpha(p, \sigma)) = p\,.$$

  The diffeomorphisms

  $$\varphi_{\alpha\beta} = h_\beta^{-1} \circ h_\alpha : U_{\alpha\beta} \times F \to U_{\alpha\beta} \times F$$

  are called the **transition functions** of the bundle.

- The transition functions are defined by, $\forall p \in U_{\alpha\beta} \subset M, \sigma \in F$,

  $$\varphi_{\alpha\beta}(p, \sigma) = (p, (R_{\alpha\beta}(p))(\sigma))\,.$$

  That is, for all $p \in U_{\alpha\beta}$ there are diffeomorphisms $R_{\alpha\beta}(p)$ of the fiber

  $$R_{\alpha\beta}(p) : F \to F\,.$$

- It is required that the set of all transformations $R_{\alpha\beta}(p) \in G$ for all α, β and $p \in U_{\alpha\beta} \subset M$ forms a group G. This group is called the **structure group** of the bundle.

- Thus, the transition functions $\varphi_{\alpha\beta}$ determine smooth maps

$$R_{\alpha\beta} : U_{\alpha\beta} \to G\,,$$

that assign to each point $p \in U_{\alpha\beta}$ an element $R_{\alpha\beta}(p) \in G$ of the structure group.

- Of course, from the definition of these maps we immediately obtain the **consistency conditions** (or **compatibility conditions**)

$$R_{\alpha\beta}(p) = (R_{\beta\alpha}(p))^{-1}\,, \qquad \forall p \in U_{\alpha\beta}\,,$$

$$R_{\alpha\beta}(p)R_{\beta\gamma}(p)R_{\gamma\alpha}(p) = \mathrm{Id}_M\,, \qquad \forall p \in U_{\alpha\beta\gamma}\,.$$

- A **principal bundle** is a fiber bundle (E, π, M) whose fiber F coincides with the structure group, that is, $F = G$.

- A fiber bundle with any fiber F is fully determined by the transition functions satisfying the consistency conditions. The fiber F does not play much role in this construction.

- Let F be a manifold and $\mathrm{Diff}(F)$ be the set of all diffeomorphisms $F \to F$. Let G be a group and $e \in G$ be the identity element of G. Then a map $T : G \to \mathrm{Diff}(F)$ such that

$$\begin{aligned} T(e) &= \mathrm{Id}_F\,, \\ T(R^{-1}) &= (T(R))^{-1}\,, \qquad \forall R \in G\,, \\ T(R_1R_2) &= T(R_1) \circ T(R_2)\,, \qquad \forall R_1, R_2 \in G\,, \end{aligned}$$

is called a **representation** of the group G.

- Given a fiber bundle (E, π, M) with a fiber F and a structure group G one can construct another fiber bundle (E', π', M) with a fiber F' and the same structure group G as follows. One takes a representation of the structure group $T : G \to \mathrm{Diff}(F')$ on the fiber F' and simply replaces the transition functions $R_{\alpha\beta}$ by $T(R_{\alpha\beta})$. Such a fiber bundle is called a bundle **associated** with the original bundle.

- Thus, every fiber bundle is an associated bundle with some principal bundle. So, all bundles can be constructed as associated bundles from principal bundles. All we need is the structure group. The fiber is not important.

- A **vector bundle** is a fiber bundle whose fiber is a vector space.

- A **section** of a bundle (E, π, M) is a map $s : M \to E$ such that the image of each point $p \in M$ is in the fiber $\pi^{-1}(p)$ over this point, that is, $s(p) \in \pi^{-1}(p)$, or

$$\pi \circ s = \mathrm{Id}_M .$$

2.2.2 Tangent Bundle

- **Definition 2.2.1** *Let M be a smooth manifold. The* **tangent bundle** *TM to M is the collection of all tangent vectors at all points of M.*

$$TM = \{(p, \mathbf{v}) \mid p \in M, \mathbf{v} \in T_pM\}$$

- Let $\dim M = n$.

- Let $p \in M$ be a point in the manifold M, (U, x) be a local chart and (x^i) be the local coordinates of the point p.

- Let $\partial_i = \partial/\partial x^i$ be the coordinate basis for T_pM.

- Let $\mathbf{v} = \sum_{i=1}^n v^i \partial_i \in T_pM$. Then the local coordinates of the point $(p, \mathbf{v}) \in TM$ are

$$(x^1, \dots, x^n, v^1, \dots, v^n) .$$

- **Remarks.**

- The coordinates (x^i) are local; they are restricted to the local chart U, that is, $(x^i) \in U \subset \mathbb{R}^n$.

- The coordinates v^i are not restricted, that is, $(v^i) \in \mathbb{R}^n$, they take any values in $\mathbb{R}^n$.

- The open set $U \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ is a local chart in the tangent bundle TM.

- Let (U_α, x_α) and (U_α, x_α) be two local charts containing the point p.

- Then the local coordinates of the point $(p, \mathbf{v})$ in overlapping local charts are related by

$$\begin{aligned} x^i_\alpha &= x^i_\alpha(x_\beta) \\ v^i_\alpha &= \sum_{j=1}^n \frac{\partial x^i_\alpha}{\partial x^j_\beta} v^j_\beta \end{aligned}$$

- This is a local diffeomorphism.

- Thus, the tangent bundle TM is a manifold of dimension $2 \times \dim M$.

- A map $\pi : TM \to M$ defined by

$$\pi(p, \mathbf{v}) = p$$

is called the **projection map**. It assigns to a vector tangent to M the point in M at which the vector sits.

- Locally, if p has coordinates $(x^1, \dots, x^n)$ and $\mathbf{v}$ has components $(v^1, \dots, v^n)$ in the coordinate basis, then

$$\pi(x^1, \dots, x^n, v^1, \dots, v^n) = (x^1, \dots, x^n)\,.$$

- Let $p \in M$ be a point in M. The set $\pi^{-1}(p) \subset TM$ is called the **fiber** of the tangent bundle.

- The fiber of the tangent bundle

$$\pi^{-1}(p) = T_pM$$

is the tangent space at p.

- **Remarks.**

- There is no global projection map $\pi' : TM \to \mathbb{R}^n$ defined by $\pi'(p, \mathbf{v}) = \mathbf{v}$.

- In general, $TM \neq M \times \mathbb{R}^n$.

- For any chart $U \in M$

$$\pi^{-1}(U) = U \times \mathbb{R}^n\,.$$

Thus, locally the tangent bundle is a product manifold.

- A vector field is a map
$$\mathbf{v} : M \to TM\,,$$
such that
$$\pi \circ \mathbf{v} = \mathrm{Id} : M \to M\,.$$

- A vector field is a **cross section** of the tangent bundle.

- The image of the manifold under a vector field is a n-dimensional submanifold of the tangent bundle TM.

- The zero vector field defines the **zero section** of the tangent bundle.

- **Definition 2.2.2** *Let (M, g) be an n-dimensional Riemannian manifold. The* **unite tangent bundle** *of M is the set T_0M of all unit vectors to M,*
$$T_0M = \{(p, \mathbf{v}) \mid p \in M, \mathbf{v} \in T_pM, ||\mathbf{v}|| = 1\}\,,$$
where, locally, $||\mathbf{v}||^2 = \sum_{i,j=1}^n g_{ij}(p)v^i v^j$.

- The unit tangent bundle is a $(2n-1)$-dimensional submanifold of the tangent bundle TM.

- **Theorem 2.2.1** *Let S^2 be the unit 2-sphere embedded in $\mathbb{R}^3$. The unit tangent bundle T_0S^2 is homeomorphic to the real projective space $\mathbb{R}P^3$ and to the special orthogonal group $SO(3)$*
$$T_0S^2 \sim \mathbb{R}P^3 \sim SO(3)\,.$$

Proof:

1.

■

2.3 The Cotangent Bundle

- **Definition 2.3.1** *Let M be a smooth manifold. The* **cotangent bundle** *T^*M to M is the collection of all covectors at all points of M*

  $$T^*M = \{(p,\sigma) \mid p \in M, \sigma \in T^*_pM\}$$

- Let $\dim M = n$.
- Let $p \in M$ be a point in the manifold M, (U, x) be a local chart and (x^i) be the local coordinates of the point p.
- Let dx^i be the coordinate basis for T^*_pM.
- Let $\alpha = \sum_{i=1}^n \alpha_i dx^i \in T^*_pM$. Then the local coordinates of the point $(p,\alpha) \in T^*M$ are

  $$(x^1, \dots, x^n, \alpha_1, \dots, \alpha_n)\,.$$

- **Remarks.**
- The open set $U \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ is a local chart in the cotangent bundle T^*M.
- Let (U_α, x_α) and (U_α, x_α) be two local charts containing the point p.
- Then the local coordinates of the point (p,σ) in overlapping local charts are related by

  $$\begin{aligned} x^i_\alpha &= x^i_\alpha(x_\beta) \\ \sigma^\alpha_i &= \sum_{j=1}^n \frac{\partial x^j_\beta}{\partial x^i_\alpha} \sigma^\beta_j\,. \end{aligned}$$

- This is a local diffeomorphism.
- Thus, the cotangent bundle T^*M is a manifold of dimension $2n$.
- The projection map $\pi : T^*M \to M$ is defined by $\pi(p,\alpha) = p$.
- A covector field (or a 1-form)is a map

  $$\alpha : M \to T^*M\,,$$

  such that

  $$\pi \circ \alpha = \mathrm{Id} : M \to M\,.$$

- A covector field is a **section** of the cotangent bundle.

2.3.1 Pull-Back of a Covector

- Let M and N be two smooth manifolds and $n = \dim M$ and $m = \dim N$.
- Let $\varphi : M \to N$ be a smooth map.
- The differential
$$\varphi_* : T_pM \to T_{\varphi(p)}N$$
is the linear transformation of the tangent spaces.
- Let x^i be a local coordinate system in a local chart about $p \in M$ and y^α be a local coordinate system in a local chart about $\varphi(p) \in N$ and ∂_i and ∂_α be the coordinate bases for T_pM and $T_{\varphi(p)}N$.
- Then the action of the differential φ_* is defined by
$$\varphi_* \left(\frac{\partial}{\partial x^j} \right) = \sum_{\alpha=1}^{m} \frac{\partial y^\alpha}{\partial x^j} \frac{\partial}{\partial y^\alpha} .$$
- Let $\mathbf{v} = \sum_{i=1}^n v^i \partial_i$. Then
$$[\varphi_*(\mathbf{v})]^\alpha = \sum_{j=1}^{n} \frac{\partial y^\alpha}{\partial x^j} v^j .$$
- **Definition 2.3.2** *The* **pullback** φ^* *is the linear transformation of the cotangent spaces*
$$\varphi^* : T^*_{\varphi(p)}N \to T^*_pM$$
taking covectors at $\varphi(p) \in N$ to covectors at $p \in M$, defined as follows. If $\alpha \in T^*_{\varphi(p)}N$, then $\varphi^*(\alpha) \in T^*_pM$ so that
$$\varphi^*(\alpha) = \alpha \circ \varphi_* : T_pM \to \mathbb{R}$$
where $\alpha : T_{\varphi(p)}N \to \mathbb{R}$. That is, for any vector $\mathbf{v} \in T_pM$
$$(\varphi^*(\alpha))(\mathbf{v}) = \alpha(\varphi_*(\mathbf{v})) .$$
- Diagram.

- In local coordinates,

$$[\varphi^*(dy^\alpha)]_j = \sum_{\alpha=1}^{m} \frac{\partial y^\alpha}{\partial x^j} dx^j \,.$$

- Let $\sigma = \sum_{\alpha=1}^{m} \sigma_\alpha dy^\alpha$. Then

$$\varphi^*(\sigma) = \sum_{j=1}^{n} \sum_{\alpha=1}^{m} \sigma_\alpha \frac{\partial y^\alpha}{\partial x^j} dx^j \,,$$

  that is, in components,

$$[\varphi^*(\sigma)]_j = \sum_{\alpha=1}^{m} \sigma_\alpha \frac{\partial y^\alpha}{\partial x^j} \,.$$

- **Remark.**

- In general, for a map $\varphi : M \to N$, the following linear transformations are well defined: the differential $\varphi_* : T_pM \to T_{\varphi(p)}N$ and the pullback $\varphi^* : T^*_{\varphi(p)}N \to T^*_pM$.

- The maps $T^*_pM \to T^*_{\varphi(p)}N$ and $T_{\varphi(p)}N \to T_pM$ are *not well defined*, in general.

- If $\dim M = \dim N$ and $\varphi : M \to N$ is a diffeomorphism, then all these maps are well defined.

- Explain.

2.3.2 Phase Space

- Let M be a **configuration space** of a dynamical system with local **generalized coordinates** $q^1, \dots, q^n$. Then $\dot{q}^i = \frac{dq^i}{dt}$ are the **generalized velocities**.

- Under a change of local coordinates

$$q^i_\alpha = q^i_\alpha(q_\beta)$$

  the velocities transform as components of a vector

$$\dot{q}^j_\alpha = \sum_{i=1}^{n} \frac{\partial q^j_\alpha}{\partial q^i_\beta} \dot{q}^i_\beta \,.$$

- Therefore, $(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)$ give local coordinates for the tangent bundle TM.
- Let $L : TM \to \mathbb{R}$ be a map. Then $L(q, \dot{q})$ is called the **Lagrangian**.
- The **generalized momenta** p_i are defined by

$$p_i = \frac{\partial L}{\partial \dot{q}^i} .$$

- The momenta are functions on TM, that is, $p : TM \to \mathbb{R}$.
- Under a change of local coordinates $q_\alpha = q_\alpha(q_\beta)$ the momenta transform as components of a covector

$$p^\alpha_j = \sum_{i=1}^n \frac{\partial q^i_\beta}{\partial q^j_\alpha} p^\beta_i ,$$

- The matrix

$$\mathcal{H}_{ik}(q, \dot{q}) = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^k}$$

  is called the **Hessian.**
- Suppose that the Hessian is non-degerate

$$\det \mathcal{H}_{ik} \neq 0 .$$

- Then the velocities can be expressed in terms of momenta

$$\dot{q}^i = \dot{q}^i(q, p) ,$$

  that is, there is a map $\dot{q} : T^*M \to \mathbb{R}$. More generally, there is a map $TM \to T^*M$.
- Thus, $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ give local coordinates for the cotangent bundle T^*M.
- The cotangent bundle is called the **phase space** in dynamics.

- The **Hamiltonian** is a smooth function on the cotangent bundle $H : T^*M \to \mathbb{R}$ defined by

$$H(q, p) = \sum_{i=1}^{n} \frac{\partial L}{\partial q^i}\dot{q}^i - L(q, \dot{q})\,.$$

- **Example.**

- One of the most important examples is the Lagrangian quadratic in velocities

$$L(q, \dot{q}) = \frac{1}{2}\sum_{i,j=1}^{n} g_{ij}(q)\dot{q}^i\dot{q}^j - V(q)\,,$$

where g_{ij} is a Riemannian metric on M and V is a smooth function on M.

- Then the Hessian is

$$g_{ik} = \frac{\partial^2 L}{\partial\dot{q}^i\partial\dot{q}^k}$$

and, therefore, nondegenerate.

- The relation between momenta and the velocities is

$$p_i = \sum_{j=1}^{n} g_{ij}(q)\dot{q}^j\,, \qquad \dot{q}^i = \sum_{j=1}^{n} g^{ij}(q)p_j\,.$$

- The Hamiltonian is given by

$$H(q, p) = \frac{1}{2}\sum_{i,j=1}^{n} g^{ij}(q)p_i p_j + V(q)\,,$$

2.3.3 The Poincaré 1-Form

- The **Poincaré 1-form** λ is a 1-form on the cotangent bundle T^*M defined in local coordinates (q, p) on T^*M by

$$\lambda = \sum_{i=1}^{n} p_i dq^i\,.$$

- **Remarks.**

- The coordinates p are not functions on M.
- The Poincaré form is not a 1-form on M.
- A general 1-form on T^*M is

$$\alpha = \sum_{i=1}^{n} \alpha_i(q, p)dq^i + \sum_{i=1}^{n} v^i(q, p)dp_i \,.$$

- **Theorem 2.3.1** *The Poincaré* 1*-form is well defined globally on the cotangent bundle of any manifold.*

  Proof:

  1. Let (q_α, p_α) and (q_β, p_β) be two overlapping coordinate patches of T^*M.
  2. Then

  $$dq^i_\alpha = \sum_{j=1}^{n} \frac{\partial q^i_\alpha}{\partial q^j_\beta} dq^j_\beta$$

  and

  $$\sum_{i=1}^{n} p^\alpha_i dq^i_\alpha = \sum_{j=1}^{n} p^\beta_j dq^j_\beta \,.$$

  ■

- We give now an intrinsic definition of the Poincaré form.
- Let $(q, p) \in T^*M$ be a point in T^*M. We want to define a 1-form $\lambda \in T^*_{(q,p)}(T^*M)$ at this point $(q, p) \in T^*M$.
- Let $\pi : T^*M \to M$ be the projection defined for any $q \in M, p \in T^*_q M$ by $\pi(q, p) = q$.
- Then the pullback is the map $\pi^* : T^*_q M \to T^*_{(q,p)}(T^*M)$. For each 1-form $p \in T^*_q M$ it defines a 1-form $\pi^*(p) \in T^*_{(q,p)}(T^*M)$. This is precisely the Poincaré 1-form λ, that is,

$$\lambda = \pi^*(p) \,.$$

- Of course, in local coordinates

$$\pi^*(p) = \sum_{i=1}^{n} p_i dq^i \,.$$

2.4 Tensors

2.4.1 Covariant Tensors

- Let E be a vector space and E^* be its dual space.
- Let $\mathbf{e}_i$ be a basis for E and σ^i be the dual basis for E^*.
- A **covariant tensor of rank** p (or a **tensor of type** $(0, p)$) is a multi-linear real-valued functional
$$Q : \underbrace{E \times \cdots \times E}_{p} \to \mathbb{R}$$
- **Remarks.**
- The function $Q(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ is linear in each argument.
- The functional Q is independent of any basis.
- A covariant vector (covector) is a covariant tensor of rank 1.
- A metric tensor is a covariant tensor of rank 2.
- The **components of the tensor** Q **with respect to the basis** $\mathbf{e}_i$ are defined by
$$Q_{i_1 \ldots i_p} = Q(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_p}) .$$
- Then for any vectors
$$\mathbf{v}_a = \sum_{j=1}^{n} v_a^j \mathbf{e}_j ,$$
where $a = 1, \ldots, p$, we have
$$Q(\mathbf{v}_1, \ldots, \mathbf{v}_p) = \sum_{j_1,\ldots,j_p=1}^{n} Q_{j_1 \ldots j_p} v_1^{j_1} \cdots v_p^{j_p} .$$
- The collection of all covariant tensors of rank p forms a vector space denoted by
$$T_p = \underbrace{E^* \otimes \cdots \otimes E^*}_{p} .$$

- The dimension of the vector space T_p is

$$\dim T_p = n^p .$$

- The **tensor product** of two covectors $\alpha, \beta \in E^*$ is a covariant tensor $\alpha \otimes \beta \in E^* \otimes E^*$ of rank 2 defined by: $\forall \mathbf{v}, \mathbf{w} \in E$

$$(\alpha \otimes \beta)(\mathbf{v}, \mathbf{w}) = \alpha(\mathbf{v})\beta(\mathbf{w}) .$$

- The components of the tensor product $\alpha \otimes \beta$ are

$$(\alpha \otimes \beta)_{ij} = \alpha_i \beta_j .$$

- The **tensor product** of a covariant tensor Q of rank p and a covariant tensor T of rank q is a covariant tensor $Q \otimes T$ of rank $(p+q)$ defined by: $\forall \mathbf{v}_1, \dots, \mathbf{v}_p, \mathbf{w}_1, \dots, \mathbf{w}_q \in E$

$$(Q \otimes T)(\mathbf{v}_1, \dots, \mathbf{v}_p, \mathbf{w}_1, \dots, \mathbf{w}_q) = Q(\mathbf{v}_1, \dots, \mathbf{v}_p) T(\mathbf{w}_1, \dots, \mathbf{w}_q) .$$

- The components of the tensor product $Q \otimes T$ are

$$(Q \otimes T)_{i_1 \dots i_p j_1 \dots j_q} = Q_{i_1 \dots i_p} T_{j_1 \dots j_q} .$$

- Thus,

$$\otimes : \; T_p \times T_q \to T_{p+q} .$$

- Tensor product is **associative**.

- The basis in the space T_p is

$$\sigma^{i_1} \otimes \cdots \otimes \sigma^{i_p} ,$$

  where $1 \le i_1, \dots, i_p \le n$.

- A covariant tensor Q of rank p has the form

$$Q = \sum_{i_1, \dots, i_p = 1}^{n} Q_{i_1 \dots i_p} \sigma^{i_1} \otimes \cdots \otimes \sigma^{i_p} .$$

2.4.2 Contravariant Tensors

- A contravariant vector can be considered as a linear real-valued functional

$$\mathbf{v} : E^* \to \mathbb{R} .$$

- A **contravariant tensor of rank** p (or a **tensor of type** $(p, 0)$) is a multi-linear real-valued functional

$$T : \underbrace{E^* \times \cdots \times E^*}_{p} \to \mathbb{R}$$

- **Remarks.**

- The function $T(\alpha_1, \cdots, \alpha_p)$ is linear in each argument.

- The functional T is independent of any basis.

- A contravariant vector (covector) is a contravariant tensor of rank 1.

- The **components of the tensor** T **with respect to the basis** σ^i are defined by

$$T^{i_1 \dots i_p} = T(\sigma^{i_1}, \dots, \sigma^{i_p}) .$$

- Then for any covectors

$$\alpha_{(a)} = \sum_{j=1}^{n} \alpha_{(a)\,j} \sigma^j ,$$

where $a = 1, \dots, p$, we have

$$T(\alpha_{(1)}, \dots, \alpha_{(p)}) = \sum_{j_1, \dots, j_p = 1}^{n} T^{j_1 \dots j_p} \alpha_{(1)\,j_1} \cdots \alpha_{(p)\,j_p} .$$

- The inverse matrix of the components of a metric tensor defines a contravariant tensor g^{-1} of rank 2 by

$$g^{-1}(\alpha, \beta) = \sum_{i,j=1}^{n} g^{ij} \alpha_i \beta_j .$$

- The collection of all contravariant tensors of rank p forms a vector space denoted by
$$T^p = \underbrace{E \otimes \cdots \otimes E}_{p} .$$

- The dimension of the vector space T^p is
$$\dim T^p = n^p .$$

- The **tensor product** of a contravariant tensor Q of rank p and a contravariant tensor T of rank q is a contravariant tensor $Q \otimes T$ of rank $(p+q)$ defined by: $\forall \alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q \in E^*$
$$(Q \otimes T)(\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q) = Q(\alpha_1, \dots, \alpha_p) T(\beta_1, \dots, \beta_q) .$$

- The components of the tensor product $Q \otimes T$ are
$$(Q \otimes T)^{i_1 \dots i_p j_1 \dots j_q} = Q^{i_1 \dots i_p} T^{j_1 \dots j_q} .$$

- Thus,
$$\otimes : \ T^p \times T^q \to T^{p+q} .$$

- Tensor product is associative.

- The basis in the space T^p is
$$\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_p} ,$$
where $1 \le i_1, \dots, i_p \le n$.

- A contravariant tensor T of rank p has the form
$$T = \sum_{i_1, \dots, i_p = 1}^{n} T^{i_1 \dots i_p} \mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_p} .$$

- The set of all tensors of type $(p, 0)$ forms a vector space T^p of dimension n^p
$$\dim T^p = n^p .$$

2.4.3 General Tensors of Type (p, q)

- A **tensor of type** (p, q) is a multi-linear real-valued functional

$$T : \underbrace{E^* \times \cdots \times E^*}_{p} \times \underbrace{E \times \cdots \times E}_{q} \to \mathbb{R}$$

- The **components of the tensor** T **with respect to the basis** $\mathbf{e}_i$**,** σ^i are defined by

$$T^{i_1 \dots i_p}{}_{j_1 \dots j_q} = T(\sigma^{i_1}, \dots, \sigma^{i_p}, \mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_q}) .$$

- Then for any covectors

$$\alpha_{(a)} = \sum_{j=1}^{n} \alpha_{(a)\,j} \sigma^j ,$$

where $a = 1, \dots, p$, and any vectors

$$\mathbf{v}_b = \sum_{j=1}^{n} v_b^k \mathbf{e}_k ,$$

where $b = 1, \dots, q$, we have

$$T(\alpha_{(1)}, \dots, \alpha_{(p)}, \mathbf{v}_1, \dots, \mathbf{v}_q) = \sum_{k_1, \dots, k_q = 1}^{n} \sum_{j_1, \dots, j_p = 1}^{n} T^{j_1 \dots j_p}{}_{k_1 \dots k_q} \alpha_{(1)\,j_1} \cdots \alpha_{(p)\,j_p} v^{k_1} \cdots v^{k_q} .$$

- The inverse matrix of the components of a metric tensor defines a contravariant tensor g^{-1} of rank 2 by

$$g^{-1}(\alpha, \beta) = \sum_{i,j=1}^{n} g^{ij} \alpha_i \beta_j .$$

- The collection of all tensors of type (p, q) forms a vector space denoted by

$$T^p_q = \underbrace{E \otimes \cdots \otimes E}_{p} \otimes \underbrace{E^* \otimes \cdots \otimes E^*}_{q} .$$

- The dimension of the vector space T_p is

$$\dim T^p_q = n^{p+q} .$$

- The **tensor product** of a tensor Q of type (p, q) and a tensor T of type (r, s) is a tensor $Q \otimes T$ of type $(p+r, q+s)$ defined by: $\forall \alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_r \in E^*$, $\mathbf{v}_1, \dots, \mathbf{v}_q, \mathbf{w}_1, \dots, \mathbf{w}_s \in E$

$$(Q \otimes T)(\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_r, \mathbf{v}_1, \dots, \mathbf{v}_q, \mathbf{w}_1, \dots, \mathbf{w}_s)$$
$$= Q(\alpha_1, \dots, \alpha_p, \mathbf{v}_1, \dots, \mathbf{v}_q) T(\beta_1, \dots, \beta_r, \mathbf{w}_1, \dots, \mathbf{w}_s) .$$

- The components of the tensor product $Q \otimes T$ are

$$(Q \otimes T)^{i_1 \dots i_p j_1 \dots j_r}{}_{k_1 \dots k_q l_1 \dots l_s} = Q^{i_1 \dots i_p}{}_{k_1 \dots k_q} T^{j_1 \dots j_r}{}_{l_1 \dots l_s} .$$

- Thus,

$$\otimes : \; T^p_q \times T^r_s \to T^{p+r}_{q+s} .$$

- We stress once again that the tensor product is associative.

- The basis in the space T^p_q is

$$\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_p} \otimes \sigma^{j_1} \otimes \cdots \otimes \sigma^{j_q} ,$$

  where $1 \leq i_1, \dots, i_p, j_1, \dots, j_q \leq n$.

- A tensor T of type (p, q) has the form

$$T = \sum_{j_1, \dots, j_q = 1}^{n} \sum_{i_1, \dots, i_p = 1}^{n} T^{i_1 \dots i_p}{}_{j_1 \dots j_q} \mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_p} \otimes \sigma^{j_1} \otimes \cdots \otimes \sigma^{j_q} .$$

- Let $p, q \geq 1$ and $1 \leq r \leq p$, $1 \leq s \leq q$. The (r, s)-**contraction** of tensors of type (p, q) is the map

$$\mathrm{tr}\,^r_s : T^p_q \to T^{p-1}_{q-1}$$

  defined by

$$(\mathrm{tr}\,^r_s\, T)^{i_1 \dots i_{p-1}}{}_{j_1 \dots j_{q-1}} = \sum_{k=1}^{n} T^{i_1 \dots i_{r-1} k i_r \dots i_{p-1}}{}_{j_1 \dots j_{s-1} k j_s \dots j_q} .$$

2.4.4 Linear Transformations and Tensors

- Let $A : E \to E$ be a linear transformation.
- Then we can define a tensor $\tilde{A}$ of type $(1, 1)$ by $\forall \alpha \in E^*, \mathbf{v} \in E$
$$\tilde{A}(\alpha, \mathbf{v}) = \alpha(A\mathbf{v}) \,.$$
- Let $(A^i{}_j)$ be the matrix of the linear transformation A, that is,
$$A\mathbf{e}_j = \sum_{i=1}^{n} A^i{}_j \mathbf{e}_i \,.$$
- Then the components of the tensor $\tilde{A}$ are
$$\tilde{A}^i{}_j = \tilde{A}(\sigma^i, \mathbf{e}_j) = \sigma^i(A\mathbf{e}_j) = A^i{}_j \,.$$
- Thus, one can identify tensors of type $(1, 1)$ and linear transformations on E (and, similarly on E^* as well).
- Then,
$$A = \sum_{i,j=1}^{n} A^i{}_j \mathbf{e}_i \otimes \sigma^j \,.$$
- The identity linear transformation I has the matrix
$$I^i{}_j = \delta^i_j$$
and defines the tensor of type $(1, 1)$
$$I = \sum_{i=1}^{n} \mathbf{e}_i \otimes \sigma^i \,.$$
- To summarize, a $(1, 1)$ tensor $\tilde{A} : E^* \times E \to \mathbb{R}$ is identified with the linear transformation $A : E \to E$.
- The covariant tensor A_{ij}, the contravariant tensor A^{ij} and the tensor $A^i{}_j$ of type $(1, 1)$ are related by
$$A_{ij} = \sum_{k=1}^{n} g_{ik} A^k{}_j \,, \qquad A^{ij} = \sum_{k=1}^{n} A^i{}_k g^{kj} \,.$$

2.4.5 Tensor Fields

- **Definition 2.4.1** *A tensor field on a manifold M is a smooth assignment of a tensor at each point of M.*

- Let $x^i_\alpha = x^i_\alpha(x_\beta)$ be a local diffeomorphism.

- Then
$$dx^i_\alpha = \sum_{j=1}^n \frac{\partial x^i_\alpha}{\partial x^j_\beta} dx^j_\beta$$
and
$$\frac{\partial}{\partial x^i_\alpha} = \sum_{j=1}^n \frac{\partial x^j_\beta}{\partial x^i_\alpha} \frac{\partial}{\partial x^j_\beta}$$

- Let T be a tensor of type (p, q). Then

$$T_{(\alpha)}{}^{i_1\dots i_p}{}_{j_1\dots j_q} = \sum_{k_1,\dots,k_p=1}^n \sum_{l_1,\dots,l_q=1}^n \frac{\partial x^{i_1}_\alpha}{\partial x^{k_1}_\beta} \cdots \frac{\partial x^{i_p}_\alpha}{\partial x^{k_p}_\beta} \frac{\partial x^{l_1}_\beta}{\partial x^{l_q}_\alpha} \cdots \frac{\partial x^{j_1}_\beta}{\partial x^{j_q}_\alpha} T_{(\beta)}{}^{k_1\dots k_p}{}_{l_1\dots l_q}$$

2.4.6 Tensor Bundles

- **Definition 2.4.2** *Let M be a smooth manifold. The* **tensor bundle of type** (p, q) $T^p_q M$ *is the collection of all tensors of type (p, q) at all points of M*
$$T^p_q M = \{(p, T) \mid p \in M, T \in T^p_{q,(x)} M\}$$

- The tensor bundle $T^p_q M$ is the tensor product of the tangent and cotangent bundles
$$T^p_q M = \underbrace{TM \otimes \cdots \otimes TM}_{p} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{q} .$$

- Let $\dim M = n$.

- Let $x \in M$ be a point in the manifold M, (U, x) be a local chart and (x^i) be the local coordinates of the point p.

- Let ∂_i and dx^i be the coordinate basis for $T_x M$ and $T^*_x M$.

- Let T be a tensor of type (p, q) and $T^{i_1 \dots i_p}_{j_1 \dots j_q}$ be the components of the tensor T in the coordinate basis. Then the local coordinates of the point $(p, T) \in T^p_q M$ are
$$(x^i, T^{i_1 \dots i_p}_{j_1 \dots j_q}),$$
where $1 \le i, i_1, \dots, i_p, j_1, \dots, j_q \le n$.

- **Remarks.**

- The open set $U \times \mathbb{R}^{n^{p+q}} \subset \mathbb{R}^{n+n^{p+q}}$ is a local chart in the tensor bundle $T^p_q M$.

- The bundle $T^p_q M$ is a manifold of dimension $n + n^{p+q}$.

- The projection map $\pi : T^p_q M \to M$ is defined by $\pi(x, T) = x$.

- A **tensor field** of type (p, q) is a map
$$T : M \to T^p_q M,$$
such that
$$\pi \circ T = \mathrm{Id} : M \to M.$$

- A tensor field of type (p, q) is a **section** of the tensor bundle $T^p_q M$.

2.4.7 Examples

- Riemannian metric $g_{\mu\nu}$.

- Energy-momentum tensor $T_{\mu\nu}$.

- Stress tensor σ_{ij}.

- Riemann curvature tensor $R^\mu{}_{\alpha\beta\gamma}$.

- Ricci curvature tensor $R_{\mu\nu}$.

- Scalar density of weight 1: $|g| = \sqrt{\det g_{ij}}$.

- Axial vectors (vector product) in $\mathbb{R}^3$.

- Strength of the electromagnetic field $F_{\mu\nu}$.

- **Theorem 2.4.1** *Let*

$$A = \sum_{i=1}^{n} A_i dx^i$$

be a covector field (1-form). Let

$$F_{ij} = \partial_i A_j - \partial_j A_i .$$

Then

$$F = \sum_{i,j=1}^{n} F_{ij} dx^i \otimes dx^j$$

is a tensor of type $(0, 2)$.

Proof:

1. Check the transformation law.

■

- A tensor is called **isotropic** if it is a tensor product of g, g^{-1} and I.
- The components of an isotropic tensor are the products of g_{ij}, g^{ij} and δ^i_j.
- Every isotropic tensor of type (p, q) has an even rank $p + q$.
- For example, the most general isotropic tensor of type $(2, 2)$ has the form

$$A^{ij}{}_{kl} = a g^{ij} g_{kl} + b \delta^i_k \delta^j_l + c \delta^i_l \delta^j_k ,$$

where a, b, c are scalars.

2.4.8 Einstein Summation Convention

- In any expression there are two types of indices: **free indices** and **repeated indices**.
- Free indices appear only once in an expression; they are assumed to take all possible values from 1 to n.
- The position of all free indices in all terms in an equation must be the same.

- Repeated indices appear twice in an expression. It is assumed that there is a summation over each repeated pair of indices from 1 to n. The summation over a pair of repeated indices in an expression is called the **contraction**.

- Repeated indices are **dummy indices**: they can be replaced by any other letter (not already used in the expression) without changing the meaning of the expression.

- Indices cannot be repeated on the same level. That is, in a pair of repeated indices one index is in upper position and another is in the lower position.

- There cannot be indices occuring three or more times in any expression.

Chapter 3

Differential Forms

3.1 Exterior Algebra

3.1.1 Permutation Group

- A **group** is a set G with an associative binary operation, $\cdot : G \times G \to G$ with identity, called the **multiplication**, such that each element has an inverse. That is, the following conditions are satisfied

  1. for any three elements $g, h, k \in G$, the **associativity law** holds: $(gh)k = g(hk)$;
  2. there exists an **identity element** $e \in G$ such that for any $g \in G$, $ge = eg = g$;
  3. each element $g \in G$ has an **inverse** g^{-1}, such that $g g^{-1} = g^{-1} g = e$

- Let X be a set. A **transformation** of the set X is a bijective map $g : X \to X$.

- The set of all transformations of a set X forms a group $\text{Aut}(X)$, with composition of maps as group multiplication.

- Any subgroup of $\text{Aut}(X)$ is a **transformation group** of the set X.

- The transformations of a finite set X are called **permutations**.

- The group S_p of permutations of the set $\mathbb{Z}_n = \{1, \dots, p\}$ is called the **symmetric group of order** p.

- **Theorem 3.1.1** *The order of the symmetric group S_p is*
$$|S_p| = p!\,.$$
- Any subgroup of S_p is called a **permutation group**.
- A permutation $\varphi : \mathbb{Z}_p \to \mathbb{Z}_p$ can be represented by
$$\begin{pmatrix} 1 & \dots & p \\ \varphi(1) & \dots & \varphi(p) \end{pmatrix}$$
- The identity permutation is
$$\begin{pmatrix} 1 & \dots & p \\ 1 & \dots & p \end{pmatrix}$$
- The inverse permutation $\varphi^{-1} : \mathbb{Z}_p \to \mathbb{Z}_p$ is represented by
$$\begin{pmatrix} \varphi(1) & \dots & \varphi(p) \\ 1 & \dots & p \end{pmatrix}$$
- The product of permutations is then defined in an obvious manner.
- An **elementary permutation** is a permutation that exchanges the order of only two elements.
- Every permutation can be realized as a product of elementary permutations.
- A permutation that can be realized by an even number of elementary permutations is called an **even permutation**.
- A permutation that can be realized by an odd number of elementary permutations is called an **odd permutation**.
- **Proposition 3.1.1** *The* **parity of a permutation** *does not depend on the representation of a permutation by a product of the elementary ones.*
- That is, each representation of an even permutation has even number of elementary permutations, and similarly for odd permutations.
- The **sign of a permutation** φ, denoted by $\mathrm{sign}(\varphi)$ (or simply $(-1)^\varphi$), is defined by
$$\mathrm{sign}(\varphi) = (-1)^\varphi = \begin{cases} +1, & \text{if } \varphi \text{ is even,} \\ -1, & \text{if } \varphi \text{ is odd} \end{cases}$$

3.1.2 Permutations of Tensors

- Let S_p be the symmetric group of order p. Then every permutation $\varphi \in S_p$ defines a map
$$\varphi : T_p \to T_p\,,$$
which assigns to every tensor T of type $(0, p)$ a new tensor $\varphi(T)$, called a **permutation of the tensor** T, of type $(0, p)$ by: $\forall \mathbf{v}_1, \dots, \mathbf{v}_p$
$$\varphi(T)(\mathbf{v}_1, \dots, \mathbf{v}_p) = T(\mathbf{v}_{\varphi(1)}, \dots, \mathbf{v}_{\varphi(p)})\,.$$

- Let $(i_1, \dots, i_p)$ be a p-tuple of integers. Then a permutation $\varphi : \mathbb{Z}_p \to \mathbb{Z}_p$ defines an action
$$\varphi(i_1, \dots, i_p) = (i_{\varphi(1)}, \dots, i_{\varphi(p)})\,.$$

- The components of the tensor $\varphi(T)$ are obtained by the action of the permutation φ on the indices of the tensor T
$$\varphi(T)_{i_1 \dots i_p} = T_{i_{\varphi(1)} \dots i_{\varphi(p)}}\,.$$

- The **symmetrization** of the tensor T of the type $(0, p)$ is defined by
$$\operatorname{Sym}(T) = \frac{1}{p!} \sum_{\varphi \in S_p} \varphi(T)\,.$$

- The symmetrization is also denoted by parenthesis. The components of the symmetrized tensor $\operatorname{Sym}(T)$ are given by
$$T_{(i_1 \dots i_p)} = \frac{1}{p!} \sum_{\varphi \in S_p} T_{i_{\varphi(1)} \dots i_{\varphi(p)}}\,.$$

- The **anti-symmetrization** of the tensor T of the type $(0, p)$ is defined by
$$\operatorname{Alt}(T) = \frac{1}{p!} \sum_{\varphi \in S_p} \operatorname{sign}(\varphi) \varphi(T)\,.$$

- The anti-symmetrization is also denoted by square brackets. The components of the anti-symmetrized tensor $\operatorname{Alt}(T)$ are given by
$$T_{[i_1 \dots i_p]} = \frac{1}{p!} \sum_{\varphi \in S_p} \operatorname{sign}(\varphi) T_{i_{\varphi(1)} \dots i_{\varphi(p)}}\,.$$

- **Examples.**
- A tensor T of type $(0, p)$ is called **symmetric** if for any permutation $\varphi \in S_p$

$$\varphi(T) = T .$$

- A tensor T of type $(0, p)$ is called **anti-symmetric** if for any permutation $\varphi \in S_p$

$$\varphi(T) = \text{sign}\,(\varphi) T .$$

- An anti-symmetric tensor of type $(0, p)$ is called a p**-form**.
- **Remarks.**
- Permutation, symmetrization, anti-symmetrization of tensors of type $(p, 0)$.
- Completely symmetric and completely anti-symmetric tensors of type $(p, 0)$.
- An anti-symmetric tensor of type $(p, 0)$ is called a p**-vector**.
- Partial permutation.
- **Examples.**
- Notation.

3.1.3 Alternating Tensors

- Let $(i_1, \dots, i_p)$ and $(j_1, \dots, j_p)$ be two p-tuples of integers $1 \le i_1, \dots, i_p, j_1, \dots, j_p \le n$. The **generalized Kronecker symbol** is defined by

$$\delta^{i_1 \dots i_p}_{j_1 \dots j_p} = \begin{cases} 1 & \text{if } (i_1, \dots, i_p) \text{ is an even permutation of } (j_1, \dots, j_p) \\ -1 & \text{if } (i_1, \dots, i_p) \text{ is an odd permutation of } (j_1, \dots, j_p) \\ 0 & \text{otherwise} \end{cases}$$

- One can easily check that

$$\delta^{i_1 \dots i_p}_{j_1 \dots j_p} = \det \begin{pmatrix} \delta^{i_1}_{j_1} & \dots & \delta^{i_1}_{j_p} \\ \vdots & \ddots & \vdots \\ \delta^{i_p}_{j_1} & \dots & \delta^{i_p}_{j_p} \end{pmatrix}$$

- Also, there holds
$$\delta^{i_1\dots i_p}_{j_1\dots j_p} = p!\delta^{i_1}_{[j_1}\cdots\delta^{i_p}_{j_p]}\,.$$
- Thus, the Kronecker symbols $\delta^{i_1\dots i_p}_{j_1\dots j_p}$ are the components of the tensors
$$p!\mathrm{Alt}(\underbrace{I\otimes\cdots\otimes I}_{p})$$
of type (p,p), which are anti-symmetric separately in upper indices and the lower indices.
- Thus, the anti-symmetrization can also be written as
$$T_{[i_1\dots i_p]} = \frac{1}{p!}\delta^{j_1\dots j_p}_{i_1\dots i_p}T_{j_1\dots j_p}\,.$$
- Notation.
- Obviously, the Kronecker symbols vanish for $p > n$
$$\delta^{i_1\dots i_p}_{j_1\dots j_p} = 0 \qquad \text{if } p > n\,.$$
- The contraction of Kronecker symbols gives Kronecker symbols with lower indices, more precisely, we have the theorem.
- **Theorem 3.1.2** *For any $p,q\in\mathbb{N}$, $1\le p,q\le n$, there holds*
$$\delta^{i_1\dots i_p l_1\dots l_q}_{j_1\dots j_p l_1\dots l_q} = \frac{(n-p)!}{(n-q)!}\delta^{i_1\dots i_p}_{j_1\dots j_p}\,.$$

Proof:

1.

■

- **Corollary 3.1.1**

  For any $q\in\mathbb{N}$, $1\le q\le n$ we have
$$\delta^{i_1\dots i_q}_{i_1\dots i_q} = \frac{n!}{(n-q)!}\,.$$
  In particular,
$$\delta^{i_1\dots i_n}_{i_1\dots i_n} = n!\,.$$

- **Lemma 3.1.1** *There holds*

$$\delta^{i_1\dots i_p\, j_1\,\dots j_r}_{l_1\dots l_p\, m_1\dots m_r}\delta^{k_1\dots k_r}_{j_1\dots j_r} = r!\delta^{i_1\dots i_p\, k_1\,\dots k_r}_{l_1\dots l_p\, m_1\dots m_r}$$

Proof:

1.

∎

- In general, let A be a p-form (an antisymmetric tensor of type $(0, p)$) and B be a p-vector (an anti-symmetric tensor of type $(p, 0)$). Then

$$A_{i_1\dots i_p}\delta^{i_1\dots i_p}_{j_1\dots j_p} = p!A_{j_1\dots j_p}\,,$$

$$B^{i_1\dots i_p}\delta^{j_1\dots j_p}_{i_1\,\dots\, i_p} = p!B^{j_1\dots j_p}\,.$$

- Let $(i_1, \dots, i_n)$ be an n-tuple of integers $1 \le i_1, \dots, i_n \le n$. The completely anti-symmetric (alternating) **Levi-Civita symbols** are defined by

$$\varepsilon_{i_1\dots i_n} = \delta^{1\,\dots\, n}_{i_1\dots i_n}\,, \qquad \varepsilon^{i_1\dots i_n} = \delta^{i_1\dots i_n}_{1\,\dots\, n}\,,$$

so that

$$\varepsilon^{i_1\dots i_n} = \varepsilon_{i_1\dots i_n} = \begin{cases} 1 & \text{if } (i_1, \dots, i_n) \text{ is an even permutation of } (1, \dots, n) \\ -1 & \text{if } (i_1, \dots, i_n) \text{ is an odd permutation of } (1, \dots, n) \\ 0 & \text{otherwise} \end{cases}$$

- **Theorem 3.1.3** *There holds the identity*

$$\begin{aligned} \varepsilon^{i_1\dots i_n}\varepsilon_{j_1\dots j_n} &= \sum_{\varphi\in S_n} \operatorname{sign}(\varphi)\, \delta^{i_1}_{j_{\varphi(1)}} \cdots \delta^{i_n}_{j_{\varphi(n)}} \\ &= n!\delta^{i_1}_{[j_1} \cdots \delta^{i_n}_{j_n]} \\ &= \delta^{i_1\dots i_n}_{j_1\dots j_n}\,. \end{aligned}$$

The contraction of this identity over k indices gives

$$\begin{aligned} \varepsilon^{i_1\dots i_{n-k} m_1\dots m_k}\varepsilon_{j_1\dots j_{n-k} m_1\dots m_k} &= k!(n-k)!\delta^{i_1}_{[j_1} \cdots \delta^{i_{n-k}}_{j_{n-k}]} \\ &= k!\delta^{i_1\dots i_{n-k}}_{j_1\dots j_{n-k}}\,. \end{aligned}$$

In particular,

$$\varepsilon^{m_1\dots m_n}\varepsilon_{m_1\dots m_n} = n!\,.$$

- It is easy to see that there holds also

$$\delta^{i_1\dots i_{n-p}}_{l_1\dots l_{n-p}}\varepsilon^{j_1\dots j_p l_1\dots l_{n-p}} = (n-p)!\varepsilon^{j_1\dots j_p i_1\dots i_{n-p}}$$

- The set of all $n\times n$ real matrices is denoted by $\mathrm{Mat}(n,\mathbb{R})$.

- The **determinant** is a map $\det : \mathrm{Mat}(n,\mathbb{R}) \to \mathbb{R}$ that assigns to each matrix $A = (A^i{}_j)$ a real number $\det A$ defined by

$$\det A = \sum_{\varphi\in S_n} \mathrm{sign}\,(\varphi) A^1{}_{\varphi(1)} \cdots A^n{}_{\varphi(n)}\,,$$

- The most important properties of the determinant are listed below:

Theorem 3.1.4 *1. The determinant of the product of matrices is equal to the product of the determinants:*

$$\det(AB) = \det A \det B\,.$$

2. The determinants of a matrix A and of its transpose A^T are equal:

$$\det A = \det A^T\,.$$

3. The determinant of the inverse A^{-1} of an invertible matrix A is equal to the inverse of the determinant of A:

$$\det A^{-1} = (\det A)^{-1}$$

4. A matrix is invertible if and only if its determinant is non-zero.

- The determinant of a matrix $A = (A^i{}_j)$ can be written as

$$\begin{aligned}\det A &= \varepsilon^{i_1\dots i_n} A^1{}_{i_1}\dots A^n{}_{i_n}\\ &= \varepsilon_{j_1\dots j_n} A^{j_1}{}_1\dots A^{j_n}{}_n\\ &= \frac{1}{n!}\varepsilon^{i_1\dots i_n}\varepsilon_{j_1\dots j_n} A^{j_1}{}_{i_1}\dots A^{j_n}{}_{i_n}\,.\end{aligned}$$

Here, as usual, a summation over all repeated indices is assumed from 1 to n.

3.1.4 Exterior p-forms

- An **exterior p-form** (or simply a **p-form**) is an anti-symmetric covariant tensor $\alpha \in T_p$ of type $(0, p)$.

- The collection of all p-forms forms a vector space Λ_p, which is a vector subspace of T_p
$$\Lambda_p \subset T_p \,.$$

- In particular,
$$\Lambda_0 = \mathbb{R} \qquad \text{and} \qquad \Lambda_1 = T_1 = E^* \,.$$

- In other words, a 0-form is a smooth function, and a 1-form is a covector field.

- Let $\alpha \in \Lambda_p$ be a p-form.

- Let $\mathbf{e}_i$ be a basis in E and σ^i be the dual basis in E^*.

- The components of the p-form α are
$$\alpha_{i_1 \dots i_p} = \alpha(\mathbf{e}_{i_1}, \dots \mathbf{e}_{i_p}) \,.$$

- The components are completely anti-symmetric in all indices, that is,
$$\alpha_{i_1 \dots i_p} = \operatorname{sign}(\varphi) \alpha_{i_{\varphi(1)} \dots i_{\varphi(p)}} \,.$$
In particular, under a permutation of any two indices the form changes sign
$$\alpha_{\dots i \dots j \dots} = -\alpha_{\dots j \dots i \dots} \,,$$
which means that the components vanish if any two indices are equal
$$\alpha_{\dots i \dots i \dots} = 0 \qquad \text{(no summation!)} \,.$$

- Thus, all non-vanishing components have different indices.

- Therefore, the values of all components $\alpha_{i_1 \dots i_p}$ are completely determined by the values of the components with the indices $i_1, \dots, i_p$ reordered in strictly increasing order
$$1 \le i_1 < \cdots < i_p \le n \,.$$

- Notation.
- To deal with forms it is convenient to introduce multi-indices. We will denote a p-tuple of integers from 1 to n by a capital letter

$$I = (i_1, \dots, i_p),$$

where $1 \le i_1, \dots, i_p \le n$. For a p-tuple of the same integers ordered in an increasing order we define

$$\hat{I} = (i_1, \dots, i_p).$$

where $1 \le i_1 < i_2 < \cdots < i_p \le n$. We call $\hat{I}$ an **increasing p-tuple** associated with I.

- Therefore, a collection of p-forms

$$p!\mathrm{Alt}\left(\sigma^{i_1} \otimes \cdots \otimes \sigma^{i_p}\right),$$

where $1 \le i_1 < i_2 < \cdots < i_p \le n$, forms a basis in the space Λ_p.

- Thus, every p-form $\alpha \in \Lambda_p$ has the form

$$\alpha = \sum_{1 \le i_1 < \cdots < i_p \le n} \alpha_{i_1 \cdots i_p} p!\mathrm{Alt}\left(\sigma^{i_1} \otimes \cdots \otimes \sigma^{i_p}\right).$$

- Therefore, the dimension of the space Λ_p is equal to the number of distinct increasing p-tuples of integers from 1 to n.
- **Theorem 3.1.5** *The dimension of the space Λ_p of p-forms is*

$$\dim \Lambda_p = \binom{n}{p} = \frac{n!}{p!(n-p)!}.$$

Proof:

1.

■

- In particular,

$$\dim \Lambda_0 = \dim \Lambda_n = 1,$$

$$\dim \Lambda_1 = \dim \Lambda_{n-1} = n,$$

etc.

- There are no p-forms with $p > n$.
- Similarly to the norm of vectors and covectors we define the **inner product of exterior p-forms** α and β in a Riemannian manifold by

$$(\alpha, \beta) = \frac{1}{p!} g^{i_1 j_1} \cdots g^{i_1 j_p} \alpha_{i_1 \dots i_p} \beta_{j_1 \dots j_p} \,.$$

- This enables one to define also the **norm of an exterior p-form** α by

$$||\alpha|| = \sqrt{(\alpha, \alpha)}$$

3.1.5 Exterior Product

- Since the tensor product of two skew-symmetric tensors is not a skew-symmetric tensor to define the algebra of antisymmetric tensors we need to define the **anti-symmetric tensor product** called the **exterior** (or **wedge**) **product**.
- If α is an p-form and β is an q-form then the exterior product of α and β is an $(p+q)$-form $\alpha \wedge \beta$ defined by

$$\alpha \wedge \beta = \frac{(p+q)!}{p!q!} \mathrm{Alt}\,(\alpha \otimes \beta) \,.$$

- In components

$$(\alpha \wedge \beta)_{i_1 \dots i_{p+q}} = \frac{(p+q)!}{p!q!} \alpha_{[i_1 \dots i_p} \beta_{i_{p+1} \dots i_{p+q}]} \,.$$

- Let $\alpha \in \Lambda_p$ be a p-form. Then

$$p = \deg(\alpha)$$

  is called the **degree** (or **rank**) of α.

- **Theorem 3.1.6** *The exterior product has the following properties*

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) \qquad \text{(associativity)}$$

$$\alpha \wedge \beta = (-1)^{\deg(\alpha)\deg(\beta)} \beta \wedge \alpha \qquad \text{(anticommutativity)}$$

$$(\alpha + \beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma \qquad \text{(distributivity)}\,.$$

  Proof:

1.

■

- The exterior square of any p-form α of odd degree p (in particular, for any 1-form) vanishes
$$\alpha \wedge \alpha = 0\,.$$

- The **exterior algebra** Λ (or **Grassmann algebra**) is the set of all forms of all degrees, that is,
$$\Lambda = \Lambda_0 \oplus \cdots \oplus \Lambda_n\,.$$

- The dimension of the exterior algebra is
$$\dim \Lambda = \sum_{p=0}^{n} \binom{n}{p} = 2^n\,.$$

- A basis of the space Λ_p is
$$\sigma^{i_1} \wedge \cdots \wedge \sigma^{i_p}\,, \qquad (1 \le i_1 < \cdots < i_p \le n)\,.$$
An p-form α can be represented in one of the following ways
$$\begin{aligned} \alpha &= \alpha_{i_1 \dots i_p} \sigma^{i_1} \otimes \cdots \otimes \sigma^{i_p} \\ &= \frac{1}{p!} \alpha_{i_1 \dots i_p} \sigma^{i_1} \wedge \cdots \wedge \sigma^{i_p} \\ &= \sum_{i_1 < \cdots < i_p} \alpha_{i_1 \dots i_p} \sigma^{i_1} \wedge \cdots \wedge \sigma^{i_p}\,. \end{aligned}$$

- The exterior product of a p-form α and a q-form β can be represented as
$$\alpha \wedge \beta = \frac{1}{p!q!} \alpha_{[i_1 \dots i_p} \beta_{i_{p+1} \dots i_{p+q}]} \sigma^{i_1} \wedge \cdots \wedge \sigma^{i_{p+q}}\,.$$

- **Theorem 3.1.7** *Let $\sigma^j \in \Lambda_1$, $1 \le j \le n$, and $\alpha^j \in \Lambda_1$, $1 \le j \le n$, be two collections of n 1-forms related by a linear transformation*
$$\alpha^j = \sum_{i=1}^{n} A^j{}_i \sigma^i\,, \qquad 1 \le j \le n\,,$$
Then
$$\alpha^1 \wedge \cdots \wedge \alpha^n = \det A^i{}_j\, \sigma^1 \wedge \cdots \wedge \sigma^n\,.$$

Proof:

1.

∎

- **Theorem 3.1.8** *Let $\alpha^j \in \Lambda_1 = E^*$, $1 \le j \le p$, be a collections of p 1-forms and $\mathbf{v}_i \in E$, $1 \le i \le p$, be a collection of p vectors. Let*

$$A^j{}_i = \alpha^j(\mathbf{v}_i)\,, \qquad 1 \le i, j \le p\,.$$

Then

$$\left(\alpha^1 \wedge \cdots \wedge \alpha^p\right)(\mathbf{v}_1, \ldots, \mathbf{v}_p) = \det A^i{}_j\,.$$

Proof:

1.

∎

- **Theorem 3.1.9** *A collections of p 1-forms $\alpha^j \in \Lambda_1 = E^*$, $1 \le j \le p$, is linearly dependent if and only if*

$$\alpha^1 \wedge \cdots \wedge \alpha^p = 0\,.$$

- **Corollary 3.1.2** *Let $x^i = x^i(x')$, $i = 1, \ldots, n$, be a local diffeomorphism. Then*

$$dx^1 \wedge \cdots \wedge dx^n = \det\left(\frac{\partial x^l}{\partial x'^m}\right) dx'^1 \wedge \cdots \wedge dx'^n\,.$$

3.1.6 Interior Product

- The **interior product** of a vector $\mathbf{v}$ and a p-form α is a $(p-1)$-form $i_{\mathbf{v}}\alpha$ defined by, for any $\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}$,

$$i_{\mathbf{v}}\alpha(\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}) = \alpha(\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_{p-1})\,.$$

In particular, if $p = 1$, then $i_{\mathbf{v}}\alpha$ is a scalar

$$i_{\mathbf{v}}\alpha = \alpha(\mathbf{v})$$

and if $p = 0$, then by definition

$$i_{\mathbf{v}}\alpha = 0\,.$$

- In components,
$$(i_{\mathbf{v}}\alpha)_{i_1\dots i_{p-1}} = v^j \alpha_{ji_1\dots i_{p-1}} \,.$$

- The interior product is a map
$$i_{\mathbf{v}} : \Lambda_p \to \Lambda_{p-1} \,,$$
or
$$i_{\mathbf{v}} : \Lambda \to \Lambda \,.$$

- A map $L : \Lambda \to \Lambda$ is called an **derivation** if for any $\alpha \in \Lambda_p, \beta \in \Lambda_q$,
$$L(\alpha \wedge \beta) = (L\alpha) \wedge \beta + \alpha \wedge L\beta \,.$$

- A map $L : \Lambda \to \Lambda$ is called an **anti-derivation** if for any $\alpha \in \Lambda_p, \beta \in \Lambda_q$,
$$L(\alpha \wedge \beta) = (L\alpha) \wedge \beta + (-1)^p \alpha \wedge L\beta \,.$$

- **Theorem 3.1.10** *Let $\mathbf{v} \in E$ be a vector. The interior product $i_{\mathbf{v}} : \Lambda \to \Lambda$ is an anti-derivation.*

3.2 Orientation and the Volume Form

3.2.1 Orientation of a Vector Space

- Let E be a vector space. Let $\{\mathbf{e}_i\} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and $\{\mathbf{e}'_j\} = \{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$ be two different bases in E related by

$$\mathbf{e}_i = \Lambda^j{}_i \mathbf{e}'_j\,,$$

where $\Lambda = (\Lambda^i{}_j)$ is a transformation matrix.

- Note that the transformation matrix is non-degenerate

$$\det\Lambda \neq 0\,.$$

- Since the transformation matrix Λ is invertible, then the determinant $\det\Lambda$ is either positive or negative.

- If $\det\Lambda > 0$ then we say that the bases $\{\mathbf{e}_i\}$ and $\{\mathbf{e}'_i\}$ have the **same orientation**, and if $\det\Lambda < 0$ then we say that the bases $\{\mathbf{e}_i\}$ and $\{\mathbf{e}'_i\}$ have the **opposite orientation**.

- If the basis $\{\mathbf{e}_i\}$ is continuously deformed into the basis $\{\mathbf{e}'_j\}$, then both bases have the same orientation.

- Since $\det I = 1 > 0$ and the function $\det : GL(n, \mathbb{R}) \to \mathbb{R}$ is continuous, then a one-parameter continuous transformation matrix $\Lambda(t)$ such that $\Lambda(0) = I$ preserves the orientation.

- This defines an equivalence relation on the set of all bases on E called the **orientation** of the vector space E.

- This equivalence relation divides the set of all bases in two equivalence classes, called the **positively oriented** and **negatively oriented** bases.

- A vector space together with a choice of what equivalence class is positively oriented is called an **oriented vector space**.

3.2.2 Orientation of a Manifold

- Let M be a manifold and let T_pM be the tangent space at a point $p \in M$.
- Let (U, x) be a local coordinate patch about a point $p \in M$.
- Then the vectors
$$\frac{\partial}{\partial x^i}, \qquad i = 1, \dots, n$$
form a basis in T_pM.
- Let (U', x') be another local coordinate system about a point p, that is, there is a local diffeomorphism $x^i = x^i(x')$.
- Then the vectors
$$\frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j}$$
form another basis in T_pM.
- The orientation of the bases $\{\partial_i\}$ and $\{\partial'_j\}$ is the same (or **consistent**) if
$$\det\left(\frac{\partial x^i}{\partial x'^j}\right) > 0\,.$$
- If it is possible to choose an orientation of all tangent spaces T_pM at all points in a continuous fashion, then the orientation of all tangent spaces is consistent.
- A manifold M is called **orientable** if there is an atlas such that the orientation of all charts of this atlas can be chosen consistently, that is, the Jacobians of all transition functions have positive determinant.
- Each connected orientable manifold has exactly two possible orientations. One orientation can be declared **positive**, then the other orientation is **negative**.
- An orientable manifold with a chosen orientation is called **oriented**.
- **Remarks**.
- If a manifold can be covered by a single coordinate chart then it is orientable.

- Not all manifolds are orientable.
- **Transport of the orientation.**
- Let $p, q \in M$ be two points in a manifold M and $C(t)$ be a curve in M connecting p and q, i.e.

$$C(0) = p, \qquad C(1) = q\,.$$

- Let $\{\mathbf{e}_i(t)\}$ be a basis in $T_{C(t)}M$ that continuously depends on $t \in [0, 1]$.
- Then the orientation of the basis $\mathbf{e}_i(1)$ is uniquely determined by the orientation of the basis $\mathbf{e}_i(0)$.
- Thus, the orientation is transported along a curve in a unique way.
- Note that the transportation of the basis is not unique, in general. Only the transportation of the orientation is!
- Given a point p and another point q, the orientation at the point q does, in general, depend on the curve $C(t)$ connecting the points p and q.
- If a manifold is orientable, then the transportation of the orientation from one point to another does not depend on the curve connecting the points.
- **Corollary 3.2.1** *Let M be a manifold. If there exist two points p and q in M and two curves $C_1(t)$ and $C_2(t)$ joining the points p and q such that the orientation at q transported from p along the curves C_1 and C_2 are different, then M is nonorientable.*

  That is, if there exists a closed curve $C(t)$ in M such that the transport of the orientation along C leads to a reversal of orientation, then M is nonorientable.

- **Example. Möbius Band.**

3.2.3 Hypersurfaces in Orientable Manifolds

- Let $V \subset W$ be a submanifold of a manifold W.
- A vector field $\mathbf{N}$ on W is **transverse** to V if $\mathbf{N}$ is nowhere tangent to V.

- That is, **N** is transverse to V if for any point $p \in V$, $\mathbf{N}_p \notin T_p V$.
- If **N** is transverse to V, then $\mathbf{N} \neq 0$ on V.
- Let W be an n-diemsnional manifold. An $(n-1)$-diemsnional submanifold M of W is called a **hypersurface** in W.
- A hypersurface M in W is **two-sided** in W if there is a continuous vector **N** in W transverse to M.
- **Examples.** Normals to surfaces in $\mathbb{R}^3$.
- **Theorem 3.2.1** *A two-sided hypersurface in an orientable manifold is orientable.*
- **Remarks.**
- Orientability of a manifold is an intrinsic property of a manifold.
- Two-sidedness of a manifold depends on the embedding of the manifold as a hypersurface in a higher-dimensional manifold.
- **Example.**
- Every manifold (even a nonorientable one) M is a two-sided hypersurface in a manifold $M \times \mathbb{R}$.

3.2.4 Projective Spaces

- The real projective space $\mathbb{R}P^2$ is the sphere S^2 with antipodal points identified.
- The sphere S^2 is two-sided in $\mathbb{R}^3$, so it is orientable.
- Let $\mathbf{e}_1, \mathbf{e}_2$ be a basis in $T_p S^2$ and **N** be an outward pointing normal vector to S^2.
- Then $\{\mathbf{N}, \mathbf{e}_1, \mathbf{e}_2\}$ is a basis in $\mathbb{R}^3$.
- We say that the basis $\mathbf{e}_1, \mathbf{e}_2$ is positively oriented in S^2 if the basis $\{\mathbf{N}, \mathbf{e}_1, \mathbf{e}_2\}$ is positive in $\mathbb{R}^3$.

- Let
$$r : \mathbb{R}^3 \to \mathbb{R}^3$$
be the reflection map defined by
$$r(\mathbf{x}) = -\mathbf{x}\,.$$

- Let
$$a : S^2 \to S^2$$
be the antipodal map in S^2 defined as the restriction of the reflection map to S^2
$$a = r\big|_{S^2}$$

- The reflection map reverses the orientation of the basis in $\mathbb{R}^3$!

- Let $p \in S^2$ be a point on S^2 and $\{\mathbf{N}, \mathbf{e}_1, \mathbf{e}_2\} \in T_pS^2$ be a positively oriented basis. Then $\{-\mathbf{e}_2, -\mathbf{e}_1, -\mathbf{N}\}$ is negatively oriented at $T_a(p)S^2$, where $a(p)$ is the antipodal point.

- The vector $-\mathbf{N}$ is the outward normal at the antipodal point $a(p)$.

- Thus the basis $\{-\mathbf{e}_1, -\mathbf{e}_2\}$ is negatively oriented at $a(p)$.

- Thus, if the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ at $p \in S^2$ is transported along a curve C on S^2 to $a(p)$, then the orientation of the transported basis $\{\mathbf{e}'_1, \mathbf{e}'_2\}$ is the opposite of the orientation of the basis $\{-\mathbf{e}_1, -\mathbf{e}_2\}$ at $a(p)$.

- Thus, the antipodal map reverses the orientation on S^2.

- In $\mathbb{R}P^2$ the basis $\{-\mathbf{e}_1, -\mathbf{e}_2\}$ at $a(p)$ represents the same basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ at p.

- Thus, the curve C on S^2 is a closed curve C' on $\mathbb{R}P^2$ going through p.

- Thus, the transportation of the basis along the closed curve C' in $\mathbb{R}P^2$ reverses the orientation.

- Thus, $\mathbb{R}P^2$ is not orientable!

- More generally, the even-dimensional projective spaces $\mathbb{R}P^{2n}$ are not orientable.

- The odd-dimensional projective spaces $\mathbb{R}P^{2n}$ are orientable.

3.2.5 Pseudotensors and Tensor Densities

- Let E be an oriented vector space and $\mathcal{B}$ be the set of all bases on E. Then the orientation is a function

$$o : \mathcal{B} \to \mathbb{R}$$

defined by

$$o(\mathbf{e}_i) = \begin{cases} +1 & \text{if } \mathbf{e}_i \text{ is positively oriented} \\ -1 & \text{if } \mathbf{e}_i \text{ is negatively oriented} \end{cases}$$

- A **pseudo-tensor** T on a vector space E assigns, for each orientation o of E a tensor T_o such that when the orientation is reversed the tensor changes sign, i.e.

$$T_{-o} = -T_o\,.$$

- That is, a pseudo-tensor is a collection of two tensors T_+ and T_-, one for each orientation.

- A **pseudo-tensor field** on a manifold is a smooth assignment of a pseudo-tensor to each point of the manifold.

- Let $x^i_\alpha = x^i_\alpha(x_\beta)$ be a local diffeomorphism and

$$J(x_\beta) = \det\left(\frac{\partial x^i_\alpha}{\partial x^j_\beta}\right).$$

- Since this transformation is a diffeomorphism $J \neq 0$. Thus, there are two cases $J > 0$ and $J < 0$.

- A **pseudo-tensor** of type (p, q) is a geometric object T whose components $T^{i_1\dots i_p}{}_{j_1\dots j_q}$ in the coordinate basis ∂_i and dx^i transform according to

$$T_{(\alpha)}{}^{i_1\dots i_p}{}_{j_1\dots j_q}(x_\alpha) = \operatorname{sign}(J) \sum_{k_1,\dots,k_p=1}^{n} \sum_{l_1,\dots,l_q=1}^{n} \frac{\partial x^{i_1}_\alpha}{\partial x^{k_1}_\beta} \cdots \frac{\partial x^{i_p}_\alpha}{\partial x^{k_p}_\beta} \frac{\partial x^{l_1}_\beta}{\partial x^{j_1}_\alpha} \cdots \frac{\partial x^{j_1}_\beta}{\partial x^{j_q}_\alpha} T_{(\beta)}{}^{k_1\dots k_p}{}_{l_1\dots l_q}(x_\beta)$$

- A **tensor density of weight** w of type (p,q) is a geometric object T whose components $T^{i_1\ldots i_p}_{j_1\ldots j_q}$ in the coordinate basis ∂_i and dx^i transform under a diffeomorphism $x^i_\alpha = x^i_\alpha(x_\beta)$ with $J > 0$ according to

$$T_{(\alpha)}{}^{i_1\ldots i_p}_{j_1\ldots j_q}(x_\alpha) = J^{-w}(x_\beta)\sum_{k_1,\ldots,k_p=1}^{n}\sum_{l_1,\ldots,l_q=1}^{n}\frac{\partial x^{i_1}_\alpha}{\partial x^{k_1}_\beta}\cdots\frac{\partial x^{i_p}_\alpha}{\partial x^{k_p}_\beta}\frac{\partial x^{l_1}_\beta}{\partial x^{l_q}_\alpha}\cdots\frac{\partial x^{j_1}_\beta}{\partial x^{j_q}_\alpha}T_{(\beta)}{}^{k_1\ldots k_p}_{l_1\ldots l_q}(x_\beta)$$

- **Theorem 3.2.2** *Let $g = (g_{ij})$ be a Riemannian metric and*

$$|g| = \det(g_{ij})\,.$$

Then $\sqrt{|g|}$ is a scalar density of weight 1.

Proof:

1. Let $x = x(x')$ be a local diffeomorphism.
2. Then
$$g'_{ij} = \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j}g_{kl}\,,$$
3. Then
$$|g'| = \det\left(\frac{\partial x}{\partial x'}\right)^2|g|\,.$$
4. Thus
$$\sqrt{|g'|} = \det\left(\frac{\partial x}{\partial x'}\right)\sqrt{|g|}\,.$$
5. Therefore, $\sqrt{|g|}$ is a scalar density of weight 1.

■

- The Levi-civita symbol are not tensors!

- **Theorem 3.2.3** *Let g_{ij} be the components of a Riemannian metric, $|g| = \det(g_{ij})$, and $\varepsilon_{i_1\dots i_n}$ and $\varepsilon^{i_1\dots i_n}$ be the Levi-Civita symbols and $E_{i_1\dots i_n}$ and $E^{i_1\dots i_n}$ be defined by*

$$E_{i_1\dots i_n} = \sqrt{|g|}\,\varepsilon_{i_1\dots i_n}, \qquad E^{i_1\dots i_n} = \frac{1}{\sqrt{|g|}}\,\varepsilon^{i_1\dots i_n}$$

Then

1. *$\varepsilon_{i_1\dots i_n}$ represents the components of a pseudo-n-form (that is, a pseudo-tensor density of type $(0,n)$) of weight (-1).*
2. *$\varepsilon^{i_1\dots i_n}$ represents the components of a pseudo-n-vector (that is, a pseudo-tensor density of type $(n,0)$) of weight 1.*
3. *$E_{i_1\dots i_n}$ represents the components of a pseudo-n-form.*
4. *$E^{i_1\dots i_n}$ represents the components of a pseudo-n-vector.*

Proof:

1. Check the transformation law.

■

3.2.6 Volume Form

- Let $\{\mathbf{v}_{(1)},\dots,\mathbf{v}_{(n)}\}$ be an ordered n-tuple of vectors. The **volume** of the parallelepiped spanned by the vectors $\{\mathbf{v}_{(1)},\dots,\mathbf{v}_{(n)}\}$ is a real number defined by

$$|\mathrm{vol}\,(\mathbf{v}_{(1)},\dots,\mathbf{v}_{(n)})| = \sqrt{\det((\mathbf{v}_{(i)},\mathbf{v}_{(j)}))}\,.$$

- If the vectors $\{\mathbf{v}_{(1)},\dots,\mathbf{v}_{(n)}\}$ are orthonormal, then

$$|\mathrm{vol}\,(\mathbf{v}_{(1)},\dots,\mathbf{v}_{(n)})| = 1\,.$$

- If the vectors $\{\mathbf{v}_{(1)},\dots,\mathbf{v}_{(n)}\}$ are linearly dependent, then

$$\mathrm{vol}\,(\mathbf{v}_{(1)},\dots,\mathbf{v}_{(n)}) = 0\,.$$

- The volume $|\mathrm{vol}\,(\mathbf{v}_{(1)},\dots,\mathbf{v}_{(n)})|$ does not depend on the orientation of vectors.

- The **signed volume** of the parallelepiped spanned by an ordered n-tuple of vectors $\{\mathbf{v}_{(1)}, \dots, \mathbf{v}_{(n)}\}$ is

$$\operatorname{vol}(\mathbf{v}_{(1)}, \dots, \mathbf{v}_{(n)}) = \operatorname{sign}(\mathbf{v}_{(1)}, \dots, \mathbf{v}_{(n)}) \, |\operatorname{vol}(\mathbf{v}_{(1)}, \dots, \mathbf{v}_{(n)})| \,,$$

where the sign of the signed volume is defined by

$$\operatorname{sign}(\mathbf{v}_{(1)}, \dots, \mathbf{v}_{(n)}) = \begin{cases} +1, & \text{if } \{\mathbf{v}_{(1)}, \dots, \mathbf{v}_{(n)}\} \text{ is positively oriented} \\ -1, & \text{if } \{\mathbf{v}_{(1)}, \dots, \mathbf{v}_{(n)}\} \text{ is negatively oriented} \end{cases}$$

- **Theorem 3.2.4** *Let $\mathbf{e}_i$ be a basis in a vector space E, σ^i be the dual basis of 1-forms in E^*, $g = (g_{ij})$ be a Riemannian metric and $\{\mathbf{v}_{(1)}, \dots, \mathbf{v}_{(n)}\}$ be a set of n vectors. Let $v^i{}_{(j)} = \sigma^i(\mathbf{v}_{(j)})$ be the contravariant components of the vectors $\mathbf{v}_{(j)}$ and $v_{i\,(j)} = (\mathbf{e}_i, \mathbf{v}_{(j)}) = g_{ik} v^k{}_{(j)}$ be the covariant components of these vectors. Then:*

$$\begin{aligned} \operatorname{vol}(\mathbf{v}_{(1)}, \dots, \mathbf{v}_{(n)}) &= \sqrt{|g|} \, \det(v^i{}_{(j)}) \\ &= E_{i_1 \dots i_n} \, v^{i_1}{}_{(1)} \cdots v^{i_n}{}_{(n)} \end{aligned}$$

and

$$\begin{aligned} \operatorname{vol}(\mathbf{v}_{(1)}, \dots, \mathbf{v}_{(n)}) &= \frac{1}{\sqrt{|g|}} \det(v_{i\,(j)}) \\ &= E^{i_1 \dots i_n} \, v_{i_1 (1)} \cdots v_{i_n (n)} \end{aligned}$$

In particular,

$$\operatorname{vol}(\mathbf{e}_1, \dots, \mathbf{e}_n) = o(\mathbf{e}_i) \sqrt{|g|} \,,$$

where $o(\mathbf{e}_i)$ is equal to $+1$ or -1 for positively oriented and negatively oriented basis $\mathbf{e}_i$.

- The n-form

$$\operatorname{vol} = \sqrt{|g|} \, \sigma^1 \wedge \cdots \wedge \sigma^n$$

is called the **Riemannian volume element** (or volume form).

- We have

$$\left(\sigma^{i_1} \wedge \cdots \wedge \sigma^{i_p}\right)(\mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_p}) = \delta^{i_1 \dots i_p}_{j_1 \cdots j_p} \,.$$

- The components of the volume form are

$$\mathrm{vol}\,(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_n}) = \sqrt{|g|}\varepsilon_{i_1\ldots i_n} = E_{i_1\ldots i_n} \,.$$

3.2.7 Star Operator and Duality

- The volume form allows one to define the **duality** of p-forms and $(n-p)$-vectors.

- For each p-form $A_{i_1\ldots i_p}$ one assigns the **dual** $(n-p)$-vector by

$$\tilde{A}^{j_1\ldots j_{n-p}} = \frac{1}{p!} E^{j_1\ldots j_{n-p} i_1\ldots i_p} A_{i_1\ldots i_p} \,.$$

- Similarly, for each p-vector $A^{i_1\ldots i_p}$ one assigns the **dual** $(n-p)$-form by

$$\tilde{A}_{j_1\ldots j_{n-p}} = \frac{1}{p!} E_{j_1\ldots j_{n-p} i_1\ldots i_p} A^{i_1\ldots i_p} \,.$$

- By lowering and raising the indices of the dual forms we can define the duality of forms and poly-vectors separately.

- The **Hodge star operator**

$$* : \Lambda_p \to \Lambda_{n-p}$$

  maps any p-form α to a $(n-p)$-form $*\alpha$ **dual** to α defined as follows.

- For each p-form α the form $*\alpha$ is the unique $(n-p)$-form such that

$$\alpha \wedge *\alpha = (\alpha, \alpha)\mathrm{vol} \,.$$

- In particular,

$$*1 = \mathrm{vol}\,, \qquad *\mathrm{vol} = 1\,.$$

- In components, this means that

$$\begin{aligned} (*\alpha)_{i_{p+1}\ldots i_n} &= \frac{1}{p!}\varepsilon_{i_1\ldots i_p i_{p+1}\ldots i_n} \sqrt{|g|} g^{i_1 j_1} \cdots g^{i_p j_p} \alpha_{j_1\ldots j_p} \\ &= \frac{1}{p!} \frac{1}{\sqrt{|g|}} g_{i_{p+1} j_{p+1}} \cdots g_{i_n j_n} \varepsilon^{j_1\ldots j_p j_{p+1}\ldots j_n} \alpha_{j_1\ldots j_p} \,. \end{aligned}$$

- **Theorem 3.2.5** *For any p-form α there holds*

  $$*^2\alpha = (-1)^{p(n-p)}\alpha\,.$$

  In particular, if n is odd, then for any p

  $$*^2 = \mathrm{Id}\,.$$

- **Theorem 3.2.6** *Let α be a 1-form and $\mathbf{v}$ be the corresponding vector, that is, $v^i = g^{ij}\alpha_j$. Then*

  $$*\alpha = i_{\mathbf{v}}\mathrm{vol}\;.$$

- A collection $\{\omega_{(1)}, \ldots, \omega_{(n-1)}\}$ of $(n-1)$ 1-forms defines a 1-form α by

  $$\alpha = *\left[\omega_{(1)} \wedge \cdots \wedge \omega_{(n-1)}\right]\;.$$

- Then

  $$*\alpha = (-1)^{n-1}\omega_{(1)} \wedge \cdots \wedge \omega_{(n-1)}$$

  and

  $$\omega_{(1)} \wedge \cdots \wedge \omega_{(n-1)} \wedge \alpha = (\alpha, \alpha)\mathrm{vol}\;.$$

- In components,

  $$\alpha_j = g_{jk}E^{i_1\ldots i_{n-1}k}\omega_{i_1\,(1)} \cdots \omega_{i_{n-1}\,(n-1)}\,.$$

- If the 1-forms $\{\omega_{(1)}, \ldots, \omega_{(n-1)}\}$ are linearly dependent, then $\alpha = 0$.

- If the collection of 1-forms $\{\omega_{(1)}, \ldots, \omega_{(n-1)}\}$ is linearly independent, then $\{\omega_{(1)}, \ldots, \omega_{(n-1)}, \alpha\}$ are linearly independent and form a basis in E^*.

- **Theorem 3.2.7** *Let $\{\sigma_1, \ldots, \sigma_n\}$ be an orthonormal basis of 1-forms. Then*

  $$\sigma_j = (-1)^{n-j} * \left[\sigma_1 \wedge \cdots \wedge \sigma_{j-1} \wedge \sigma_{j+1} \wedge \cdots \wedge \sigma_n\right]\,,$$

  In particular,

  $$\sigma_n = *\left[\sigma_1 \wedge \cdots \wedge \sigma_{n-1}\right]\;.$$

- Similarly, a collection $\{\mathbf{v}_{(1)}, \ldots, \mathbf{v}_{(n-1)}\}$ of $(n-1)$ vectors defines a covector α by: for any vector $\mathbf{v}$

  $$\alpha(\mathbf{v}) = \mathrm{vol}\,(\mathbf{v}_{(1)}, \ldots, \mathbf{v}_{(n-1)}, \mathbf{v})$$

or, in components,

$$\alpha_j = E_{i_1 \dots i_{n-1} j} v^{i_1}{}_{(1)} \cdots v^{i_{n-1}}{}_{(n-1)} .$$

Theorem 3.2.8 *Let* $\{\mathbf{v}_{(1)}, \dots, \mathbf{v}_{(n-1)}\}$ *be an ordered* $(n-1)$*-tuple of linearly independent vectors and* $\{\omega_{(1)}, \dots, \omega_{(n-1)}\}$ *be the corresponding* 1*-forms. Let* α *be* 1*-form defined by*

$$\alpha = * \left[\omega_{(1)} \wedge \cdots \wedge \omega_{(n-1)}\right] ,$$

and $\mathbf{N}$ *be the corresponding vector, that is,*

$$N^i = g^{ik} \sqrt{|g|} \varepsilon_{i_1 \dots i_{n-1} k} v^{i_1}{}_{(1)} \cdots v^{i_{n-1}}{}_{(n-1)}$$

•

Then:

1. *The vector* $\mathbf{N}$ *is orthogonal to all vectors* $\{\mathbf{v}_{(1)}, \dots, \mathbf{v}_{(n-1)}\}$*, that is,*

$$(\mathbf{N}, \mathbf{v}_{(j)}) = g_{ik} N^i v^k{}_{(j)} = 0 , \qquad (j = 1, \dots, n-1) .$$

2. *The n-tuple* $\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{N}\}$ *forms a positively oriented basis.*

3. *The volume of the parallelepiped spanned by the vectors* $\{\mathbf{v}_{(1)}, \dots, \mathbf{v}_{(n-1)}, \mathbf{N}\}$ *is determined by the norm of* $\mathbf{N}$

$$\mathrm{vol}\,(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{N}) = (\mathbf{N}, \mathbf{N}) .$$

3.3 Exterior Derivative

- From now on, if not specified otherwise, we will denote the derivatives by

$$\partial_i = \frac{\partial}{\partial x^i}\,.$$

- The exterior derivative of a 0-form (that is, a function) f is a 1-form df defined by: for any vector $\mathbf{v}$

$$(df)(\mathbf{v}) = \mathbf{v}(f)\,.$$

- In local cordinates

$$df = \partial_j f\; dx^j\,.$$

- The exterior derivative of a 1-form (that is, a covector) A is a 2-form dA defined by: for any vectors $\mathbf{v}, \mathbf{w}$

$$(dA)(\mathbf{v}, \mathbf{w}) = \mathbf{v}(A(\mathbf{w})) - \mathbf{w}(A(\mathbf{v})) - A([\mathbf{v}, \mathbf{w}])\,.$$

- In local cordinates

$$dA = \frac{1}{2}\left(\partial_i A_j - \partial_j A_i\right) dx^i \wedge dx^j\,.$$

- Let α be a p-form

$$\alpha = \frac{1}{p!}\alpha_{i_1\dots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}\,.$$

- The **exterior derivative** of α is a $(p+1)$-form $d\alpha$ defined by

$$\begin{aligned} d\alpha &= \frac{1}{p!} d\alpha_{i_1\dots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\ &= \frac{1}{p!}\partial_{i_1}\alpha_{i_2\dots i_{p+1}} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{p+1}}\,. \end{aligned}$$

- In components

$$\begin{aligned} (d\alpha)_{i_1 i_2 \dots i_{p+1}} &= (p+1)\partial_{[i_1}\alpha_{i_2\dots i_{p+1}]} \\ &= \sum_{k=1}^{p+1} (-1)^{k-1} \partial_{i_k}\alpha_{i_1\dots i_{k-1} i_{k+1}\dots i_{p+1}} \end{aligned}$$

- **Theorem 3.3.1** *The exterior derivative is a linear map*

$$d : \Lambda_p \to \Lambda_{p+1} .$$

Proof: Show that $d\alpha$ is a form whose value does not depend on the coordinate system.

∎

- **Theorem 3.3.2** *Let $\alpha \in \Lambda_p$ be a p-form and $\{\mathbf{v}_1, \dots, \mathbf{v}_{p+1}\}$ be a collection of $(p+1)$ vectors. Then*

$$(d\alpha)(\mathbf{v}_1, \dots, \mathbf{v}_{p+1}) = \sum_{k=1}^{p+1} (-1)^{k-1} \mathbf{v}_k(\alpha(\mathbf{v}_1, \dots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \dots, \mathbf{v}_{p+1}))$$
$$- \sum_{k=1}^{p+1} \sum_{i=1}^{k-1} (-1)^{i+k-1} \alpha([\mathbf{v}_i, \mathbf{v}_k], \mathbf{v}_1, \dots, \mathbf{v}_i, \mathbf{v}_{i+1}, \dots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \dots, \mathbf{v}_{p+1}))$$

Proof: Calculation.

∎

- **Remark.** This formula can be taken as the intrinsic definition of the exterior derivative.

- **Theorem 3.3.3** *For any p-form*

$$d^2 = 0 .$$

Proof: Easy.

∎

- **Theorem 3.3.4** *The exterior derivative $d : \Lambda \to \Lambda$ is an anti-derivation on the exterior algebra.*

  That is, for any p-form $\alpha \in \Lambda_p$ and any q-form $\beta \in \Lambda_q$ there holds

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge (d\beta) .$$

Proof: Calculation.

∎

- **Examples.**

3.3.1 Coderivative

- Given a Riemannian metric $g_{\mu\nu}$ we also define the co-derivative of p-forms.
- The **coderivative** is a linear map
$$\delta : \Lambda_p \to \Lambda_{p-1}$$
defined by
$$\delta = *^{-1} d* = (-1)^{(n-p+1)(p-1)} * d*$$
- That is the coderivative of a p-form α is the $(p-1)$-form $\delta\alpha$ defined by
$$\begin{aligned}(\delta\alpha)_{i_1\dots i_{p-1}} &= \frac{1}{(n-p+1)!}\varepsilon_{i_1\dots i_{p-1}i_p\dots i_n}\sqrt{|g|}g^{ji_p}g^{j_{p+1}i_{p+1}}\cdots g^{j_n i_n}\\ &\quad (n-p+1)\partial_j\left(\frac{1}{p!}\varepsilon_{j_1\dots j_p j_{p+1}\dots j_n}\sqrt{|g|}g^{j_1k_1}\cdots g^{j_pk_p}\alpha_{k_1\dots k_p}\right)\end{aligned}$$
- **Theorem 3.3.5** *For any p-form*
$$\delta^2 = 0\,.$$

Proof: Follows from $*^2 = \pm 1$ and $d^2 = 0$.

∎

- From this definition, we can also see that, for any 0-form f (a function) $*f$ is an n-form and, therefore, $d * f = 0$, i.e. a coderivative of any 0-form is zero
$$\delta f = 0\,.$$
- For a 1-form α, $\delta\alpha$ is a 0-form
$$\delta\alpha = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\alpha_j\right)\,.$$
- More generally, one can prove that for a p-form α
$$(\delta\alpha)_{i_1\dots i_{p-1}} = g_{i_1j_1}\dots g_{i_{p-1}j_{p-1}}\frac{1}{\sqrt{|g|}}\partial_j\left(\sqrt{|g|}g^{jk}g^{j_1k_1}\cdots g^{j_{p-1}k_{p-1}}\alpha_{kk_1\dots k_{p-1}}\right)\,.$$
- **Examples.**

3.4 Pullback of Forms

- Let M be a n-dimensional manifold and W be a r-dimensional manifold.
- Let $F : M \to W$ be a smooth map of a manifold M to a manifold W.
- Let $p \in M$ be a point in M and $q = F(p) \in W$ be the image of p in W.
- Let x^i, $(i = 1, \dots, n)$, be a local coordinate system about p and y^μ, $(\mu = 1, \dots, r)$, be a local coordinate system about q so that
$$y^\mu = y^\mu(x)\,.$$
- Let $f : W \to \mathbb{R}$ be a smooth function on W.
- The **pullback** of f to M is a function $F^* f : M \to \mathbb{R}$ on M defined by
$$F^* f = f \circ F\,,$$
that is, for any x
$$(F^* f)(x) = f(y(x))\,.$$
- Suppose that $n = r$ and the map F is bijective.
- Let $h : M \to \mathbb{R}$ be a smooth function on M.
- The **pushforward** of h to W is a function $F_* h : W \to \mathbb{R}$ on W defined by
$$F_* h = h \circ F^{-1}\,,$$
that is, for any y
$$(F_* h)(y) = h(x(y))\,.$$
- **Remarks.** The pullback is well defined for an arbitrary map F.
- The pushforward is only defined for bijections!
- The **pullback** is the map
$$F^* : \Lambda_p W \to \Lambda_p M$$
defined as follows.

  Let $\alpha \in \Lambda_p W$ be a p-form on W. The **pullback** of α is a p-form $F^*\alpha$ on M defined by: for any vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$
$$(F^*\alpha)(\mathbf{v}_1, \dots, \mathbf{v}_p) = \alpha(F_*\mathbf{v}_1, \dots, F_*\mathbf{v}_p)\,.$$

- In local cordinates

$$
\begin{aligned}
F^*\alpha &= \frac{1}{p!}\alpha_{\mu_1\dots\mu_p}(y(x))dy^{\mu_1}\wedge\cdots\wedge dy^{\mu_p} \\
&= \frac{1}{p!}\frac{\partial y^{\mu_1}}{\partial x^{i_1}}\cdots\frac{\partial y^{\mu_p}}{\partial x^{i_p}}\alpha_{\mu_1\dots\mu_p}(y(x))dx^{i_1}\wedge\cdots\wedge dx^{i_p}
\end{aligned}
$$

- In components

$$
(F^*\alpha)_{i_1\dots i_p} = \frac{\partial y^{\mu_1}}{\partial x^{i_1}}\cdots\frac{\partial y^{\mu_p}}{\partial x^{i_p}}\alpha_{\mu_1\dots\mu_p}(y(x))
$$

- **Remark.** The pullback is well-defined only for covariant tensors and the pushforward is well defined only for contravariant tensors.

- **Theorem 3.4.1** *Let $F : M \to W$. The pullback $F^* : \Lambda_p W \to \Lambda_p M$ has the properties:*

  1. *F^* is linear.*
  2. *For any two forms α and β*

  $$F^*(\alpha\wedge\beta) = (F^*\alpha)\wedge(F^*\beta)$$

  3. *F^* commutes with exterior derivative. That is, for any p-form α*

  $$F^*(d\alpha) = d(F^*\alpha)\,.$$

***Proof*:** Direct calculation.

■

3.5 Vector Analysis in $\mathbb{R}^3$

3.5.1 Vector Algebra in $\mathbb{R}^3$

- In the case of three-dimensional Euclidean space the metric in Cartesian coordinates is $g_{ij} = \delta_{ij}$.

- The bases of p-forms are:

$$1, \quad dx, \quad dy, \quad dz, \quad dx \wedge dy, \quad dx \wedge dz, \quad dy \wedge dz, \quad dx \wedge dy \wedge dz\,.$$

- The star operator acts on this forms by

$$*1 = dx \wedge dy \wedge dz,$$

$$*dx = dy \wedge dz, \quad *dy = -dx \wedge dz, \quad *dz = dx \wedge dy,$$

$$*(dx \wedge dy) = dz, \quad *(dy \wedge dz) = dx, \quad *(dx \wedge dz) = -dy,$$

$$*(dx \wedge dy \wedge dz) = 1\,.$$

- So, any 2-form

$$\alpha = \alpha_{12} dx \wedge dy + \alpha_{13} dx \wedge dz + \alpha_{23} dy \wedge dz$$

is represented by the dual 1-form

$$*\alpha = \alpha_{12} dz - \alpha_{13} dy + \alpha_{23} dx\,,$$

- That is

$$(*\alpha)_i = \frac{1}{2}\varepsilon_{ijk}\alpha^{jk}$$

$$(*\alpha)_1 = \alpha_{23}\,, \qquad (*\alpha)_2 = \alpha_{31}\,, \qquad (*\alpha)_3 = \alpha_{12}\,,$$

- Any 3-form α

$$\alpha = \alpha_{123} dx \wedge dy \wedge dz$$

is represented by the dual 0-form

$$*\alpha = \frac{1}{3!}\varepsilon_{ijk}\alpha^{ijk} = \alpha_{123}\,.$$

- Now, let α and β be two 1-forms

$$\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz\,, \qquad \beta = \beta_1 dx + \beta_2 dy + \beta_3 dz\,.$$

Then

$$*\beta = \beta_1 dy \wedge dz + \beta_2 dz \wedge dx + \beta_3 dx \wedge dz$$

and

$$\alpha\wedge\beta = (\alpha_1\beta_2 - \alpha_2\beta_1) dx\wedge dy + (\alpha_1\beta_3 - \alpha_3\beta_1) dx\wedge dz + (\alpha_2\beta_3 - \alpha_3\beta_2) dy\wedge dz\,,$$

$$\alpha\wedge(*\beta) = (\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3) dx\wedge dy\wedge dz\,.$$

Therefore,

$$*(\alpha\wedge\beta) = (\alpha_1\beta_2 - \alpha_2\beta_1) dz - (\alpha_1\beta_3 - \alpha_3\beta_1) dy + (\alpha_2\beta_3 - \alpha_3\beta_2) dx\,,$$

$$*[\alpha\wedge(*\beta)] = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3\,,$$

or

$$*(\alpha\wedge\beta) = \alpha\times\beta\,,$$

$$*[\alpha\wedge(*\beta)] = \alpha\cdot\beta\,.$$

3.5.2 Vector Analysis in $\mathbb{R}^3$

- **Zero-Forms.**

For a 0-form f we have

$$(df)_i = \partial_i f\,,$$

so that

$$df = \text{grad } f\,.$$

- **One-Forms.**

For a 1-form

$$\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz$$

we have

$$(d\alpha)_{ij} = \partial_i\alpha_j - \partial_j\alpha_i$$

that is,

$$d\alpha = (\partial_1\alpha_2 - \partial_2\alpha_1) dx\wedge dy + (\partial_2\alpha_3 - \partial_3\alpha_2) dy\wedge dz + (\partial_3\alpha_1 - \partial_1\alpha_3) dz\wedge dx\,.$$

- Therefore
$$(*d\alpha)^i = \varepsilon^{ijk}\partial_j\alpha_k\,,$$
so that
$$*d\alpha = (\partial_2\alpha_3 - \partial_3\alpha_2)dx + (\partial_3\alpha_1 - \partial_1\alpha_3)dy + (\partial_1\alpha_2 - \partial_2\alpha_1)dz\,.$$

- We see that
$$*d\alpha = \mathbf{curl}\,\alpha\,.$$

- **Two-Forms.**
For a 2-form β there holds
$$(d\beta)_{ijk} = \partial_i\beta_{jk} + \partial_j\beta_{ki} + \partial_k\beta_{ij}\,,$$
or
$$d\beta = (\partial_1\beta_{23} + \partial_2\beta_{31} + \partial_3\beta_{12})dx \wedge dy \wedge dz\,.$$

- Hence,
$$*d\beta = \frac{1}{2}\varepsilon^{ijk}\partial_i\beta_{jk} = \partial_1\beta_{23} + \partial_2\beta_{31} + \partial_3\beta_{12}\,.$$

- Now let α be a 1-form
$$\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz\,.$$
Then
$$*\alpha = \alpha_1 dy \wedge dz - \alpha_2 dx \wedge dz + \alpha_3 dx \wedge dy\,,$$
and
$$d * \alpha = (\partial_1\alpha_1 + \partial_2\alpha_2 + \partial_3\alpha_3)dx \wedge dy \wedge dz\,,$$
or
$$*d * \alpha = \partial_1\alpha_1 + \partial_2\alpha_2 + \partial_3\alpha_3\,.$$
So,
$$*d * \alpha = \operatorname{div}\alpha\,.$$

Chapter 4

Integration of Differential Forms

4.1 Integration over a Parametrized Subset

4.1.1 Integration of n-Forms in $\mathbb{R}^n$

- Let $U \subset \mathbb{R}^n$ be a closed ball in $\mathbb{R}^n$ and u^i, $i = 1, \ldots, n$ be the Cartesian coordinates in $\mathbb{R}^n$.

- Let $f : U \to \mathbb{R}$ be a continuous real-valued function on U.

- Then the integral of f over U is the multiple integral

$$\int_U f = \int_U f(u)\, du^1 \cdots du^n \,.$$

- Let o be an orientation of U so that $o(u) = +1$ if the coordinate basis of covectors $(du^1, \ldots, du^n)$ has the same orientation as o and $o(u) = -1$ otherwise.

- Let

$$\alpha = f(u)\, du^1 \wedge \cdots \wedge du^n$$

  be an n-form.

- Then the **integral of α over U** is defined by

$$\int_U \alpha = o(u) \int_U f(u)\, du^1 \ldots du^n \,.$$

- The integral of the form α over U reverses sign if the orientation of U is reversed.

4.1.2 Integration over Parametrized Subsets

- Let M be an n-dimensional manifold with local coordinates x^i, $i = 1, \ldots, n$.
- Let $0 \le p \le n$ and U be an oriented region in $\mathbb{R}^p$ with orientation o and coordinates u^μ, $\mu = 1, \ldots, p$.
- Let $F : U \to M$ be a smooth map given locally by
$$x^i = F^i(u)\,.$$
- Then the image $F(U) \subset M$ of the set U is called a p**-subset of** M and the collection (U, o, F) is called an **oriented parametrized** p**-subset of** M.
- A parametrized 1-subset is called a **curve** in M.
- A parametrized 2-subset is called a **surface** in M.
- **Remarks.**
- A p-subset is not a submanifold, in general.
- A p-subset could have different dimensions at different points.
- Usually, the differential F_* has rank p (that is, $F(U)$ is a submanifold) **almost everywhere**.
- Let $\alpha \in \Lambda_p$ be a p-form on M
$$\alpha = \frac{1}{p!}\alpha_{i_1\ldots i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}\,.$$
- The **integral of** α **over an oriented parametrized** p**-subset** $F(U)$ is defined by
$$\int_{F(U)} \alpha = \int_U F^*\alpha\,.$$
- In more detail,
$$\begin{aligned}
\int_{F(U)} \alpha &= o(u) \int_U (F^*\alpha)\left(\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^p}\right) du^1 \cdots du^p \\
&= o(u) \int_U \alpha\left(F_*\frac{\partial}{\partial u^1}, \ldots, F_*\frac{\partial}{\partial u^p}\right) du^1 \cdots du^p \\
&= \frac{1}{p!} o(u) \int_U \alpha_{i_1\ldots i_p}(x(u)) \frac{\partial x^{i_1}}{\partial u^1} \cdots \frac{\partial x^{i_p}}{\partial u^p}\, du^1 \cdots du^p\,.
\end{aligned}$$

4.1.3 Line Integrals

- Let $U = [a, b] \subset \mathbb{R}$ be an interval.
- Then a map $F : U \to M$ defines an oriented curve $C = F(U)$ in M

$$x^i = F^i(t)\,.$$

- Let α be a 1-form in M

$$\alpha = \alpha_i(x)\, dx^i\,.$$

- Then the integral of α over C is called the **line integral**.
- In more detail

$$\int_C \alpha = \int_a^b \alpha \left[F_*\left(\frac{d}{dt}\right)\right] = \int_a^b \alpha_i(x(t))\frac{dx^i}{dt}\, dt\,.$$

4.1.4 Surface Integrals

- Let $U \subset \mathbb{R}^2$ be an oriented region in the plane, for example, $U = [a, b] \times [c, d]$.
- Then a map $F : U \to M$ defines an oriented parametrized surface $S = F(U)$ in M

$$x^i = F^i(u^1, u^2)\,.$$

- Let α be a 2-form in M

$$\alpha = \frac{1}{2}\alpha_{ij}(x)\, dx^i \wedge dx^j\,.$$

- Then the integral of α over C is called the **surface integral**.
- In more detail

$$\int_S \alpha = \int_U \alpha \left[F_*\left(\frac{d}{du^1}\right), F_*\left(\frac{d}{du^2}\right)\right] du^1\, du^2 = \int_U \frac{1}{2}\alpha_{ij}(x(u))\frac{dx^i}{du^1}\frac{dx^j}{du^2}\, du^1\, du^2\,.$$

- **Example in $\mathbb{R}^3$.**

4.1.5 Independence of Parametrization

- Let $U, V \subset \mathbb{R}^p$ be regions in $\mathbb{R}^p$ with coordinates u^μ, $\mu = 1, \dots, p$, and v^ν, $\nu = 1, \dots, p$, respectively.

- Let $H : U \to V$ be an diffeomorphism so that $V = H(U)$ defined by
$$v^\mu = H^\mu(u)\,.$$

- Then for any function $f : V \to \mathbb{R}$ there holds a formula for the change of variables in multiple integrals
$$\int_V f(v)\, dv^1 \dots dv^p = \int_U f[v(u)] \left| \frac{\partial(v^1, \dots, v^p)}{\partial(u^1, \dots, u^p)} \right| du^1 \dots du^p$$

- Let M be an n-dimensional manifold with $n \geq p$.

- Let U and V be oriented regions, u^μ and v^ν be positively-oriented coordinates on U and V, and the diffeomorphism H be orientation-preserving, that is, the Jacobian
$$\frac{\partial(v^1, \dots, v^p)}{\partial(u^1, \dots, u^p)} > 0$$
is positive.

- Let $F : U \to M$ and $G : V \to M$ be a smooth maps so that
$$F = G \circ H\,.$$

- Then $F(U)$ is an oriented parametrized p-subset of M and G is a reparametrization of this subset
$$x^i = F^i(u) = G^i(H(u))\,.$$

- Let $\alpha \in \Lambda_p$ be a p-form on M
$$\alpha = \frac{1}{p!} \alpha_{i_1 \dots i_p}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

- Then

$$\begin{aligned}
\int_{G(V)} \alpha &= \int_V G^* \alpha \\
&= \frac{1}{p!} \int_V \alpha_{i_1 \dots i_p}(G(v)) \frac{\partial x^{i_1}}{\partial v^1} \cdots \frac{\partial x^{i_p}}{\partial v^p} \, dv^1 \cdots dv^p \\
&= \frac{1}{p!} \int_U \alpha_{i_1 \dots i_p}(G(H(u))) \frac{\partial x^{i_1}}{\partial v^1} \cdots \frac{\partial x^{i_p}}{\partial v^p} \, \frac{\partial(v^1, \dots, v^p)}{\partial(u^1, \dots, u^p)} du^1 \cdots du^p \\
&= \frac{1}{p!} \int_U \alpha_{i_1 \dots i_p}(F(u)) \frac{\partial x^{i_1}}{\partial u^1} \cdots \frac{\partial x^{i_p}}{\partial u^p} \, du^1 \cdots du^p \\
&= \int_U \alpha
\end{aligned}$$

- Thus, the integral is independent of the parametrization of a p-subset.

4.1.6 Integrals and Pullbacks

- Let M be an n-dimensional manifold and W be an r-dimensional manifold.

- Let $\varphi : M \to W$ be a smooth map.

- Let $U \subset \mathbb{R}^p$ be an oriented region in $\mathbb{R}^n$ and $F : U \to M$ be an oriented parametrized p-subset of M.

- Then $\psi = \varphi \circ F : U \to W$ is an oriented parametrized p-subset of W.

- Let $\alpha \in \Lambda_p W$ be a p-form on W.

- Then

$$\int_{\psi(U)} \alpha = \int_U \psi^* \alpha = \int_U (F^* \circ \varphi^*) \alpha = \int_U F^*(\varphi^* \alpha) = \int_{F(U)} \varphi^* \alpha$$

- Let $S = F(U)$ be an oriented subset of M. Then $\psi(U) = \varphi(F(U)) = \varphi(S)$ is an oriented subset of W.

- The general pullback formula takes the form

$$\int_{\varphi(S)} \alpha = \int_S \varphi^* \alpha \, .$$

- **Remarks.**
- We have only defined integrals of forms over subsets of M covered by a single coordinate chart.
- We need to define the integrals over general submanifolds, covered by multiple charts.

4.2 Integration over Manifolds

4.2.1 Partition of Unity

- Let M be a manifold, $p_0 \in M$ be a point in M and (U, x) be a local coordinate chart about p_0.
- Let $x^i = x^i(p)$ be the local coordinates of the point p and $x^i_0 = x^i(p_0)$ be the local coordinates of the point p_0.
- Let
$$||x - x_0|| = \sqrt{\sum_{i=1}^{n}(x^i - x^i_0)^2}$$
- A neighborhood of p_0 is a subset of M defined by
$$B_\varepsilon(p_0) = \{p \in M \mid ||x - x_0|| < \varepsilon\}\,.$$
- Every neighborhood of a point in M is an **open** set in M.
- Let $A \subset M$ be a subset of M. A point $p \in M$ is called an **accumulation point** (or a **limit point**) of A if every neighborhood of p contains at least one point in A other than p.
- A subset of M is **closed** if and only if it contains all of its limit points.
- A **closure** of A, denoted by $\bar{A}$, is a set obtained by adding to A all its accumulation points.
- A closure of any set is a closed set.
- A function $f : M \to \mathbb{R}$ is **continuous** if the inverse image of every open set in $\mathbb{R}$ is open in M.
- Obviously, the set $\mathbb{R} \setminus \{0\}$ of non-zero real numbers is open.
- Thus, the subset of M where f is not equal to zero, that is, the set $f^{-1}(\mathbb{R}\setminus\{0\})$, is open.
- The **support** of f is the closure of the set $f^{-1}(\mathbb{R} \setminus \{0\})$,
$$\text{supp } f = \overline{f^{-1}(\mathbb{R} \setminus \{0\})}\,.$$

- Thus, f vanishes outside its support supp f.
- There could be points in supp f where f vanishes.
- A **bump function** is a smooth function $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that

$$0 \le \varphi_\varepsilon(t) \le 1 \qquad \forall t \in \mathbb{R}$$

  and

$$\varphi_\varepsilon(t) = \begin{cases} 1, & \text{if } |t| < \frac{\varepsilon}{4} \\ 0, & \text{if } |t| > \frac{\varepsilon}{2} \end{cases}$$

- Similarly, we define support of any tensor field.
- For example, an n-form $\omega \in \Lambda_n$ on M defined by

$$\omega = \varphi_\varepsilon(\|x - x_0\|)\, dx^1 \wedge \cdots \wedge dx^n$$

  has a support inside the neighborhood $B_\varepsilon(p_0)$,

$$\operatorname{supp}\omega \subset B_\varepsilon(p_0)\,.$$

  Such an n-form is called a **bump form**.
- Let $\{U_\alpha\}_{\alpha=1}^N$ be a (finite) atlas for the manifold M.
- A **partition of unity** is a set $\{\varphi_\alpha\}_{\alpha=1}^N$ of functions $\varphi_\alpha : M \to \mathbb{R}$ with the properties

$$\begin{aligned} & 0 \le \varphi_\alpha(p) \le 1 \qquad \forall \alpha, p \in M \\ & \operatorname{supp}\varphi_\alpha \subset U_\alpha \qquad \forall \alpha \\ & \sum_{\alpha=1}^N \varphi_\alpha = 1 \end{aligned}$$

- Notice that for any α, supp φ_α is closed and φ_α vanishes outside U_α.
- A general theorem from analysis says that every manifold has a partition of unity.
- **Example.**

4.2.2 Integration over Submanifolds

- A manifold M is **compact** if every open cover of M has a finite subcover.
- Thus, very compact manifold has a finite atlas.
- A subset of $\mathbb{R}^n$ is compact if it is closed and bounded.
- Let V be a p-dimensional compact oriented manifold.
- Let $\{U_\alpha\}_{\alpha=1}^N$ be a finite atlas of V.
- Let each U_α be positively oriented.
- Let $\{\varphi_\alpha\}$ be a partition of unity on V.
- Let β be a p-form over V.
- The **integral** of β over V is defined by

$$\int_V \beta = \sum_{\alpha=1}^N \int_{U_\alpha} \varphi_\alpha \beta$$

- This integral does not depend on the atlas and the partition of unity.
- Now, let V be a p-dimensional compact oriented submanifold of an n-dimensional manifold M described by the **inclusion map**

$$i : V \to M\,.$$

- Let $\beta \in \Lambda_p M$ be a p-form on M.
- The **integral** of a p-form β on M over $V \subset M$ is defined by

$$\int_V \beta = \int_V i^*\beta\,.$$

4.2.3 Manifolds with boundary

- Recall that an open ball in $\mathbb{R}^n$ is the set

$$B_\epsilon(x_0) = \{x \in \mathbb{R}^n \mid ||x - x_0|| < \varepsilon\}$$

- Let us consider also sets

$$H_\epsilon(x_0) = \{x \in \mathbb{R}^n \mid ||x - x_0|| < \varepsilon, x^n - x_0^n \geq 0\}\,.$$

  Such sets are called **half-open**.

- A n-dimensional **manifold with boundary** consists of the the **interior** M^o and the **boundary** ∂M.

- The interior M^o is a genuine n-dimensional manifold such that all its points have neighborhoods diffeomorphic to open balls in $\mathbb{R}^n$.

- The boundary ∂M is a subset of M such that all its points have neighborhoods diffeomorphic to half-open sets.

- Usually, the boundary ∂M is itself an $(n-1)$-dimensional submanifold of M (without boundary).

- Boundary may be disconnected. It can also be not smooth.

- Local coordinates $(x^1, \dots, x^{n-1}, x^n)$ in M in the neighborhoods of points on the boundary can always be chosen in such a way that $(x^1, \dots, x^{n-1})$ are the coordinates along the boundary and $0 \leq x^n < \delta$ with some δ.

- A compact manifold is a manifold which is closed and bounded (say, as a submanifold of some $\mathbb{R}^N$).

- A **closed manifold** is a manifold which is compact and does not have a boundary.

4.3 Stokes's Theorem

4.3.1 Orientation of the Boundary

- Let M be an n-dimensional orientable manifold with boundary ∂M, which is an $(n-1)$-dimensional manifold without boundary.
- Let M be oriented.
- Then an orientation on M naturally induces an orientation on ∂M.
- Let $p \in \partial M$ and $\{\mathbf{e}_2, \dots, \mathbf{e}_n\}$ be a basis in $T_p\partial M$.
- Let $\mathbf{N} \in T_pM$ be a tangent vector at p that is transverse to ∂M and points out of M.
- Then $\{\mathbf{N}, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ forms a basis in T_pM.
- Then, by definition, the basis $\{\mathbf{e}_2, \dots, \mathbf{e}_n\}$ has the same orientation as the basis $\{\mathbf{N}, \mathbf{e}_2, \dots, \mathbf{e}_n\}$. That is, $\{\mathbf{e}_2, \dots, \mathbf{e}_n\}$ is positively oriented in ∂M if $\{\mathbf{N}, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is positively oriented in M.

4.3.2 Stokes' Theorem

- **Theorem 4.3.1** *Let M be an n-dimensional manifold and V be a p-dimensional compact oriented submanifold with boundary ∂V in M. Let $\omega \in \Lambda_{p-1}M$ be a smooth $(p-1)$-form in M. Then*

$$\int_V d\omega = \int_{\partial V} \omega\,.$$

Proof:

1. Let
$$i : V \to M$$
be the inclusion map of the submanifold V.
2. Then
$$\int_V d\omega = \int_V i^*(d\omega) = \int d(i^*\omega)$$
and
$$\int_{\partial V} \omega = \int_{\partial V} i^*\omega$$

3. Let
$$\beta = i^*\omega$$

4. Then we need to show
$$\int_V d\beta = \int_{\partial V} \beta\,.$$

5. Let $\{V_\alpha\}$ be a finite cover of V and $\{\varphi_\alpha\}$ be the corresponding partition of unity.

6. We have
$$\int_V d\beta = \sum_\alpha \int_{V_\alpha} d(\varphi_\alpha\beta)$$
and
$$\int_{\partial V} \beta = \sum_\alpha \int_{\partial V} \varphi_\alpha\beta\,.$$

7. Thus, we need to show that
$$\int_{V_\alpha} d(\varphi_\alpha\beta) = \int_{\partial V} \varphi_\alpha\beta\,.$$

8. The charts V_α can be of two different kinds: I) those that are fully in the interior of V, that is, $V_\alpha \cap \partial V = \emptyset$, which are diffeomorphic to the open balls in $\mathbb{R}^p$, and II) those which contain points of the boundary, which are diffeomorphic to half-open balls in $\mathbb{R}^p$.

9. Case I. Let V_α be open sets in V fully in the interior of V.

10. Let $U_\alpha \subset \mathbb{R}^p$ be the open sets in $\mathbb{R}^p$ such that $f_\alpha : U_\alpha \to V_\alpha$ be the local coordinate diffeomorphisms.

11. Then
$$\int_{V_\alpha} d(\varphi_\alpha\beta) = \int_{U_\alpha} f_\alpha^*(d(\varphi_\alpha\beta)) = \int_{U_\alpha} d(f_\alpha^*(\varphi_\alpha\beta))$$

12. Let
$$\gamma_\alpha = f_\alpha^*(\varphi_\alpha\beta)\,.$$

13. Then
$$\gamma_\alpha = \sum_{i=1}^p (-1)^{i-1}\gamma_{\alpha,i} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^p$$

and

$$d\gamma_\alpha = \sum_{i=1}^{p} \frac{\partial \gamma_{\alpha,i}}{\partial x^i} dx^1 \wedge \cdots \wedge dx^p$$

14. Therefore,

$$\int_{U_\alpha} d\gamma_\alpha = \sum_{i=1}^{p} \int_{U_\alpha} \frac{\partial \gamma_{\alpha,i}}{\partial x^i} dx^1 \wedge \cdots \wedge dx^p = \sum_{i=1}^{p} \int_{\mathbb{R}^p} \frac{\partial \gamma_{\alpha,i}}{\partial x^i} dx^1 \wedge \cdots \wedge dx^p = 0\,.$$

15. Hence

$$\int_{V_\alpha} d(\varphi_\alpha \beta) = 0\,.$$

16. Also, since U_α is disjoint from the boundary

$$\int_{\partial V} \varphi_\alpha \beta = 0\,.$$

17. Thus, for each chart disjoint from the boundary

$$\int_{V_\alpha} d(\varphi_\alpha \beta) = \int_{\partial V} \varphi_\alpha \beta = 0\,.$$

18. Case II. Now, let us consider the half-open charts V_α at the boundary.
19. Let $U_\alpha \subset \mathbb{R}^p$ be the half-open sets in $\mathbb{R}^p$ such that $f_\alpha : U_\alpha \to V_\alpha$ be the local coordinate diffeomorphisms.
20. Let

$$W_\alpha = V_\alpha \cap \partial V$$

and

$$Y_\alpha = f_\alpha^{-1}(W_\alpha)\,.$$

21. Notice that for any point on Y_α, $x^p = 0$.
22. Then

$$\int_{V_\alpha} d(\varphi_\alpha \beta) = \sum_{i=1}^{p} \int_{\mathbb{R}^p} \frac{\partial \gamma_{\alpha,i}}{\partial x^i} dx^1 \wedge \cdots \wedge dx^p$$

23. We have

$$\int_{-\infty}^{\infty} \frac{\partial \gamma_{\alpha,i}}{\partial x^i} dx^i = 0$$

for any $i \neq p$, and

$$\int_{0}^{\infty} \frac{\partial \gamma_{\alpha,p}}{\partial x^p} dx^p = -\gamma_{\alpha,p}\Big|_{x^p=0}\,.$$

24. Therefore,

$$\int_{V_\alpha} d(\varphi_\alpha\beta) = -\int_{\mathbb{R}^{p-1}} \gamma_{\alpha,p}(x^1,\dots,x^{p-1},0)\, dx^1\cdots dx^p$$

25. Further,

$$\int_{\partial V} \varphi_\alpha\beta = \int_{W_\alpha} \varphi_\alpha\beta = \int_{Y_\alpha} f_\alpha^*(\varphi_\alpha\beta) = \int_{Y_\alpha} \gamma_\alpha$$

26. Since on Y_α, $x^p = 0$, then $dx^p = 0$. Therefore,

$$\int_{Y_\alpha} \gamma_\alpha = (-1)^{p-1}\gamma_{\alpha,p}(x^1,\dots,x^{p-1},0)dx^1\wedge\cdots\wedge dx^{p-1}$$

27. We have that $\{\partial_1,\dots,\partial_p\}$ is positively oriented on V and $-\partial_p$ is the outward normal to ∂V.

28. Thus, $\{\partial_1,\dots,\partial_{p-1}\}$ has orientation $(-1)^p$ on ∂V.

29. Thus

$$\int_{\partial V} \varphi_\alpha\beta = -\int_{Y_\alpha} \gamma_{\alpha,p}(x^1,\dots,x^{p-1},0)dx^1\cdots dx^{p-1}$$

30. So,

$$\int_{V_\alpha} d(\varphi_\alpha\beta) = \int_{\partial V} \varphi_\alpha\beta$$

which establishes the theorem.

∎

- **Examples.**

- **Newton formula**. Let C be an oriented curve, $\partial C = Q - P$ be its boundary and f be a smooth real-valued function. Then

$$\int_C df = \int_{\partial C} f = f(Q) - f(P)$$

- **Green formula**. Let S be an oriented surface in a 2-dimensional manifold M with parameters (u^1,u^2) and ∂S be its boundary with the induced orientation. Let U be the preimage of S on the plane $\mathbb{R}^2$ with the boundary $\partial U = [a,b]$. Let A be a one-form in M. Then

$$\int_U \left(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}\right)\frac{\partial(x^1,x^2)}{\partial(u^1,u^2)}du^1\wedge du^2 = \int_a^b \left(A_1\frac{dx^1}{dt} + A_2\frac{dx^2}{dt}\right)dt$$

- **Gauss formula.** Let D be a region in a 3-dimensional manifold M and U be the parameter preimage of D in $\mathbb{R}^3$. Let ∂D be the boundary of D and ∂U be the boundary of U. Let F be a 2-form in M. Then

$$\int_U \left(\frac{\partial F_{12}}{\partial x^3} + \frac{\partial F_{23}}{\partial x^1} + \frac{\partial F_{31}}{\partial x^2} \right) \frac{\partial(x^1, x^2, x^3)}{\partial(u^1, u^2, u^3)} du^1 \wedge du^2 \wedge du^3$$
$$= \int_{\partial U} \left(F_{12} \frac{\partial(x^1, x^2)}{\partial(z^1, z^2)} + F_{23} \frac{\partial(x^2, x^3)}{\partial(z^1, z^2)} + F_{31} \frac{\partial(x^3, x^1)}{\partial(z^1, z^2)} \right) dz^1 \wedge dz^2$$

- **Stokes' formula.** Let S be a surface in a 3-dimensional manifold M and ∂S be its boundary. Let U be the preimage of S in the parameter plane $\mathbb{R}^2$ and $\partial U = [a, b]$ be its boundary. Let A be a 1-form in M. Then

$$\int_U \left\{ \left(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2} \right) \frac{\partial(x^1, x^2)}{\partial(u^1, u^2)} + \left(\frac{\partial A_3}{\partial x^1} - \frac{\partial A_1}{\partial x^3} \right) \frac{\partial(x^1, x^3)}{\partial(u^1, u^2)} \right.$$
$$\left. + \left(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3} \right) \frac{\partial(x^2, x^3)}{\partial(u^1, u^2)} \right\} du^1 \wedge du^2$$
$$= \int_a^b \left(A_1 \frac{dx^1}{dt} + A_2 \frac{dx^2}{dt} + A_3 \frac{dx^3}{dt} \right) dt$$

4.4 Poincaré Lemma

•

Definition 4.4.1 *Let M be a manifold.*
A p-form α on M is called **closed** *if $d\alpha = 0$.*

A p-form α on M is called **exact** *if there is a $(p-1)$-form β such that $\alpha = d\beta$. The form β is called a* **potential** *of α.*

•

Theorem 4.4.1

1. *Every exact form is closed (***Poincaré Lemma***).*
   That is, if $\alpha = d\beta$, then $d\alpha = 0$.
2. *The exterior product of closed forms is closed.*
   That is, if $d\alpha = 0$ and $d\beta = 0$, then $d(\alpha \wedge \beta) = 0$.
3. *The exterior product of a closed form and an exact form is exact.*
   That is, if $d\alpha = 0$ and $\beta = d\gamma$, then there is σ such that $\alpha\wedge\beta = d\sigma$.
4. *Let M be an n-dimensional orientable compact manifold without boundary and α be an exact n-form on M, that is, $\alpha = d\beta$ for some $(n-1)$-form β. Then*
$$\int_M \alpha = 0\,.$$
5. *Let M be an n-dimensional oriented compact manifold with boundary ∂M and α be a closed $(n-1)$-form on M, that is, $d\alpha = 0$. Then*
$$\int_{\partial M} \alpha = 0\,.$$

Proof: Use Stokes theorem.

■

•

Definition 4.4.2 *Let M be a manifold. Suppose that for every closed oriented smooth curve C there is a smooth oriented 2-dimensional surface S and a map $F : S \to M$ such that $\partial F(V) = C$, that is, the curve C is the boundary of the surface S. Then the manifold M is said to have* **first Betti number** *equal to zero.*

• **Theorem 4.4.2** *Let M be a manifold with first Betti number equal to zero. Then every closed 1-form on M is exact. That is, if α is a 1-form such that $d\alpha = 0$, then there is a function f such that $\alpha = df$.*

Proof:

1. Let α be a closed form on M.
2. Let x and y be two points in M and C_{xy} be an oriented curve with the initial point y and the final point x.
3. Let f be defined by
$$f(x) = \int_{C_{xy}} \alpha\,.$$
4. Then f is independent on the curve C_{xy} and so is well defined.
5. Finally, we show that
$$df = \alpha\,.$$

■

• **Theorem 4.4.3** *Let α be a closed p-form in $\mathbb{R}^n$. Then there is $(p-1)$-form β in $\mathbb{R}^n$ such that $\alpha = d\beta$.*

That is, every closed form in $\mathbb{R}^n$ is exact.

Proof:

1. Let α be a closed p-form in $\mathbb{R}^n$.
2. We define a $(p-1)$-form β by
$$\beta_{i_1\dots i_{p-1}}(x) = \int_0^1 d\tau\ \tau^{p-1}\ x^j \alpha_{j i_1\dots i_{p-1}}(\tau x)$$
3. We can show that
$$d\beta = \alpha\,.$$

■

• **Corollary 4.4.1** *Let M be a manifold and α be a closed p-form on M. Then for every point x in M there is a neighborhood U of x and a $(p-1)$-form β on M such that $\alpha = d\beta$ in U.*

Proof:

1. Use the fact that a sufficiently small neighborhood of a point in M is diffeomorphic to an open ball in $\mathbb{R}^n$.
2. Pullback the form α from M to $\mathbb{R}^n$ by the pullback F^* of the diffeomorphism $F : V \to U$, where $U \subset M$ and $V \subset \mathbb{R}^n$.
3. Use the previous theorem.

■

Chapter 5

Lie Derivative

5.1 Lie Derivative of a Vector Field

5.1.1 Lie Bracket

- Let M be a manifold.
- Let $\mathbf{X}$ be a vector field on M.
- Let $\varphi_t : M \to M$ be the flow generated by $\mathbf{X}$.
- Let $x \in M$. Then $\varphi_t(x)$ is the point on the integral curve of the vector field $\mathbf{X}$ going through x and such that

  $$\varphi_0(x) = x$$

  and

  $$\frac{d\varphi_t(x)}{dt} = \mathbf{X}_{\varphi_t(x)} .$$

- Let $\varphi_{t*} : T_xM \to T_{\varphi_t(x)}M$ be the differential of the diffeomorphism φ_t.
- Notice that

  $$\varphi_{0\,*} = \mathrm{Id}$$

  is the identity and

  $$\varphi_{-t*} = (\varphi_{t*})^{-1}$$

  is the inverse transformation.

- In local coordinates for small t we have
$$\varphi_t^i(x) = x^i + tX^i(x) + O(t^2)\,,$$
so that
$$(\varphi_{t*})^i{}_j = \frac{\partial \varphi_t^i(x)}{\partial x^j} = \delta^i_j + t\frac{\partial X^i(x)}{\partial x^j} + O(t^2)$$
- Thus
$$\left(\frac{d}{dt}\varphi_{t*}\right)^i{}_j\Bigg|_{t=0} = \frac{\partial X^i(x)}{\partial x^j}\,.$$
- Let f be a smooth function on M.
- The flow φ_t naturally defines a new function
$$(\varphi_{t*} f)(x) = (f \circ \varphi_t)(x) = f(\varphi_t(x))\,.$$
- Then for small t
$$f \circ \varphi = f + t\mathbf{X}(f) + O(t^2)\,.$$
- Thus
$$\frac{d}{dt}(f \circ \varphi_t)\Big|_{t=0} = \mathbf{X}(f)\,.$$
- Let $\mathbf{Y}$ be another vector field on M.
- Then
$$\frac{d}{dt}\mathbf{Y}((f \circ \varphi_t))\Big|_{t=0} = \mathbf{Y}(\mathbf{X}(f))\,.$$
- Then at the point $\varphi_t(x)$ we have two different well defined vectors, $\mathbf{Y}_{\varphi_t(x)}$ and $\varphi_{t*}\mathbf{Y}_x$.
- **Diagram.**
- **Definition 5.1.1** *A vector field* $\mathbf{Y}$ *is* **invariant** *under the flow* φ_t *generated by a vector field* $\mathbf{X}$ *if*
$$\mathbf{Y}_{\varphi_t(x)} = \varphi_{t*}\mathbf{Y}_x\,.$$
An invariant vector field $\mathbf{Y}$ *is also called a* **Jacobi field**.

- **Definition 5.1.2** *The* **Lie derivative** *of the vector field* $\mathbf{Y}$ *with respect to the vector field* $\mathbf{X}$ *is the vector field* $L_{\mathbf{X}}\mathbf{Y}$ *defined at a point* x *by*

$$(L_{\mathbf{X}}\mathbf{Y})_x = \lim_{t\to 0}\frac{1}{t}\left(\mathbf{Y}_{\varphi_t(x)} - \varphi_{t*}\mathbf{Y}_x\right)$$

- **Remark**. Notice that this can also be written as

$$(L_{\mathbf{X}}\mathbf{Y})_x = \lim_{t\to 0}\frac{1}{t}\left(\varphi_{-t*}\mathbf{Y}_{\varphi_t(x)} - \mathbf{Y}_x\right)$$

or

$$(L_{\mathbf{X}}\mathbf{Y})_x = \frac{d}{dt}\left(\varphi_{-t*}\mathbf{Y}_{\varphi_t(x)}\right)\Big|_{t=0}.$$

- **Proposition 5.1.1**

$$L_{\mathbf{X}}\mathbf{Y} = [\mathbf{X}, \mathbf{Y}].$$

Proof:

1. We compute in local coordinates

$$\begin{aligned}
(L_{\mathbf{X}}\mathbf{Y})^i &= \frac{d}{dt}\left(\varphi_{-t*}\mathbf{Y}_{\varphi_t(x)}\right)^i\Big|_{t=0}\\
&= \frac{d}{dt}\left[(\varphi_{-t*}))^i{}_j Y^j(\varphi_t(x))\right]\Big|_{t=0}\\
&= \frac{d}{dt}(\varphi_{-t*}))^i{}_j\Big|_{t=0} Y^j(x) + \delta^i_j\frac{d}{dt}Y^j(\varphi_t(x))\Big|_{t=0}\\
&= -\frac{\partial X^i}{\partial x^j}Y^j(x) + \frac{\partial Y^i}{\partial x^j}X^j(x)\\
&= [\mathbf{X}, \mathbf{Y}]^i.
\end{aligned}$$

■

- **Definition 5.1.3** *The* **Lie bracket** *of two vector fields* $\mathbf{X}$ *and* $\mathbf{Y}$ *is a vector field* $[\mathbf{X}, \mathbf{Y}]$ *such that for any smooth function* f *on* M

$$[\mathbf{X}, \mathbf{Y}] = \mathbf{X}(\mathbf{Y}(f)) - \mathbf{Y}(\mathbf{X}(f)).$$

- Notice that

$$[\mathbf{X}, \mathbf{Y}] = -[\mathbf{Y}, \mathbf{X}].$$

- In particular,

$$L_{\mathbf{X}}\mathbf{X} = 0\,.$$

- In local coordinates the Lie bracket is given by

$$[\mathbf{X}, \mathbf{Y}]^i = X^j \partial_j Y^i - Y^j \partial_j X^i\,.$$

- The linear ordinary differential equation

$$[\mathbf{X}, \mathbf{Y}]^i = \frac{dY^i(t)}{dt} - (\partial_j X^i)(\varphi_t(x))Y^j(t) = 0$$

  for $Y^i(t)$ is called the **Jacobi equation**. For a given vector field **X** and given initial conditions for Y^i it defines a unique Jacobi field along the flow φ_t.

- In particular,

$$L_{\partial_i}\partial_j = 0\,.$$

5.1.2 Flow generated by the Lie Bracket

- **Theorem 5.1.1** *Let M be a manifold. Let* **X** *and* **Y** *be vector fields on M and φ_t^X and φ_t^Y be the flows generated by* **X** *and* **Y** *respectively. Let $\sigma_t : M \to M$ be a diffeomorphism defined by*

$$\sigma_t = \varphi_{-t}^Y \circ \varphi_{-t}^X \circ \varphi_t^Y \circ \varphi_t^X\,.$$

  Let f be a smooth function on M. Then

$$\begin{aligned} [\mathbf{X}, \mathbf{Y}]_x(f) &= \lim_{t\to 0} \frac{1}{t^2}\left[f(\sigma_t(x)) - f(x)\right] \\ &= \frac{d}{dt} f(\sigma_{\sqrt{t}})\Big|_{t=0}\,. \end{aligned}$$

  which means

$$[\mathbf{X}, \mathbf{Y}] = \frac{d}{dt}\sigma_{\sqrt{t}}\Big|_{t=0}\,.$$

  ***Proof*:**

  1. Diagram.

2. Use Taylor expansion in local coordinates.

■

• **Corollary 5.1.1** *Let M be a manifold and W be a submanifold of M. Let $\mathbf{X}$ and $\mathbf{Y}$ be vector fields on M tangent to W. Then the Lie bracket $[\mathbf{X}, \mathbf{Y}]$ is also tangent to W.*

5.2 Lie Derivative of Forms and Tensors

- Let $\mathbf{X}$ be a vector field on a manifold M.
- Let $\varphi_t : M \to M$ be the flow generated by $\mathbf{X}$ and $\varphi_t^* : T_{\varphi_t(x)}M \to T_xM$ be the corresponding pullback.
- **Definition 5.2.1** *Let f be a function (0-form) on M. Then the* **Lie derivative** *of f with respect to $\mathbf{X}$ is a function $L_\mathbf{X} f$ defined by*
$$(L_\mathbf{X} f)_x = \frac{d}{dt}(\varphi_t^* f)_x\Big|_{t=0} = \frac{d}{dt} f(\varphi_t(x))\Big|_{t=0} .$$
- **Proposition 5.2.1** *The Lie derivative of a function f with respect to a vector field $\mathbf{X}$ is equal to*
$$L_\mathbf{X} f = \mathbf{X}(f) .$$
- In local coordinates
$$L_\mathbf{X} f = X^i \partial_i f .$$
- **Definition 5.2.2** *Let α be a 1-form on M. The Lie derivative of α with respect to $\mathbf{X}$ is a 1-form $L_\mathbf{X}\alpha$ defined by*
$$(L_\mathbf{X}\alpha)_x = \lim_{t\to 0}\frac{1}{t}\left(\varphi_t^*\alpha_{\varphi_t(x)} - \alpha_x\right) = \frac{d}{dt}(\varphi_t^*\alpha)_x\Big|_{t=0}$$
- We can immediately generalize this to p-forms.
- **Definition 5.2.3** *Let α be a p-form on M. The Lie derivative of α with respect to $\mathbf{X}$ is a p-form $L_\mathbf{X}\alpha$ defined by*
$$(L_\mathbf{X}\alpha)_x = \lim_{t\to 0}\frac{1}{t}\left(\varphi_t^*\alpha_{\varphi_t(x)} - \alpha_x\right) = \frac{d}{dt}(\varphi_t^*\alpha)_x\Big|_{t=0}$$

- **Proposition 5.2.2** *The Lie derivative of a p-form α with respect to a vector field $\mathbf{X}$ is given by*

$$(L_{\mathbf{X}}\alpha)_{i_1\dots i_p} = X^j\partial_j\alpha_{i_1\dots i_p} + \alpha_{ji_2\dots i_p}\partial_{i_1}X^j + \cdots + \alpha_{i_1\dots i_{p-1}j}\partial_{i_p}X^j$$

Proof:

1. Use that

$$(\varphi_t^*\alpha)_{i_1\dots i_p}(x) = \frac{\partial\varphi_t^{j_1}(x)}{\partial x^{i_1}}\cdots\frac{\partial\varphi_t^{j_p}(x)}{\partial x^{i_p}}\alpha_{j_1\dots j_p}(\varphi_t(x))$$

■

- In particular, for a 1-form α we have

$$(L_{\mathbf{X}}\alpha)_i = X^j\partial_j\alpha_i + \alpha_j\partial_i X^j\,.$$

- More generally, since the flow $\varphi_t : M \to M$ is a diffeomorphism it naturally acts on general tensor bundles of type (p,q), that is,

$$\varphi_t^* : (T^p_q)_{\varphi_t(x)}M \to (T^p_q)_x M$$

For a general tensor field T of type (p,q) on M, φ_t^*T is a tensor field of type (p,q) defined by

$$(\varphi_t^*T)^{k_1\dots k_p}_{i_1\dots i_q}(x) = \frac{\partial\varphi_t^{j_1}(x)}{\partial x^{i_1}}\cdots\frac{\partial\varphi_t^{j_q}(x)}{\partial x^{i_q}}\frac{\partial x^{k_1}}{\partial\varphi_t^{m_1}(x)}\cdots\frac{\partial x^{k_p}}{\partial\varphi_t^{m_p}(x)}T^{m_1\dots m_p}{}_{j_1\dots j_q}(\varphi_t(x))$$

- **Definition 5.2.4** *Let T be a tensor field of type (p,q) on M. The Lie derivative of T with respect to $\mathbf{X}$ is a tensor field $L_{\mathbf{X}}T$ of type (p,q) defined by*

$$\begin{aligned}(L_{\mathbf{X}}T)_x &= \lim_{t\to 0}\frac{1}{t}\left(\varphi_t^*T_{\varphi_t(x)} - T_x\right)\\ &= \frac{d}{dt}\left(\varphi_t^*T\right)_x\Big|_{t=0}\end{aligned}$$

- **Proposition 5.2.3** *The Lie derivative of a tensor field T of type (p,q) with respect to a vector field $\mathbf{X}$ is given in local coordinates by*

$$\begin{aligned}(L_{\mathbf{X}}T)^{k_1\dots k_p}_{i_1\dots i_q} &= X^j\partial_j T^{k_1\dots k_p}_{i_1\dots i_q} + T^{k_1\dots k_p}_{ji_2\dots i_q}\partial_{i_1}X^j + \cdots + T^{k_1\dots k_p}_{i_1\dots i_{q-1}j}\partial_{i_q}X^j\\ &\quad -T^{jk_2\dots k_p}_{i_1 i_2\dots i_q}\partial_j X^{k_1} - \cdots - T^{k_1\dots k_{p-1}j}_{i_1\dots i_q}\partial_j X^{k_p}\end{aligned}$$

Proof:

1. Use the definition of the action φ_t^*.

■

- In particular, for a tensor g_{ij} of type $(0,2)$ we obtain

$$(L_{\mathbf{X}}g)_{ij} = X^k\partial_k g_{ij} + g_{ik}\partial_j X^k + g_{kj}\partial_i X^k\,.$$

If g_{ij} is a Riemannian metric, then this tensor is called the **strain tensor**.

5.2.1 Properties of Lie Derivative

- **Theorem 5.2.1** *For any two tensors T and R and a vector field $\mathbf{X}$ the Leibnitz rule holds*

$$L_{\mathbf{X}}(T \otimes R) = (L_{\mathbf{X}}T) \otimes R + T \otimes (L_{\mathbf{X}}R)$$

Proof:

1. Use the Leibnitz rule for

$$\frac{d}{dt}\left[\varphi_t^*(T \otimes R)\right] = \frac{d}{dt}\left[(\varphi_t^* T) \otimes (\varphi_t^* R)\right]$$

■

- **Theorem 5.2.2** *Let α be a p-form, β be a q-form and $\mathbf{X}$ be a vector field on M. Then the Leibnitz rule holds*

$$L_{\mathbf{X}}(\alpha \wedge \beta) = (L_{\mathbf{X}}\alpha) \wedge \beta + \alpha \wedge (L_{\mathbf{X}}\beta)$$

Proof:

1. Follows from the previous theorem and the definition of the wedge product.

■

- **Theorem 5.2.3** *For any 1-form ω and vector fields $\mathbf{X}$ and $\mathbf{Y}$ the Leibnitz rule holds*

$$L_{\mathbf{X}}(\omega(\mathbf{Y})) = (L_{\mathbf{X}}\omega)(\mathbf{Y}) + \omega(L_{\mathbf{X}}\mathbf{Y})$$

Proof:

1. Computation in local coordinates.

■

- The Lie derivative of a 1-form ω with respect to a vector field $\mathbf{X}$ can be defined in an intrinsic way. $L_{\mathbf{X}}\omega$ is a 1-form whose value on any vector field $\mathbf{Y}$ is

$$(L_{\mathbf{X}}\omega)(\mathbf{Y}) = \mathbf{X}(\omega(\mathbf{Y})) - \omega([\mathbf{X},\mathbf{Y}])\,.$$

- More generally, we have

Theorem 5.2.4 *Let $\mathbf{X}$ and $\mathbf{Y}_1,\dots,\mathbf{Y}_p$ be vector fields on a manifold M and $\alpha \in \Lambda_p$ be a p-form. Then*

$$L_{\mathbf{X}}\left(\alpha(\mathbf{Y}_1,\dots,\mathbf{Y}_p)\right) = (L_{\mathbf{X}}\alpha)(\mathbf{Y}_1,\dots,\mathbf{Y}_p) + \sum_{i=1}^{p}\alpha(\mathbf{Y}_1,\dots,L_{\mathbf{X}}\mathbf{Y}_i,\dots,\mathbf{Y}_p)\,.$$

Proof: By induction or direct calculation.

■

- **Remark.** This can be used as an intrinsic definition of $L_{\mathbf{X}}\alpha$.

- **Theorem 5.2.5** *Let $\mathbf{X}$ and $\mathbf{Y}$ be any two vector fields and $c \in \mathbb{R}$. Then*

  1. $L_{\mathbf{X}+\mathbf{Y}} = L_{\mathbf{X}} + L_{\mathbf{Y}}$,
  2. $L_{c\mathbf{X}} = cL_{\mathbf{X}}$,
  3. $L_{\mathbf{X}}\mathbf{Y} = -L_{\mathbf{Y}}\mathbf{X}$,
  4. $[L_{\mathbf{X}}, L_{\mathbf{Y}}] = L_{[\mathbf{X},\mathbf{Y}]}$.

Proof:

1.

■

- **Theorem 5.2.6** *Let g_{ij} be a Riemannian metric on an n-dimensional manifold M and*

$$\text{vol} = \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n$$

be the Riemannian volume form. Let $\mathbf{X}$ *be a vector field on M. Then*

$$L_{\mathbf{X}} \text{vol} = (\operatorname{div} \mathbf{X}) \text{vol} ,$$

where $\operatorname{div} \mathbf{X}$ *is a scalar function defined by*

$$\operatorname{div} \mathbf{X} = *L_X \text{vol} = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} X^i \right) .$$

Proof:

1. Direct calculation. Use $L_{\mathbf{X}} g_{ij}$.

■

- The scalar $\operatorname{div} \mathbf{X}$ is called the **divergence** of the vector field $\mathbf{X}$.

- **Remark.** Let $\mathbf{Y}_1, \dots, \mathbf{Y}_n$ be vector fields invariant under the vector field $\mathbf{X}$. Then

$$\begin{aligned} \operatorname{div} \mathbf{X} &= \frac{(L_{\mathbf{X}} \text{vol})(\mathbf{Y}_1, \dots, \mathbf{Y}_n)}{\text{vol}(\mathbf{Y}_1, \dots, \mathbf{Y}_n)} \\ &= \frac{d}{dt} \log \text{vol}(\mathbf{Y}_1, \dots, \mathbf{Y}_n) \Big|_{t=0} . \end{aligned}$$

- Thus, $\operatorname{div} \mathbf{X}$ is the logarithmic rate of change of the volume along the flow.

- **Remark.** Let $\beta \in \Lambda_{n-1}$ be an $(n-1)$-form defined by

$$\beta = i_{\mathbf{X}} \text{vol} .$$

In components

$$\beta_{i_1 \dots i_{n-1}} = X^j \sqrt{|g|} \varepsilon_{j i_1 \dots i_{n-1}}$$

Then

$$d\beta = (\operatorname{div} \mathbf{X}) \text{vol} .$$

To prove this compute in local coordinates

$$(d\beta)_{i_1 \dots i_n} = n \partial_{[i_1} \left(X^j \sqrt{|g|} \varepsilon_{j i_2 \dots i_n} \right)$$

- Recall that a linear map
$$A : \Lambda_p \to \Lambda_{p+r}$$
is a **derivation** if r is even and for any $\alpha \in \Lambda_p$ and $\beta \in \Lambda_q$
$$A(\alpha \wedge \beta) = (A\alpha) \wedge \beta + \alpha \wedge A\beta\,.$$
and an **anti-derivation** if r is odd and
$$A(\alpha \wedge \beta) = (A\alpha) \wedge \beta + (-1)^p \alpha \wedge A\beta\,.$$

- **Examples.**

- The Lie derivative $L_{\mathbf{X}}\Lambda_p \to \Lambda_p$ is a derivation.

- The exterior derivative $d : \Lambda_p \to \Lambda_{p+1}$ and the interior product $i_{\mathbf{X}} : \Lambda_p \to \Lambda_{p-1}$ are anti-derivations.

- Recall the following theorem about exterior derivative

Theorem 5.2.7 *Let $\mathbf{Y}_1, \dots, \mathbf{Y}_{p+1}$ be vector fields on a manifold M and $\alpha \in \Lambda_p$ be a p-form. Then*
$$(d\alpha)(\mathbf{Y}_1, \dots, \mathbf{Y}_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \mathbf{Y}_i \left(\alpha(\mathbf{Y}_1, \dots, \hat{\mathbf{Y}}_i, \dots, \mathbf{Y}_{p+1})\right)$$
$$+ \sum_{1 \le i < j \le p+1} (-1)^{i+j} \alpha([\mathbf{Y}_i, \mathbf{Y}_j], \dots, \hat{\mathbf{Y}}_i, \dots, \hat{\mathbf{Y}}_j, \dots, \mathbf{Y}_{p+1})\,.$$

- In particular, for $p = 1$ this takes the form

Theorem 5.2.8 *Let $\mathbf{X}$ and $\mathbf{Y}$ be vector fields on a manifold M and $\alpha \in \Lambda_1$ be a 1-form. Then*
$$(d\alpha)(\mathbf{X}, \mathbf{Y}) = \mathbf{X}(\alpha(\mathbf{Y})) - \mathbf{Y}(\alpha(X)) - \alpha([\mathbf{X}, \mathbf{Y}])\,.$$

- **Theorem 5.2.9** *The Lie derivative commutes with the exterior derivative, that is,*
$$L_{\mathbf{X}} d = d L_{\mathbf{X}}\,.$$
In other words, for any p-form α and a vector field $\mathbf{X}$ there holds
$$L_{\mathbf{X}} d\alpha = d L_{\mathbf{X}} \alpha$$

***Proof*:**

1. Check for $p = 0$ and $p = 1$ explicitly.
2. Generalize for $p > 1$.

■

Theorem 5.2.10 **(Cartan Formula)** *Let* $\mathbf{X}$ *be a vector field on a manifold* M *and* $\alpha \in \Lambda_p$ *be a* p*-form. Then*

$$L_{\mathbf{X}}\alpha = i_{\mathbf{X}}d\alpha + di_{\mathbf{X}}\alpha\,.$$

That is,

$$L_{\mathbf{X}} = i_{\mathbf{X}}d + di_{\mathbf{X}}$$

Proof:

1. Verify it for $p = 0$ and $p = 1$.

■

Theorem 5.2.11 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be vector fields on a manifold* M *and* $\alpha \in \Lambda_p$ *be a* p*-form. Then*

$$[L_{\mathbf{X}}, i_{\mathbf{Y}}]\alpha = L_{\mathbf{X}}i_{\mathbf{Y}}\alpha - i_{\mathbf{Y}}L_{\mathbf{X}}\alpha = i_{[\mathbf{X},\mathbf{Y}]}\alpha\,.$$

That is,

$$[L_{\mathbf{X}}, i_{\mathbf{Y}}] = i_{[\mathbf{X},\mathbf{Y}]}\,.$$

Proof: Exercise (Use Cartan formula).

■

Theorem 5.2.12 *Let* M *be a* n*-dimensional manifold,* $\mathbf{X}$ *be a vector field on* M *and* $\varphi_t : M \to M$ *be the corresponding flow of* $\mathbf{X}$*. Let* W *be a* p*-dimensional oriented compact submanifold of* M *and* $W_t = \varphi_t W$ *be the image of* W *under the flow* φ_t*. Let* α *be a* p*-form on* M*. Then*

$$\frac{d}{dt}\int_{W_t}\alpha = \int_{W_t} L_{\mathbf{X}}\alpha$$

5.3 Frobenius Theorem

5.3.1 Distributions

- **Definition 5.3.1** *Let M be a n-dimensional manifold. A k-dimensional* **distribution** *(or a* **tangent subbundle***) $\Delta : M \to \Delta_x \subset T_xM$ is a smooth assignment to each point $x \in M$ a k dimensional subspace Δ_x of the tangent space T_xM.*

  An submanifold V of M that is everywhere tangent to the distribution is called an **integral manifold** *of the distribution.*

  A k-didmensional distribution Δ is called **integrable** *if at each point $x \in M$ there is a k-dimensional integral submanifold of Δ.*

  In other words, the distribution Δ is integrable if everywhere in M there exist local coordinates $(x^1, \dots, x^k, y^1, \dots, y^{n-k})$ such that the coordinate surfaces $y^a = c^a$, $a = 1, \dots, n-k$, c^a being some constants, are integral manifolds of the distribution Δ. Such a coordinate system is called a **Frobenius chart**.

- **Examples** in $\mathbb{R}^3$.

- A one-dimensional distribution in $\mathbb{R}^3$ is a family of lines at every point in $\mathbb{R}^3$ (that is, a vector field). It is always integrable. Integral manifolds of the distribution are families of integral curves of the vector field.

- A two-dimensional distribution is a smooth family of planes at every point in $\mathbb{R}^3$. It is integrable if there exist nonintersecting surfaces that are everywhere tangent to the planes (and fill up a region in $\mathbb{R}^3$). Not every two-dimensional distribution is integrable!

- Let α be a 1-form in $\mathbb{R}^3$
$$\alpha = \alpha_i(x) dx^i \,,$$
  where $i = 1, 2, 3$. A two-dimensional distribution can be described by
$$\alpha = 0 \,.$$

- It is integrable if there exists a diffeomorphism $x^i = x^i(t, u)$, $u = (u^\mu)$, $\mu = 1, 2$, such that for a fixed t, $x^i = x^i(t, u)$ describes a smooth one-parameter family of surfaces S_t and

$$(F^*\alpha)_\mu = \alpha_i(x(t,u))\frac{\partial x^i}{\partial u^\mu} = 0$$

and for fixed $u = (u^1, u^2)$, $x^i = x^i(t,u)$ describes a smooth two-parameter family of curves C_u transversal to the surfaces S_t.

- Let $t = t(x)$ and $u^\mu = u^\mu(x)$ be the inverse diffeomorphism. Then the level surfaces $t(x) = C$ are the integral surfaces S_t of the distribution, so that the coordinate system (t, u^1, u^2) is the Frobenius chart.

- Therefore, we have

$$\begin{aligned}
\alpha &= \alpha_i \frac{\partial x^i}{\partial u^\mu} du^\mu + \alpha_i \frac{\partial x^i}{\partial t} dt \\
&= \alpha_i \frac{\partial x^i}{\partial t} dt \\
&= f\ dt\,,
\end{aligned}$$

where

$$f = \alpha_i \frac{\partial x^i}{\partial t}\,.$$

- Therefore,

$$d\alpha = df \wedge dt\,.$$

and

$$\alpha \wedge d\alpha = 0\,.$$

- Thus the distribution is integrable if

$$\alpha \wedge d\alpha = 0\,.$$

This is called the **Euler's integrability condition**.

- In local coordinates

$$\alpha_{[i}\partial_j\alpha_{k]} = 0\,.$$

In $\mathbb{R}^3$ this means

$$\boldsymbol{\alpha} \cdot \textbf{curl}\, \boldsymbol{\alpha} = 0\,.$$

- **Definition 5.3.2** *Let Δ be a distribution on M. It is said to be* **in involution** *if it is closed under Lie brackets, that is, for any two vector field* $\mathbf{X}$ *and* $\mathbf{Y}$ *in* Δ *the Lie bracket* $[\mathbf{X}, \mathbf{Y}]$ *is also in the distribution, or*

$$[\Delta, \Delta] \subset \Delta\,.$$

- Let Δ be an integrable distribution. Let $\mathbf{X}$ and $\mathbf{Y}$ be two vector fields in Δ. Then $\mathbf{X}$ and $\mathbf{Y}$ are tangent to the integral manifold of Δ. Therefore, the Lie bracket $[\mathbf{X}, \mathbf{Y}]$ is also tangent to the integral manifold and is in Δ.

- **Proposition 5.3.1** *Every integral distribution is in involution.*

- **Definition 5.3.3** *Let α be a 1-form on M. Let $x \in M$ be a point such that $\alpha_x \neq 0$. The* **null space** *of the form α at x is the $(n-1)$-dimensional subspace of T_xM spanned by the vectors $\mathbf{X} \in T_xM$ such that*

$$\alpha(\mathbf{X}) = 0\,.$$

- **Remark.** A 1-form is also called a **Pfaffian**.

- Let $\alpha_1, \dots, \alpha_{n-k}$ be $(n-k)$ linearly independent 1-forms such that

$$\alpha_1 \wedge \cdots \wedge \alpha_{n-k} \neq 0$$

- Let $N_1, \dots N_{n-k}$ be their null spaces. Then the intersection of the null spaces forms a k-dimensional distribution Δ

$$\Delta = \bigcap_{\mu=1}^{n-k} N_\mu\,.$$

- Locally this distribution is described by $(n-k)$ Pfaffian equations

$$\alpha_1 = \cdots = \alpha_{n-k} = 0\,.$$

- **Remarks.**

- Let Δ be in involution. Then $d\alpha_\mu|_\Delta = 0$, $\mu = 1, \dots, (n-k)$, that is, for any $\mathbf{X}, \mathbf{Y} \in \Delta$

$$(d\alpha_\mu)(\mathbf{X}, \mathbf{Y}) = 0\,.$$

- Now, suppose that $d\alpha_\mu|_\Delta = 0$, $\mu = 1, \dots, (n-k)$. Then the distribution Δ is in involution.

- **Theorem 5.3.1** *Let M be a n-dimensional manifold and $\alpha_1, \dots, \alpha_{n-k}$ be $(n-k)$ one-forms and ω be a $(n-k)$-form defined by*

$$\omega = \alpha_1 \wedge \cdots \wedge \alpha_{n-k} .$$

Suppose that the forms $\alpha_1, \dots, \alpha_{n-k}$ are linearly independent and $\omega \neq 0$. Let Δ be the k-dimensional distribution defined by the intersection of the null spaces of the 1-forms $\alpha_1, \dots, \alpha_{n-k}$. Then the following conditions are locally equivalent

1. *the distribution Δ is in involution,*
2. *$d\alpha_\mu|_\Delta = 0$, that is, for any $\mathbf{X}, \mathbf{Y} \in \Delta$*

$$(d\alpha_\mu)(\mathbf{X}, \mathbf{Y}) = 0 .$$

3. *$d\alpha_\mu \wedge \omega = 0$,*
4. *there exist 1-forms $\gamma_{\mu\nu}$ such that*

$$d\alpha_\mu = \sum_{\nu=1}^{n-k} \gamma_{\mu\nu} \wedge \alpha_\nu .$$

Proof:

1. (1) $\Leftrightarrow$ (2). Use the formula

$$d\alpha(\mathbf{X}, \mathbf{Y}) = \mathbf{X}(\alpha(\mathbf{Y})) - \mathbf{Y}(\alpha(\mathbf{X})) - \alpha([\mathbf{X}, \mathbf{Y}]) .$$

2. (4) $\Longrightarrow$ (2) and (4) $\Longrightarrow$ (3). Trivial.
3. (2) $\Longrightarrow$ (4). Suppose $d\alpha_\mu|_\Delta = 0$.
4. Let $\beta_1, \dots, \beta_k$ be 1-forms such that $\alpha_1, \dots, \alpha_{n-k}, \beta_1, \dots, \beta_k$ is a basis in $T^*_x M$.
5. Let $\mathbf{e}_1, \dots \mathbf{e}_{n-k}, \mathbf{v}_1, \dots, \mathbf{v}_k$ be the dual basis in $T_x M$.
6. Then for $\mu = 1, \dots, (n-k)$, $i = 1, \dots, k$

$$\alpha_\mu(\mathbf{v}_i) = 0 .$$

Thus, span $\{\mathbf{v}_i\} = \Delta$.

7. We have

$$d\alpha_\mu = \sum_{\mu<\nu} C_{\mu\lambda\nu}\alpha_\lambda \wedge \alpha_\nu + \sum_{i,\nu} A_{\mu\lambda i}\beta_i \wedge \alpha_\nu + \sum_{i<j} B_{\mu ij}\beta_i \wedge \beta_j$$
$$= \sum_{\nu} \gamma_{\mu\nu} \wedge \alpha_\nu + \sum_{i<j} B_{\mu ij}\beta_i \wedge \beta_j \,.$$

8. Thus,

$$B_{\mu ij} = (d\alpha_\mu)(\mathbf{v}_i, \mathbf{v}_j) = 0\,.$$

9. (3) $\implies$ (4). Suppose that $d\alpha_\mu \wedge \omega = 0$.

10. Then we have

$$0 = d\alpha_\mu \wedge \omega = \sum_{\nu} \gamma_{\mu\nu} \wedge \alpha_\nu \wedge \omega + \sum_{i<j} B_{\mu ij}\beta_i \wedge \beta_j \wedge \omega\,.$$
$$= \sum_{i<j} B_{\mu ij}\beta_i \wedge \beta_j \wedge \alpha_1 \wedge \cdots \wedge \alpha_{n-k}\,.$$

11. Thus, $B_{\mu ij} = 0$.

■

- **Remarks.**
- A k-dimensional distribution Δ can be described locally by either k linearly independent vector fields that span Δ or by $(n-k)$ linearly independent 1-forms whose common null space is Δ.
- If $d\alpha_\mu = \sum_{\nu=1}^{n-k} \gamma_{\mu\nu} \wedge \alpha_\nu$ for some 1-forms $\gamma_{\mu\nu}$, then we write

$$d\alpha_\mu = 0 \pmod{\alpha}\,.$$

5.3.2 Frobenius Theorem

- **Definition 5.3.4** *Let M be an n-dimensional manifold, W be a k-dimensional manifold and $F : W \to M$ be a smooth map.*

  Then F is an **immersion** *if for each $x \in W$ the differential $F_* : T_xW \to T_{F(x)}M$ is injective, that is,* $\operatorname{Ker} F_* = 0$.

  The image $F(W)$ of the manifold W is called an **immersed submanifold**.

- Let M be an n-dimensional manifold.
- Let Δ be a k-dimensional distribution on M.
- Let $\mathbf{X}_1, \dots, \mathbf{X}_k$ be vector fields that span Δ and $\varphi^1_t, \dots, \varphi^k_t$ be the corresponding flows.
- Let $x \in M$ be a point in M.
- Let $B \in \mathbb{R}^k$ be a sufficiently small open ball in $\mathbb{R}^k$ around the origin.
- We define
$$F : B \to M$$
for any $t = (t^1, \dots, t^k) \in B$ by
$$F(t) = \left(\varphi^k_{t_k} \circ \cdots \circ \varphi^1_{t_1}\right)(x)\,.$$
- Note that $F(0) = x$.
- Then for the differential at the origin
$$F_* T_0 B = \mathbb{R}^k \to T_x M$$
we have
$$F_* \frac{\partial}{\partial t^\mu}\Big|_{t=0} = \mathbf{X}_\mu\big|_x\,,$$
where $\mu = 1, \dots, k$.
- Thus,
$$F_* T_0 B = \Delta_x\,.$$
- So, F_* is injective at the origin and, hence, in some neighborhood of the origin.
- Therefore, $F(B)$ is tangent to Δ at the single point x.
- Thus, $F(B)$ is an immersed submanifold of M.

Theorem 5.3.2 Frobenius Theorem. *Let M be an n-dimensional manifold, Δ be a k-dimensional distribution in involution on M, $\mathbf{X}_1, \dots, \mathbf{X}_k$ be vector fields that span Δ and $\varphi^1_t, \dots, \varphi^k_t$ be the corresponding flows. Let $x \in M$, B be a sufficiently small ball around the origin in $\mathbb{R}^k$ and $F : B \to M$ be defined by*

•

$$F(t) = \left(\varphi^k_{t_k} \circ \cdots \circ \varphi^1_{t_1}\right)(x) .$$

Then:

1. *$F(B)$ is an immersed submanifold of M,*
2. *$F(B)$ is an integral manifold of Δ,*
3. *the distribution Δ is integrable.*

Proof:

1. (I) done earlier.
2. (II). We need to show that Δ is tangent to $F(B)$ at each point of $F(B)$.
3. We have for $\mu = 1, \dots, k$

$$F_* \frac{\partial}{\partial t^\mu} = \left(\varphi^k_{t_k *} \circ \cdots \circ \varphi^\mu_{t_\mu *}\right) \mathbf{X}_\mu \left[\left(\varphi^{k-1}_{t_{k-1}} \circ \cdots \circ \varphi^1_{t_1}\right)(x)\right] ,$$

4. Thus, the tangent space $T_{F(t)}F(B)$ has a basis

$$\begin{aligned}
&\left(\varphi^k_{t_k *} \circ \cdots \circ \varphi^2_{t_2 *}\right) \mathbf{X}_1 \left(\varphi^1_{t_1}(x)\right) \\
&\left(\varphi^k_{t_k *} \circ \cdots \circ \varphi^3_{t_3 *}\right) \mathbf{X}_2 \left[\left(\varphi^2_{t_2} \circ \varphi^1_{t_1}\right)(x)\right] \\
&\dots \\
&\varphi^k_{t_k *} \mathbf{X}_{k-1} \left[\left(\varphi^{k-1}_{t_{k-1}} \circ \cdots \circ \varphi^1_{t_1}\right)(x)\right] \\
&\mathbf{X}_k \left[\left(\varphi^k_{t_k} \circ \cdots \circ \varphi^1_{t_1}\right)(x)\right] .
\end{aligned}$$

5. Therefore, we need to show that for each $\mu = 1, \dots, k$, the differentials φ^μ_{t*} map the distribution Δ into itself, that is,

$$\varphi^\mu_{t*}(\Delta) \subset \Delta .$$

6. Claim: This follows from the fact that Δ is in involution.

7. Let $y \in F(B)$ and $\mathbf{Y} \in \Delta_y$.
8. Let $\mathbf{Y}_{\mu t} = \varphi^\mu_{t*}\mathbf{Y}$, $\mu = 1, \dots, k$.
9. We will show that $\mathbf{Y}_{\mu t} \in \Delta_{\varphi^\mu_t(y)}$ for $\mu = 1, \dots, k$.
10. Let Δ be defined by the 1-forms
$$\alpha_1 = \cdots = \alpha_{n-k} = 0\,.$$
11. The vector field $\mathbf{Y}_{\mu t}$ is invariant under the flow of $\mathbf{X}_\mu$, so along the orbits $\varphi^\mu_t(y)$ we have
$$L_{\mathbf{X}_\mu}\mathbf{Y}_{\mu t} = 0\,.$$
12. Let $f^\mu_1, \dots, f^\mu_{n-k}$ be real valued functions defined by
$$f^\mu_i(t) = \alpha_i(\mathbf{Y}_{\mu t})\,, \qquad (i = 1, \dots, n-k)\,.$$
13. Then at $t = 0$ we have the initial conditions
$$f_i(0) = \alpha_i(\mathbf{Y}) = 0\,.$$
14. Further,
$$\frac{d}{dt} f^\mu_i(t) = i_{\mathbf{Y}_{\mu t}} i_{\mathbf{X}_\mu} d\alpha_i\,.$$
15. Since Δ is in involution, by using $d\alpha_i = \sum_k \beta_{ik} \wedge \alpha_k$, we obtain
$$\frac{d}{dt} f^\mu_i(t) = \sum_{j=1}^{n-k} \gamma_{ij}(\mathbf{X}_\mu) f^\mu_j(t)\,.$$
16. Thus, the above system of the differential equations has the unique solution
$$f^\mu_i(t) = 0\,.$$
17. Thus, $\mathbf{Y}_{\mu t}$ is in Δ for all t and Δ is tangent to the immersed ball $F(B)$ at each point of $F(B)$.
18. (III). By constructing Frobenius chart.
19. That is, we construct coordinates $x^1, \dots, x^k, y^1, \dots, y^{n-k}$ so that the immersed balls $F(B)$ are described locally by
$$y^1 = c^1, \dots, y^{n-k} = c^{n-k}\,,$$
where c^i are constants.

20. We define a transversal to $F(B)$ $(n-k)$-dimensional submanifold W with local coordinates $y^1, \ldots, y^{n-k}$.
21. If the integral balls are sufficiently small, then for different points of W the integral balls at those points are disjoint.

■

5.3.3 Foliations

- Let M be an n-dimensional manifold and Δ be a k-dimensional distribution on M.
- Let Δ be in involution, and, therefore, integrable.
- Then, at each point x of M there exists an integral manifold of Δ.
- The integral manifold may return to the Frobenius coordinate patch around the point x infinitely many times.
- **Definition 5.3.5** *The integral manifolds of an integrable distribution define a* **foliation** *of M.*

  Each connected integral manifold is called a **leaf** *of the foliation.*

  A leaf that is not properly contained in another leaf is called a **maximal leaf**.
- A maximal leaf is not necessarily an embedded submanifold.
- An immersed submanifold does not have to be an embedded submanifold.
- **Theorem 5.3.3** *A maximal leaf of a foliated manifold is an immersed submanifold.*

  More precisely, for each maximal leaf V there is an injective immersion $F : V \to M$ that realizes V globally.
- **Examples.**

5.4 Degree of a Map

5.4.1 Gauss-Bonnet Theorem

- Let us consider a compact oriented two-dimensional manifold M (a surface embedded in $\mathbb{R}^3$).

- Let x^i, $i = 1, 2, 3$, be the Cartesian coordinates in $\mathbb{R}^3$ and u^μ, $\mu = 1, 2$, be the local coordinates on M.

- Let $F : M \to \mathbb{R}^3$ be the embedding map defined locally by

$$x^i = F^i(u) .$$

- The differential of the map F is given by the matrix

$$(F_*)^i{}_\mu = \frac{\partial x^i}{\partial u^\mu} .$$

- We assume that F_* is onto, that is, $\operatorname{rank} F_* = 2$.

- The tangent space T_xM is spanned by the vectors $\mathbf{e}_\mu$, $\mu = 1, 2$, with components

$$e^i_\mu = \frac{\partial x^i}{\partial u^\mu} .$$

- Let δ_{ij} be the Euclidean metric in $\mathbb{R}^3$.

- Then the **induced metric** on M is defined by

$$g_{\mu\nu} = (\mathbf{e}_\mu, \mathbf{e}_\nu) = \delta_{ij} e^i_\mu e^j_\nu .$$

  This matrix is also called the **first fundamental form**. It describes the **intrinsic geometry** of the surface.

- Let $\mathbf{N}$ be the vector in $\mathbb{R}^3$

$$\mathbf{N} = \mathbf{e}_1 \times \mathbf{e}_2$$

  with components

$$N_i = \varepsilon_{ijk} e^j_1 e^k_2 = \varepsilon_{ijk} \frac{\partial x^j}{\partial u^1} \frac{\partial x^k}{\partial u^2} .$$

- Then **N** is a vector field that is everywhere normal to M.

- Notice that the norm of the normal vector is

$$||\mathbf{N}||^2 = ||\mathbf{e}_1||^2 \; ||\mathbf{e}_2||^2 - (\mathbf{e}_1, \mathbf{e}_2)^2 \,.$$

- The **second fundamental form**, or the **extrinsic curvature** is defined by the matrix

$$b_{\mu\nu} = \frac{1}{||\mathbf{N}||}\left(\frac{\partial \mathbf{e}_\mu}{\partial u^\nu}, \mathbf{N}\right) = \frac{1}{||\mathbf{N}||}\frac{\partial^2 x^i}{\partial u^\mu \partial u^\nu} N_i \,.$$

- The second fundamental form describes the **extrinsic geometry** of the surface M.

- The **mean curvature** of M is defined by

$$H = g^{\mu\nu} b_{\mu\nu} \,.$$

- The **Gauss curvature** of M is defined by

$$K = \frac{\det b_{\mu\nu}}{\det g_{\alpha\beta}} \,.$$

- Gauss has shown that the K is an intrinsic invariant. In fact,

$$K = R^{12}{}_{12} = \frac{1}{2} R \,,$$

  where $R^{12}{}_{12}$ is the only non-vanishing components of the Riemann curvature of the metric g and R is the scalar curvature. This will be discussed later.

- The **Gauss map** is the map $\varphi : M \to S^2$ from M to S^2 defined by

$$\varphi(x) = \frac{\mathbf{N}(x)}{||\mathbf{N}(x)||} \,,$$

  that is, it associates to every point x in M the unit normal vector at that point.

- **Theorem 5.4.1** *Let M be a closed (compact without boundary) oriented 2-dimensional surface embedded in R^3. Let $g_{\mu\nu}$ be the induced Riemannian metric, $d\text{vol} = \sqrt{|g|}dx$ be the Riemannian volume element on M, K be the Gaussian curvature of M and $\varphi : M \to S^2$ be the Gauss normal map. Then*

$$\frac{1}{4\pi}\int_M d\text{vol}\ K = \deg(\varphi)$$

is an integer that does not depend on the metric and does not change under smooth deformations of the surface.

Proof: Later.

- **Remark**. For a 2-surface of genus g (with g holes)

$$\deg(\varphi) = 1 - g\,.$$

- The **Euler characteristic** of the surface M is a topological invariant equal to

$$\chi = 1 - g\,.$$

5.4.2 Laplacian

- Let g_{ij} be a Riemannian metric on a manifold V. Let h be a function and $\mathbf{X}$ be its gradient vector field defined by

$$X^i = g^{ij}\partial_j h\,.$$

- The **Laplacian** operator Δ on the scalar function h is defined by

$$\Delta h = \operatorname{div}\mathbf{X}\,.$$

- In local coordinates

$$\Delta h = g^{-1/2}\partial_i(g^{1/2}g^{ij}\partial_j h)\,,$$

where $g = \det g_{ij}$.

- The operator $(-\Delta) : C^\infty(V) \to C^\infty(V)$ acting on smooth functions on a compact manifold V without boundary can be extended to a self-adjoint operator on the Hilbert space $L^2(V)$.

- It has a discrete non-negative real spectrum $\{\lambda\}_{k=0}^{\infty}$ bounded from below by zero
$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \cdots$$
with finite multiplicities.

- The eigenfunctions $\{h_k\}_{k=0}^{\infty}$ form an orthonormal basis in $L^2(V)$, that is,
$$(h_k, h_l) = \int_V d\text{vol}\ \ h_k h_l = \delta_{kl}\,.$$

- For each $f \in L^2(V)$ there is a Fourier series
$$f = \sum_{k=0}^{\infty} a_k h_k\,,$$
where
$$a_k = (h_k, f) = \int_V f h_k\,.$$

- The lowest eigenvalue is 0. It is simple (has multiplicity 1). The corresponding eigenfunction is the constant $h_0 = [\text{vol}\,(V)]^{-1/2}$.

- **Lemma 5.4.1** *Let f be a function on a closed manifold M such that $\int_M f = 0$. Let $\{\lambda_k, h_k\}_{k=1}^{\infty}$ be the spectral resolution of the operator $(-\Delta)$. Then the equation*
$$\Delta h = f$$
has a unique solution given by the Fourier series
$$h = -\sum_{k=1}^{\infty} \frac{1}{\lambda_k} a_k h_k\,,$$
where
$$a_k = (h_k, f) = \int_M f h_k\,.$$

Proof:

1.

■

• **Lemma 5.4.2** *Let V be a closed (compact without boundary) oriented n-dimensional manifold and ω be an n-form on V. Then ω is exact if and only if*

$$\int_V \omega = 0\,.$$

Proof:

1. (I). If ω is exact, then $\omega = d\sigma$ for some $(n-1)$-form σ.
2. Then by Stokes theorem
$$\int_V \omega = 0\,.$$
3. (II). Suppose that
$$\int_V \omega = 0\,.$$
4. Let us fix a Riemannian metric on V and let vol be a Riemannian volume element.
5. Then
$$\omega = f\text{vol}$$
for some function f which satisfies
$$\int_V f = 0\,.$$
6. Let h be the solution of the equation
$$\Delta h = f\,.$$
7. Let $\mathbf{X}$ be the gradient of the function h.
8. We define an $(n-1)$-form σ by
$$\sigma = i_{\mathbf{X}}\text{vol}\,.$$
9. Then
$$d\sigma = \omega\,.$$

■

5.4.3 Brouwer Degree

- Let M and V be two closed (compact without boundary) oriented n-dimensional manifolds.
- Let $\varphi : M \to V$ be a smooth map.
- Let $\omega \in \Lambda_n V$ be an n-form on V.
- Suppose that
$$\int_V \omega \neq 0\,.$$
Then we can normalize it so that
$$\int_V \omega = 1\,.$$
- Then we can consider the quantity
$$\frac{\int_M \varphi^* \omega}{\int_V \omega}\,,$$
which can also be written as
$$\frac{\int_{\varphi(M)} \omega}{\int_V \omega}\,.$$
- This quantity counts how many times the image of M wraps around V.
- **Corollary 5.4.1** *Let M and V be n-dimensional manifolds and $\varphi : M \to V$ be a smooth map. Let α and β be n-forms on V such that $\int_V \alpha \neq 0$ and $\int_V \beta \neq 0$. Then*
$$\frac{\int_M \varphi^* \alpha}{\int_V \alpha} = \frac{\int_M \varphi^* \beta}{\int_V \beta}\,.$$

Proof:

1. Let
$$\omega = \left(\frac{\alpha}{\int_V \alpha} - \frac{\beta}{\int_V \beta} \right).$$

2. We have
$$\int_V \omega = 0\,.$$
3. Therefore, the form ω is exact.
4. Hence, the form $\varphi^*\omega$ is exact.
5. Thus,
$$\int_M \varphi^*\omega = 0\,.$$

■

- The quantity
$$\deg(\varphi) = \frac{\int_M \varphi^*\omega}{\int_V \varphi}$$
does not depend on the choice of the form ω but only on the map φ. It is called the **Brouwer degree** of the map φ.

- **Example.**

- In the case of one-dimensional manifolds the degree of the map $\varphi : M \to S^1$ is called the **winding number**.

- Picture.

- **Theorem 5.4.2** *Let V and M be n-dimensional compact oriented manifolds without boundary. Let $\varphi : M \to V$ be a smooth map. Let $y \in V$ be a regular value of φ so that the differential $\varphi_* : T_xM \to T_yV$ at any point $x \in \varphi^{-1}(y)$ is bijective (isomorphism). Then*
$$\deg(\varphi) = \sum_{x\in\varphi^{-1}(y)} \operatorname{sign}(\varphi(x))\,,$$
where
$$\operatorname{sign}(\varphi(x)) = \operatorname{sign}(\det(\varphi_*))\,.$$

Proof:

1. (I). Claim: the preimage $\varphi^{-1}(y)$ of a regular value is a finite set, that is,
$$\varphi^{-1}(y) = \{x_i \in M \mid \varphi(x_i) = y,\ i = 1, 2, \ldots, N\}\,.$$

2. Suppose $\varphi^{-1}(y)$ is infinite.
3. Then by compactness argument $\varphi^{-1}(y)$ has a limit point $x_0 \in M$, which is a regular point.

   Indeed, every sequence has a convergent subsequence. Thus, there is a sequence (x_k), such that $x_k \in \varphi^{-1}(y)$, converging to some x_0, $x_k \to x_0$, so that $\varphi(x_0) = y$. Thus x_0 is a regular point of M.
4. Since $\varphi_* : T_{x_0}M \to T_yV$ is bijective at x_0, the map $\varphi : M \to V$ is a *diffeomorphism* in a neighborhood of x_0.

   That is, $\det \varphi_*|_{x_0} \neq 0$.
5. This contradicts the fact that there is sequence of points $x_k \to x_0$ such that $\varphi(x_k) = y$.

   Since, otherwise there exist infinitely many points $x_k \in M$ such that $\varphi(x_k) = \varphi(x_0) = y$ contradicting the fact that φ is bijective.
6. (II). Claim: The point $y \in V$ has a neighborhood whose inverse image is a disjoint union of neighborhoods of the preimages of y, each of which is diffeomorphic to V_y.
7. Let W_i be disjoint neighborhoods of $x_i \in M$ such that $\varphi : W_i \to V_i = \varphi(W_i)$ are diffeomorphisms.
8. Let $S = \varphi(M - \cup_{i=1}^N W_i)$.
9. Since $M - \cup_{i=1}^N W_i$ is compact, then S is compact (and closed in V).
10. Let $O = V - S$.
11. Then O is open and is a neighborhood of y.
12. Now, we define
$$V_y = O \cap \cap_{i=1}^N V_i\,.$$
13. Then
$$V_y \subset O, \qquad V_y \subset V_i\,.$$
14. Let
$$U_i = \varphi^{-1}(O) \cap W_i\,.$$
15. Then
$$U_i \subset W_i \subset M\,.$$

16. Then
$$\varphi^{-1}(V_y) = \cup_{i=1}^N U_i\,,$$
and $\varphi : U_i \to V_y$ are diffeomorphisms.
17. (III). Let ω be an n-form on V with support in V_y such that $\int_V \omega = 1$.
18. Let y^i, $i = 1, \dots, n$, be local coordinates in V_y.
19. Since each $\varphi : U_i \to V_y$ are diffeomorphisms, one can use y^i as local coordinates on U_i.
20. In such coordinates the diffeomorphism $\varphi : U_i \to V_y$ is the identity map.
21. We notice that the orientation of U_i is described by $\operatorname{sign}\varphi(x_i)$.
22. Thus
$$\deg(\varphi) = \int_M \varphi^*\omega = \sum_{i=1}^N \int_{U_i} \varphi^*\omega = \sum_{i=1}^N \int_{\varphi(U_i)} \omega$$
$$= \sum_{i=1}^N \operatorname{sign}\varphi(x_i) \int_{V_y} \omega = \sum_{i=1}^N \operatorname{sign}(\varphi(x_i))\,.$$

■

- **Corollary 5.4.2**

  1. *The Brouwer degree* $\deg(\varphi)$ *of any map* $\varphi : M \to V$ *is an integer.*
  2. *The sum*
  $$\sum_{x\in\varphi^{-1}(y)} \operatorname{sign}(\varphi(x))$$
  is independent on the regular value $y \in V$*.*
  3. *The Brouwer degree* $\deg(\varphi)$ *remains constant under continuous deformations of the map* φ*.*

- **Problem**. Let ω be the volume form on S^n in $\mathbb{R}^{n+1}$ given by
$$\omega = \sum_{k=1}^{n+1} (-1)^k dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^{n+1}\,.$$
Show that the antipodal map $\varphi : S^n \to S^n$ has degree $\deg(\varphi) = (-1)^{n+1}$.

- **Problem**. Let M be a closed oriented n-dimensional manifold and $\varphi : M \to S^n$ be a smooth map. We identify $\varphi(x)$ with a unit vector field $\mathbf{v}$ on S^n in $\mathbb{R}^{n+1}$. Let vol be the volume $(n+1)$-form in $\mathbb{R}^{n+1}$. Let $\mathrm{vol}\,(S^n)$ be the volume of the unit sphere S^n. Let u^μ, $\mu = 1, \dots, n$, be local coordinates on M. Show that

$$\deg(\varphi) = \frac{1}{\mathrm{vol}\,(S^n)} \int_M \mathrm{vol}\left(\mathbf{v}, \frac{\partial \mathbf{v}}{\partial u^1}, \cdots, \frac{\partial \mathbf{v}}{\partial u^n}\right) du^1 \wedge \cdots \wedge du^n$$

5.4.4 Index of a Vector Field

- Let M be a closed (compact without boundary) n-dimensional submanifold of $\mathbb{R}^{n+1}$ that is a boundary of a compact region $U \subset \mathbb{R}^{n+1}$, that is,

$$M = \partial U\,.$$

- Let $\mathbf{N}$ be an outward-pointing normal to M.

- Then $\mathbf{N}$ induces an orientation on M.

- Let $\mathbf{v}$ be a unit vector field on M.

- Let S^n be the unit n-sphere embedded in $\mathbb{R}^n$ centered at the origin.

- We identify the unit vectors in $\mathbb{R}^{n+1}$ with points in S^n.

- Let $\varphi : M \to S^n$ be a map defined for every $x \in M$ by

$$\varphi(x) = \mathbf{v}(x)\,.$$

- The **Kronecker index** of the vector field $\mathbf{v}$ on M is defined as the degree of the map φ:

$$\mathrm{Index}(\mathbf{v}) = \deg(\varphi)\,.$$

- In general, if $\mathbf{v}$ is a nonvanishing vector field on M, then its index is defined as the Kronecker index of the unit vector field $\mathbf{v}/||\mathbf{v}||$.

- **Example.** $M = S^1$ in $\mathbb{R}^2$.

- **Problem.** Let $M = \partial U$, where $U \subset \mathbb{R}^{n+1}$ is a compact region as above. Show that if a vector field $\mathbf{v}$ on M can be extended to a nonvanishing vector field on all of the interior region U, then $\mathrm{Index}(\mathbf{v}) = 0$.

  In other words, a vector field on M with a non-zero index has a singularity (that is, it vanishes) at least at one point inside M.

- **Problem.** Suppose that a unit vector field $\mathbf{v}$ in $\mathbb{R}^{n+1}$ is such that it is smooth in U except for a finite number of points x_a, $a = 1, \dots, N$. Let B_a be sufficiently small balls around the points x_a so that they are in U. Then $\mathbf{v}$ is non-vanishing in $U \setminus \cup_{a=1}^N B_a$. Then ∂B_a are small spheres with outward-poiting normals. Let $\mathrm{Index}(\mathbf{v}|_{\partial B_a})$ be the indices of $\mathbf{v}$ on ∂B_a. Show that

$$\mathrm{Index}(\mathbf{v}) = \sum_{a=1}^{N} \mathrm{Index}(\mathbf{v}|_{\partial B_a}) .$$

  That is, the index of a vector field on a closed manifold M is equal to the sum of indices inside M.

- **Theorem 5.4.3** *Let $\mathbf{v}_t$, $t \in [0, 1]$, be a smooth family of non-vanishing vector field on a closed manifold M. Then*

$$\mathrm{Index}(\mathbf{v}_0) = \mathrm{Index}(\mathbf{v}_1) .$$

  Proof:

  1. Follows from the fact that the index is an integer.

  ■

- **Problem**. Show that if a vector field $\mathbf{v}$ on a sphere S^n in $\mathbb{R}^{n+1}$ never points to the origin, then

$$\mathrm{Index}(\mathbf{v}) = 1 .$$

- **Corollary.** The index of every non-vanishing vector field tangent to S^n is equal to

$$\mathrm{Index}(\mathbf{v}) = 1 .$$

- **Theorem 5.4.4 Brouwer Fixed Point Theorem.** *Let B be a closed unit ball in $\mathbb{R}^{n+1}$ centered at the origin. Let $\varphi : B \to B$ be a smooth map. Then φ has a fixed point.*

  Proof:

1. Let $\mathbf{v}$ be a vector field in $\mathbb{R}^{n+1}$ defined by

$$\mathbf{v}(x) = \varphi(\mathbf{x}) - \mathbf{x}\,.$$

2. Then $\mathbf{v}$ never points to the origin. So, $\text{Index}(\mathbf{v}) = 1$, and, therefore, φ has a fixed point in B.
3. Alternatively, suppose that φ does not have a fixed point.
4. Then $\mathbf{v}$ is non-vanishing.
5. Let $\psi : B \to \partial B = S^n$ be a map defined as follows. The point $\psi(x)$ is the point on the sphere S^n where the line from the $\varphi(x)$ to the point x intersects S^n.
6. Then $\psi(x) = x$ for any $x \in S^n$.
7. Let α be an n-form on S^n normalized by $\int_{S^n} \alpha = 1$.
8. Since ψ is an identity map on S^n, the form $\psi^*\alpha$ is an n-form on B whose restriction to S^n is equal to α.
9. Since also $S^n = \partial B$ and $d\alpha = 0$, we have

$$1 = \int_{S^n} \alpha = \int_{\partial B} \psi^*\alpha = \int_B d(\psi^*\alpha) = 0\,,$$

which is a contradiction.

■

- **Problem.** Let M be a closed n-dimensional submanifold of $\mathbb{R}^{n+1}$ Let $\mathbf{v}$ be a unit vector field on M. Let vol be the volume $(n+1)$-form in $\mathbb{R}^{n+1}$. Let $\text{vol}\,(S^n)$ be the volume of the unit sphere S^n. Let u^μ, $\mu = 1, \dots, n$, be local coordinates on M. Show that

$$\text{Index}(\mathbf{v}) = \frac{1}{\text{vol}\,(S^n)} \int_M \text{vol}\left(\mathbf{v}, \frac{\partial \mathbf{v}}{\partial u^1}, \cdots, \frac{\partial \mathbf{v}}{\partial u^n}\right) du^1 \wedge \cdots \wedge du^n$$

- **Problem.** If $\mathbf{v}$ is a non-vanishing vector field, not necessarily unit, then

$$\text{Index}(\mathbf{v}) = \frac{1}{\text{vol}\,(S^n)} \int_M \frac{1}{||\mathbf{v}||^{n+1}} \text{vol}\left(\mathbf{v}, \frac{\partial \mathbf{v}}{\partial u^1}, \cdots, \frac{\partial \mathbf{v}}{\partial u^n}\right) du^1 \wedge \cdots \wedge du^n$$

•

Corollary 5.4.3 *Let M be a closed n-dimensional submanifold of $\mathbb{R}^{n+1}$ such that M is the boundary of a compact region $U \subset \mathbb{R}^{n+1}$. Let f_i, $i = 1, \dots, (n+1)$, be smooth functions on U. Let*

$$\|f\|^2 = \sum_{i=1}^{n+1} |f_i|^2 .$$

Suppose the functions f_i do not have common zeros on M, that is, $\|f\| \neq 0$ on M. Let u^μ, $\mu = 1, \dots, n$, be local coordinates on M. Let

$$\begin{aligned}\det(f, df) &= \varepsilon^{i_1 i_2 \dots i_n i_{n+1}} f_{i_1} \frac{\partial f_{i_2}}{\partial u^1} \cdots \frac{\partial f_{i_{n+1}}}{\partial u^n} \\ &= \det \begin{pmatrix} f_1 & \frac{\partial f_1}{\partial u^1} & \cdots & \frac{\partial f_1}{\partial u^n} \\ \vdots & \vdots & \ddots & \vdots \\ f_n & \frac{\partial f_n}{\partial u^1} & \cdots & \frac{\partial f_n}{\partial u^n} \end{pmatrix}\end{aligned}$$

Suppose that

$$\int_M \frac{1}{\|f\|^{n+1}} \det(f, df) du^1 \wedge \cdots \wedge du^n \neq 0 .$$

Then the functions f_i have a common zero in U, that is, the system of $(n+1)$ equations

$$f_1 = \cdots = f_{n+1} = 0$$

has a solution in U.

Proof: Follows from above.

■

5.4.5 Linking Number

- Let $C_\mu : S^1 \to \mathbb{R}^3$, $\mu = 1, 2$, be two nonintersecting smooth closed curves (loops) in $\mathbb{R}^3$ described by

$$\mathbf{x} = \mathbf{x}_1(\theta_1) , \qquad \mathbf{x} = \mathbf{x}_2(\theta_2) .$$

- Let $T^2 = S^1 \times S^1$ be the torus with the local coordinates θ_1, θ_2.

- Let $\varphi : T \to S^2$ be a smooth map defined by

$$\varphi(\theta) = \frac{\mathbf{x}_1(\theta_1) - \mathbf{x}_2(\theta_2)}{||\mathbf{x}_1(\theta_1) - \mathbf{x}_2(\theta_2)||}\,.$$

- The **Gauss linking number** of the loops C_1 and C_2 is defined by

$$\mathrm{Link}(C_1, C_2) = \deg(\varphi)\,.$$

- **Problem.** Let

$$\mathbf{x}_{12} = \mathbf{x}_2 - \mathbf{x}_1\,.$$

  Show that

$$\begin{aligned}
\mathrm{Link}(C_1, C_2) &= \frac{1}{4\pi}\int_{C_1}\left(\int_{C_2}\frac{1}{||\mathbf{x}_{12}||^3}\mathbf{x}_{12}\times d\mathbf{x}_{12}\right)\cdot d\mathbf{x}_1 \\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} d\theta_1\, d\theta_2\, \frac{\{[\mathbf{x}_2(\theta_2) - \mathbf{x}_1(\theta_1)]\times\dot{\mathbf{x}}_2(\theta_2)\}\cdot\dot{\mathbf{x}}_1(\theta_1)}{||\mathbf{x}_2(\theta_2) - \mathbf{x}_1(\theta_1)||^3}
\end{aligned}$$

- Let V be an orientable surface in $\mathbb{R}^3$ such that $\partial V = C_1$.

- Let $\mathbf{N}$ be the normal vector to V consistent with the orientation of C_1.

- Let the curve C_2 intersect V transversally.

- The **intersection number** $V \circ C_2$ of the curve C_2 and the surface V is the signed number of intersections of C_2 and V, with an intersection being positive if the tangent vector to C_2 at the point of the intersection has the same direction as $\mathbf{N}$, that is,

$$V \circ C_2 = \sum_{x_i \in V} \mathrm{sign}\,(\mathbf{N}, \dot{\mathbf{x}})\Big|_{x_i}\,,$$

  where the sum goes over all intersection points x_i.

- **Problem.** Show that

$$\mathrm{Link}(C_1, C_2) = V \circ C_2\,.$$

Chapter 6

Connection and Curvature

6.1 Affine Connection

6.1.1 Covariant Derivative

- **Definition 6.1.1** *Let M be an n-dimensional manifold. An* **affine connection** *is an operator*

  $$\nabla : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$$

  that assigns to two vector fields $\mathbf{X}$ *and* $\mathbf{Y}$ *a new vector field* $\nabla_{\mathbf{X}}\mathbf{Y}$*, that is linear in both variables, that is, for any* $a, b \in \mathbb{R}$ *and any vector fields* $\mathbf{X}$*,* $\mathbf{Y}$ *and* $\mathbf{Z}$*,*

  $$\nabla_{\mathbf{X}}(a\mathbf{Y} + b\mathbf{Z}) = a\nabla_{\mathbf{X}}\mathbf{Y} + b\nabla_{\mathbf{X}}\mathbf{Z}$$
  $$\nabla_{a\mathbf{X}+b\mathbf{Y}}\mathbf{Z} = a\nabla_{\mathbf{X}}\mathbf{Z} + b\nabla_{\mathbf{Y}}\mathbf{Z}$$

  and satisfies the Leibnitz rule, that is, for any smooth function $f \in C^\infty(M)$ *and any two vector fields* $\mathbf{X}$ *and* $\mathbf{Y}$*,*

  $$\begin{aligned}\nabla_{\mathbf{X}}(f\mathbf{Y}) &= \mathbf{X}(f)\mathbf{Y} + f\nabla_{\mathbf{X}}\mathbf{Y} \\ &= (df)(\mathbf{X})\mathbf{Y} + f\nabla_{\mathbf{X}}\mathbf{Y}\end{aligned}$$

- Let x^μ, $\mu = 1, \dots, n$, be local coordinates and ∂_μ be the basis of vector fields. It will be called a **coordinate frame** for the tangent bundle.

- A basis of vector fields $\mathbf{e}_i = e^\mu_i \partial_\mu$, $i = 1, \dots, n$, for the tangent bundle TM is called a **frame**.

- We will denote partial derivatives by

$$\partial_i f = f_{,i}$$

and the action of frame vector fields on functions by

$$\mathbf{e}_i(f) = e_i^\mu \partial_\mu = f_{|i} .$$

- For any frame the commutator of the frame vector fields defines the **commutation coefficients**

$$[\mathbf{e}_i, \mathbf{e}_j] = C^k{}_{ij}\mathbf{e}_k .$$

- The commutation coefficients are scalar functions, in general.

- **Theorem 6.1.1** *A frame* $\mathbf{e}_i$ *is a coordinate frame if and only if for any* $i, j = 1, \ldots, n$,

$$[\mathbf{e}_i, \mathbf{e}_j] = 0 .$$

Proof :

1. Need to show that there is a local coordinate system x^i such that $\mathbf{e}_i = \partial_i$.
2. Use the dual basis of 1-forms to prove that they are exact.

■

- **Definition 6.1.2** *Let* $\{\mathbf{e}_i\}$ *be a frame of vector fields and* σ^j *be the dual frame of* 1*-forms. The symbols* $\omega^i{}_{jk}$ *defined by*

$$\omega^i{}_{kj} = \sigma^i(\nabla_{\mathbf{e}_j}\mathbf{e}_k)$$

are called the **coefficients of the affine connection**.

- We denote

$$\nabla_i = \nabla_{\mathbf{e}_i} .$$

- Then

$$\nabla_i \mathbf{e}_j = \omega^k{}_{ji}\mathbf{e}_k ,$$

- Then, if $\mathbf{X} = X^i \mathbf{e}_i$ is a vector field, then

$$\nabla_{\mathbf{X}} = X^i \nabla_i .$$

- If $\mathbf{Y} = Y^j \mathbf{e}_j$ is another vector field then

$$\begin{aligned} \nabla_{\mathbf{X}} \mathbf{Y} &= X^i \left\{ \mathbf{e}_i(Y^k) + \omega^k{}_{ji} Y^j \right\} \mathbf{e}_k \\ &= \left\{ \left[dY^k + \omega^k{}_{ji} Y^j \sigma^i \right] (\mathbf{X}) \right\} \mathbf{e}_k . \end{aligned}$$

- That is,

$$\nabla_{\mathbf{X}} \mathbf{Y} = X^i (\nabla_i \mathbf{Y})^k \mathbf{e}_k$$

  where

$$(\nabla_i \mathbf{Y})^k = \mathbf{e}_i(Y^k) + \omega^k{}_{ji} Y^j$$

- We will often write simply $\nabla_i Y^k$ meaning $(\nabla_{\mathbf{e}_i} \mathbf{Y})^k$. This should not be confused with the covariant derivative of the scalar functions Y^k.

- The tensor field $\nabla \mathbf{Y}$ of type $(1, 1)$ with components $\nabla_i Y^k$, that is,

$$\nabla \mathbf{Y} = \sigma^i \otimes \nabla_i \mathbf{Y} = \nabla_i Y^k \sigma^i \otimes \mathbf{e}_k .$$

  is called the **covariant derivative** of the vector field $\mathbf{Y}$.

- The components of the covariant derivative are also denoted by $Y^k{}_{;i}$, in contrast to partial derivatives $\partial_i Y^k$, which are also denoted by $Y^k{}_{,i}$.

- In the coordinate frame $\mathbf{e}_i = \partial_i$ the covariant derivative takes the form

$$\nabla_i Y^k = \partial_i Y^k + \omega^k{}_{ji} Y^j .$$

6.1.2 Curvature, Torsion and Levi-Civita Connection

- **Definition 6.1.3** *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be vector fields on a manifold* M*. Then the vector field*

$$\mathcal{T}(\mathbf{X},\mathbf{Y}) = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X},\mathbf{Y}]$$

defines a tensor field T *of type* $(1,2)$*, called the* **torsion***, so that for any* 1*-form* σ

$$T(\sigma,\mathbf{X},\mathbf{Y}) = \sigma(\mathcal{T}(\mathbf{X},\mathbf{Y}))\,.$$

The affine connection is called **torsion-free** *(or* **symmetric***) if the torsion vanishes, that is, for any* $\mathbf{X}$, $\mathbf{Y}$,

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} = [\mathbf{X},\mathbf{Y}]$$

- The components of the torsion tensor are defined by

$$T^i{}_{jk} = \sigma^i(\mathcal{T}(\mathbf{e}_j,\mathbf{e}_k))\,.$$

- In the coordinate frame the components of the torsion tensor are given by

$$T^i{}_{jk} = \omega^i{}_{kj} - \omega^i{}_{jk}\,.$$

- **Definition 6.1.4** *Let* $\mathbf{X}$, $\mathbf{Y}$ *and* $\mathbf{Z}$ *be vector fields on a manifold* M*. Then the vector field*

$$\mathcal{R}(\mathbf{X},\mathbf{Y})\mathbf{Z} = \left\{[\nabla_{\mathbf{X}},\nabla_{\mathbf{Y}}] - \nabla_{[\mathbf{X},\mathbf{Y}]}\right\}\mathbf{Z}$$

defines a tensor field R *of type* $(1,3)$*, called the* **Riemann curvature***, so that for any* 1*-form* σ

$$R(\sigma,\mathbf{Z},\mathbf{X},\mathbf{Y}) = \sigma(\mathcal{R}(\mathbf{X},\mathbf{Y})\mathbf{Z})\,.$$

The affine connection is called **flat** *if the curvature vanishes, that is, for any* $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$,

$$[\nabla_{\mathbf{X}},\nabla_{\mathbf{Y}}]\mathbf{Z} = \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}\,.$$

- The components of the curvature tensor are defined by

$$R^i{}_{jkl} = \sigma^i(\mathcal{R}(\mathbf{e}_k,\mathbf{e}_l)\mathbf{e}_j)\,.$$

- **Theorem 6.1.2** *The components of the curvature tensor have the form*

$$R^i{}_{jkl} = \omega^i{}_{jl|k} - \omega^i{}_{jk|l} + \omega^i{}_{mk}\omega^m{}_{jl} - \omega^i{}_{ml}\omega^m{}_{jk} - C^m{}_{kl}\omega^i{}_{jm} .$$

Proof:

1.

■

- In the coordinate frame the components of the curvature tensor are given by

$$R^i{}_{jkl} = \partial_k\omega^i{}_{jl} - \partial_l\omega^i{}_{jk} + \omega^i{}_{mk}\omega^m{}_{jl} - \omega^i{}_{ml}\omega^m{}_{jk} .$$

- For a Riemannian manifold (M, g) the metric tensor g has the components

$$g_{ij} = g(\mathbf{e}_i, \mathbf{e}_j) = g_{\mu\nu}e^\mu_i e^\nu_j .$$

- This metric is used to lower and raise the frame indices.

- **Definition 6.1.5** *Let (M, g) be a Riemannian manifold and ∇ be an affine connection on M. Then the connection ∇ is called* **compatible with the metric** *g if for any vector fields $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ it satisfies the condition*

$$\mathbf{Z}(g(\mathbf{X}, \mathbf{Y})) = g(\nabla_{\mathbf{Z}}\mathbf{X}, \mathbf{Y}) + g(\mathbf{X}, \nabla_{\mathbf{Z}}\mathbf{Y}) .$$

An affine connection that is torsion-free and compatible with the metric is called the **Levi-Civita connection**.

- Wed define

$$\omega_{ijk} = g_{im}\omega^m{}_{jk} \qquad C_{ijk} = g_{im}C^m{}_{jk} .$$

- The coefficients of the Levi-Civita connection satisfy the equation

$$g_{ij|k} = \omega_{ijk} + \omega_{jik} .$$

- **Theorem 6.1.3** *Then*

$$\omega_{ijk} = \frac{1}{2}\left(g_{ij|k} + g_{ik|j} - g_{jk|i} + C_{kij} + C_{jik} - C_{ijk}\right) .$$

Proof:

1. Direct calculation.

■

- The coefficients of the Levi-Civita connection in a coordinate frame are called **Christoffel symbols** and denoted by $\Gamma^i{}_{jk}$.

- **Corollary 6.1.1** *The coefficients of the Levi-Civita connection in a coordinate frame (Christoffel symbols) have the form*

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{im} \left(\partial_j g_{mk} + \partial_k g_{jm} - \partial_m g_{jk} \right) .$$

Christoffel symbols have the following symmetry property

$$\Gamma^i{}_{jk} = \Gamma^i{}_{kj} .$$

Proof:

1. Direct calculation.

■

- **Corollary 6.1.2** *The coefficients of the Levi-Civita connection in an orthonormal frame have the form*

$$\omega_{ijk} = \frac{1}{2} \left(C_{kij} + C_{jik} - C_{ijk} \right) .$$

They have the following symmetry properties

$$\omega_{ijk} = -\omega_{jik} .$$

Proof:

1. Direct calculation.

■

- **Theorem 6.1.4** *Each Riemannian manifold has a unique Levi-Civita connection.*

Proof:

1. By construction.

■

6.1.3 Parallel Transport

- Let x_0 and x_1 be two points on a manifold M and C be a smooth curve connecting these points described locally by $x^i = x^i(t)$, where $t \in [0, 1]$ and $x(0) = x_0$ and $x(1) = x_1$. The tangent vector to C is defined by

$$\mathbf{X} = \dot{x}(t),$$

  where the dot denotes the derivative with respect to t.

- Let $\mathbf{Y}$ be a vector field on M. We say that $\mathbf{Y}$ is parallel transported along C if

$$\nabla_{\mathbf{X}}\mathbf{Y} = 0\,.$$

- The vector field $\mathbf{Y}$ is parallel transported along C if its components satisfy the linear ordinary differential equation

$$\frac{d}{dt}Y^i(x(t)) + \omega^i{}_{jk}(x(t))\dot{x}^k(t)Y^j(x(t)) = 0\,.$$

- **Problem.** Solve the equation of parallel transport in terms of a Taylor series up to terms qubic in the connection coefficients.

- A curve C such that the tangent vector to C is trasported parallel along C, that is,

$$\nabla_{\dot{x}}\dot{x} = 0,$$

  is called the **geodesics**.

- The coordinates of the geodesics $x = x(t)$ satisfy the non-linear second-order ordinary differential equation

$$\ddot{x}^i + \omega^i{}_{jk}(x(t))\dot{x}^k\dot{x}^j = 0\,.$$

6.2 Tensor Analysis

6.2.1 Covariant Derivative of Tensors

- We extend the definition of the affine connection from the tangent bundle TM to arbitrary tensor bundles $T^p_q M = \otimes^p TM \otimes^q T^*M$.

- The **affine connection** on a tensor bundle $T^p_q M$ is an operator

$$\nabla : C^\infty(TM) \times C^\infty(T^p_q M) \to C^\infty(T^p_q M)$$

  that assigns to a vector field $\mathbf{X}$ and a tensor field T of type (p, q) a new tensor field $\nabla_\mathbf{X} T$ of type (p, q).

- The **covariant derivative** of a tensor field T of type (p, q) is a linear operator

$$\nabla : C^\infty(T^p_q M) \to C^\infty(T^p_{q+1} M)$$

  that assigns to a tensor field T of type (p, q) a new tensor field ∇T of type $(p, q+1)$.

- First of all, the covariant derivative of a 1-form α on a manifold M is a tensor $\nabla\alpha$ of type $(0, 2)$ such that for any two vector fields $\mathbf{X}$ and $\mathbf{Y}$

$$(\nabla\alpha)(\mathbf{X}, \mathbf{Y}) = (\nabla_\mathbf{X}\alpha)(\mathbf{Y}) = \mathbf{X}[\alpha(\mathbf{Y})] - \alpha(\nabla_\mathbf{X}\mathbf{Y}) .$$

- Then the covariant derivative of a tensor T of type (p, q) is a tensor ∇T of type $(p, q+1)$ such that for any vector fields $\mathbf{X}, \mathbf{Y}_1, \dots, \mathbf{Y}_q$ and 1-forms ω_1, ω_p

$$\begin{aligned}
(\nabla T)(\mathbf{X}, \mathbf{Y}_1, \dots, \mathbf{Y}_q, \omega_1, \dots, \omega_p) &= (\nabla_\mathbf{X} T)(\mathbf{X}, \mathbf{Y}_1, \dots, \mathbf{Y}_q, \omega_1, \dots, \omega_p)\\
&= \mathbf{X}[T(\mathbf{Y}_1, \dots, \mathbf{Y}_q, \omega_1, \dots, \omega_p)]\\
&\quad - \sum_{i=1}^q T(\mathbf{Y}_1, \dots, \nabla_\mathbf{X}\mathbf{Y}_i, \dots \mathbf{Y}_q, \omega_1, \dots, \omega_p)\\
&\quad - \sum_{j=1}^p T(\mathbf{Y}_1, \dots, \mathbf{Y}_q, \omega_1, \dots, \nabla_\mathbf{X}\omega_j, \dots, \omega_p) .
\end{aligned}$$

- Let $\mathbf{e}_i$ be a basis of vector fields and σ^i be the dual basis of 1-forms.

- The covariant derivative of a 1-form has the form

$$(\nabla_i \alpha)_k = \mathbf{e}_i(\alpha_k) - \omega^l{}_{ki}\alpha_l$$

  In the coordinate frame this simplifies to

$$(\nabla_i \alpha)_k = \partial_i \alpha_k - \omega^l{}_{ki}\alpha_l \,.$$

- The covariant derivative of a tensor of type (p, q) in a coordinate basis has the form

$$\nabla_i T^{j_1 \dots j_p}_{k_1 \dots k_q} = \partial_i T^{j_1 \dots j_p}_{k_1 \dots k_q} + \sum_{m=1}^{p} \omega^{j_m}{}_{li} T^{j_1 \dots j_{m-1} l j_{m+1} \dots j_p}_{k_1 \dots k_q} - \sum_{n=1}^{q} \omega^l{}_{k_n i} T^{j_1 \dots j_p}_{k_1 \dots k_{n-1} l k_n k_q}$$

- The parallel transport of tensor fields is defined similarly to vector fields. Let C be a smooth curve on a manifold M described locally by $x^i = x^i(t)$, where $t \in [0, 1]$, with the tangent vector $\mathbf{X} = \dot{x}(t)$.

- Let T be a tensor field on M. We say that T is **parallel transported** along C if

$$\nabla_{\mathbf{X}} T = 0 \,.$$

6.2.2 Ricci Identities

- The commutators of covariant derivatives of tensors are expressed in terms of the curvature and the torsion.

- In a coordinate basis for a torsion-free connection we have the following identities (called the Ricci identities):

$$[\nabla_i, \nabla_j] Y^k = R^k{}_{lij} Y^l$$

$$[\nabla_i, \nabla_j] \alpha_k = -R^l{}_{kij} \alpha_l$$

$$[\nabla_i, \nabla_j] T^{j_1 \dots j_p}_{k_1 \dots k_q} = \sum_{m=1}^{p} R^{j_m}{}_{lij} T^{j_1 \dots j_{m-1} l j_{m+1} \dots j_p}_{k_1 \dots k_q} - \sum_{n=1}^{q} R^l{}_{k_n ij} T^{j_1 \dots j_p}_{k_1 \dots k_{n-1} l k_n k_q}$$

6.2.3 Normal Coordinates

- Let (M, g) be a Riemannian manifold and $\Gamma^i{}_{jk}$ be the Christoffel coefficients defining the Levi-Civita connection in a coordinate basis.

- Let x_0 be a point in M and x^i be a local coordinate system in a coordinate patch about x_0.

- We expand the metric in a Taylor series at the point x_0

$$g_{ij}(x) = g_{ij}(x_0)+[\partial_k g_{ij}](x_0)(x^k-x_0^k)+\frac{1}{2}[\partial_k\partial_l g_{ij}](x_0)(x^k-x_0^k)(x^l-x_0^l)+O((x-x_0)^3)$$

- The matrix $g_{ij}(x_0)$ is a constant real symmetric matrix with real eigenvalues.

- **Theorem 6.2.1** *There exists a coordinate system such that*

$$g_{ij}(x_0) = \delta_{ij}\,, \qquad [\partial_k g_{ij}](x_0) = 0\,,$$

so that the Taylor series has the form

$$g_{ij}(x) = \delta_{ij} + \frac{1}{2}[\partial_k\partial_l g_{ij}](x_0)(x^k - x_0^k)(x^l - x_0^l) + O((x - x_0)^3)\,.$$

Such coordinates are called **Riemann normal coordinates**.

Proof:

1.

■

- **Corollary 6.2.1** *In Riemann normal coordinates the Christoffel symbols vanish at x_0*

$$\Gamma^i{}_{jk}(x_0) = 0$$

and the curvature of the Levi-Civita connection at x_0 is expressed in terms of second derivatives of the metric at x_0,

$$R_{ijkl}(x_0) = \frac{1}{2}\left\{\partial_k\partial_j g_{il} - \partial_l\partial_j g_{ik} - \partial_i\partial_l g_{kj} + \partial_k\partial_i g_{jl}\right\}\Big|_{x_0}\,,$$

so that the Taylor series of the metric at x_0 has the form

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}R_{ikjl}(x_0)(x^k - x_0^k)(x^l - x_0^l) + O((x - x_0)^3)\,.$$

Proof:

1.

■

6.2.4 Properties of the Curvature Tensor

- Let (M, g) be an n-dimensional Riemannian manifold. We will restrict ourselves to the Levi-Civita connection below. We define some new curvature tensors.

- The **Ricci tensor**

$$R_{ij} = R^k{}_{ikj} .$$

- The **scalar curvature**

$$R = g^{ij} R_{ij} = g^{ij} R^k{}_{ikj} .$$

- The **Einstein tensor**

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R .$$

- The **trace-free Ricci tensor**

$$E_{ij} = R_{ij} - \frac{1}{n} g_{ij} R .$$

- The **Weyl tensor** (for $n > 2$)

$$\begin{aligned} C^{ij}{}_{kl} &= R^{ij}{}_{kl} - \frac{4}{n-2} R^{[i}{}_{[k} \delta^{j]}{}_{l]} + \frac{2}{(n-1)(n-2)} R \delta^{[i}{}_{[k} \delta^{j]}{}_{l]} \\ &= R^{ij}{}_{kl} - \frac{4}{n-2} E^{[i}{}_{[k} \delta^{j]}{}_{l]} - \frac{2}{n(n-1)} R \delta^{[i}{}_{[k} \delta^{j]}{}_{l]} \end{aligned}$$

•

Theorem 6.2.2 *The Riemann curvature tensor of the Levi-Civita connection has the following symmetry properties*

1. $R_{ijkl} = -R_{ijlk}$
2. $R_{ijkl} = -R_{jikl}$
3. $R_{ijkl} = R_{klij}$
4. $R^i{}_{[jkl]} = R^i{}_{jkl} + R^i{}_{klj} + R^i{}_{ljk} = 0$
5. $R_{ij} = R_{ji}$

***Proof*:**

1.

■

•

Theorem 6.2.3 *The Weyl tensor has the same symmetry properties as the Riemann tensor and all its contractions vanish, that is,*

$$C^i{}_{jik} = 0\,.$$

***Proof*:**

1.

■

•

Theorem 6.2.4 *The number of algebraically independent components of the Riemann tensor of the Levi-Civita connection is equal to*

$$\frac{n^2(n^2-1)}{12}\,.$$

***Proof*:**

1.

■

- **Corollary 6.2.2** *In dimension $n = 2$ the Riemann tensor has only one independent component determined by the scalar curvature*

$$R^{12}{}_{12} = \frac{1}{2}R\,.$$

The trace-free Ricci tensor E_{ij} vanishes, that is

$$R^{ij}{}_{kl} = R\delta^{[i}{}_{[k}\delta^{j]}{}_{l]}$$

$$R_{ij} = \frac{1}{2}Rg_{ij}\,.$$

Proof:

1.

■

- **Corollary 6.2.3** *In dimension $n = 3$ the Riemann tensor has six independent components determined by the Ricci tensor R_{ij}. The Weyl tensor C_{ijkl} vanishes, that is,*

$$R^{ij}{}_{kl} = 4R^{[i}{}_{[k}\delta^{j]}{}_{l]} + R\delta^{[i}{}_{[k}\delta^{j]}{}_{l]}\,.$$

Proof:

1.

■

6.2.5 Bianci Identities

- Let (M, g) be an n-dimensional Riemannian manifold. We will restrict ourselves to the Levi-Civita connection below.

- **Theorem 6.2.5** *The Riemann tensor satisfies the following identities*

$$\nabla_{[m}R^{ij}{}_{kl]} = \nabla_m R^{ij}{}_{kl} + \nabla_k R^{ij}{}_{lm} + \nabla_l R^{ij}{}_{mk} = 0\,.$$

These identities are called the **Bianci identities**.

Proof:

1.

■

- **Corollary 6.2.4** *The divergences of the Riemann tensor and the Ricci tensor have the form*

$$\nabla_i R^{ij}{}_{kl} = \nabla_k R^j_l - \nabla_l R^j_k \,,$$

$$\nabla_i R^i_j = \frac{1}{2}\nabla_j R \,.$$

The divergence of the Einstein tensor vanishes

$$\nabla_i G^i_j = 0 \,.$$

Proof:

1.

■

- **Problem.** By using the Bianci identities simplify the Laplacian of the Riemann tensor, $\Delta R^{ij}{}_{kl} = \nabla^m \nabla_m R^{ij}{}_{kl}$.

6.3 Cartan's Structural Equations

- Let ∂_μ be a coordinate basis for the tangent bundle TM and $\mathbf{e}_i = e^\mu_i \partial_\mu$ be an orthonormal frame of vector fields.

- Then
$$g_{ij} = g(\mathbf{e}_i, \mathbf{e}_k) = g_{\mu\nu} e^\mu_i e^\nu_j = \delta_{ij} .$$

- We use this metric to lower and raise the frame indices.

- Let dx^μ be a coordinate basis for the cotangent bundle TM and $\sigma^i = \sigma^i_\mu dx^\mu$ be an orthonormal frame of 1-forms dual to $\mathbf{e}_i$.

- Then
$$\sigma^i(\mathbf{e}_j) = \sigma^i_\mu e^\mu_j = \delta^i_j$$
and
$$g^{\mu\nu}\sigma^i_\mu \sigma^j_\nu = \delta^{ij}$$
$$\sigma^i_\mu e^\nu_i = \delta^\nu_\mu .$$

- The commutators of the frame vector fields define the **commutation coefficients** $C^i{}_{jk}$ by
$$[\mathbf{e}_i, \mathbf{e}_j] = C^k{}_{ij}\mathbf{e}_k ,$$
or, in components,
$$e^\nu_{j|i} - e^\nu_{i|j} = e^\mu_i \partial_\mu e^\nu_j - e^\mu_j \partial_\mu e^\nu_i = C^k{}_{ij} e^\nu_k .$$
That is,
$$C^k{}_{ij} = \sigma^k([\mathbf{e}_i, \mathbf{e}_j])$$
or
$$C^k{}_{ij} = \sigma^k_\nu \left(e^\mu_i \partial_\mu e^\nu_j - e^\mu_j \partial_\mu e^\nu_i \right) .$$

- **Proposition 6.3.1** *There holds*
$$d\sigma^i = -\frac{1}{2} C^i{}_{jk} \sigma^j \wedge \sigma^k .$$
or
$$(d\sigma^i)(\mathbf{e}_j, \mathbf{e}_k) = -\frac{1}{2} C^i{}_{jk} .$$

***Proof*:**

1. Direct calculation using the duality condition.

∎

- Let $\omega^i{}_{jk}$ be the coefficients of the Levi-Civita connection in the orthonormal frame.

- Then
$$\nabla_i \mathbf{e}_j = \omega^k{}_{ji} \mathbf{e}_k$$
and
$$\nabla_i \sigma^j = -\omega^j{}_{ki} \sigma^k ,$$
where $\nabla_i = \nabla_{\mathbf{e}_i}$.

- This can also be written as
$$\omega^k{}_{ji} = \sigma^k_\mu e^\nu_i e^\mu_{j;\nu} = -\sigma^k_{\mu;\nu} e^\nu_i e^\mu_j .$$

- Thus
$$\begin{aligned}
\omega^k_{ji} - \omega^k{}_{ij} &= \left(\sigma^k_{\nu;\mu} - \sigma^k_{\mu;\nu}\right) e^\nu_i e^\mu_j \\
&= \left(\sigma^k_{\nu,\mu} - \sigma^k_{\mu,\nu}\right) e^\nu_i e^\mu_j \\
&= (d\sigma^k)(\mathbf{e}_j, \mathbf{e}_i) .
\end{aligned}$$

- **Proposition 6.3.2** *There holds*
$$d\sigma^i = \omega^i{}_{jk} \sigma^j \wedge \sigma^k .$$

***Proof*:**

1. Follows from above.

∎

- Since the torsion of the Levi-Civita connection is zero, we have
$$\nabla_i \mathbf{e}_j - \nabla_j \mathbf{e}_i = [\mathbf{e}_i, \mathbf{e}_j] .$$

- Therefore,
$$\omega^k{}_{ji} - \omega^k{}_{ij} = C^k{}_{ij} .$$

- Since the Levi-Civita connection is compatible with the metric, we have

$$0 = \nabla_i g(\mathbf{e}_j, \mathbf{e}_k) = g(\nabla_i \mathbf{e}_j, \mathbf{e}_k) + g(\mathbf{e}_j, \nabla_i \mathbf{e}_k)\,.$$

- Thus,

$$\omega_{kji} + \omega_{jki} = 0\,.$$

- Finally, we obtain,

- **Proposition 6.3.3** *The coefficients of the Levi-Civita connection in an orthonormal frame are given in terms of the commutation coefficients by*

$$\omega_{ijk} = \frac{1}{2}\left(C_{kij} + C_{jik} - C_{ijk}\right)\,.$$

Proof:

1. Use the equations

$$\omega_{kij} + \omega_{ikj} = 0$$
$$\omega_{ijk} + \omega_{jik} = 0$$
$$\omega_{jki} + \omega_{kji} = 0$$

and

$$\omega^k{}_{ji} - \omega^k{}_{ij} = C^k{}_{ij}\,.$$

■

- Now we define the **connection** 1**-forms**

$$\mathcal{A}^i{}_j = \omega^i{}_{jk}\sigma^k$$

and the **curvature** 2**-forms**

$$\mathcal{F}^i{}_j = \frac{1}{2}R^i{}_{jkl}\sigma^k \wedge \sigma^l\,.$$

- Then the equation

$$d\sigma^i = \omega^i{}_{jk}\sigma^j \wedge \sigma^k$$

can be written as

$$d\sigma^i + \mathcal{A}^i{}_j \wedge \sigma^j = 0\,.$$

This is called **Cartan's first structural equation**.

- The curvature 2-forms are obtained from the connection 1-forms by **Cartan's second structural equation**

$$\mathcal{F}^i{}_j = d\mathcal{A}^i{}_j + \mathcal{A}^i{}_k \wedge \mathcal{A}^k{}_j\,.$$

This equation is equivalent to the expression for the curvature components and can be obtained from that.

- **Cartan's third structural equation**

$$d\mathcal{F}^i{}_j + \mathcal{A}^i{}_k \wedge \mathcal{F}^k{}_j - \mathcal{F}^i{}_k \wedge \mathcal{A}^k{}_j = 0\,.$$

is equivalent to Bianci identities.

- Cartan structural equations can be written in a very compact way by introducing the **covariant exterior derivative** acting on vector valued and matrix valued forms.

- Let α be a p-form valued in a vector space V (in our case $V = \mathbb{R}^n$). Such a form is called a **twisted form**. Let α^i be the components of this form in a fixed basis in V. That is, α^i is a p-form for each $i = 1, \ldots, n$. Then the covariant exterior derivative

$$\mathcal{D} : \Lambda_p \otimes V \to \Lambda_{p+1} \otimes V$$

is defined by

$$(\mathcal{D}\alpha)^i = d\alpha^i + \mathcal{A}^i{}_j \wedge \alpha^j$$

or, in matrix form,

$$\mathcal{D}\alpha = d\alpha + \mathcal{A} \wedge \alpha$$

with obvious notation.

- Let V^* be the dual vector space to V (in our case it is again $\mathbb{R}^n$). We consider p-forms valued in V^* (covectors) and naturally extend the operator $\mathcal{D}$ to such forms

$$\mathcal{D} : \Lambda_p \otimes V^* \to \Lambda_{p+1} \otimes V^*$$

by

$$(\mathcal{D}\alpha)_i = d\alpha_i - (-1)^p \alpha_j \wedge \mathcal{A}^j{}_i$$

or, in matrix form,

$$\mathcal{D}\alpha = d\alpha - (-1)^p \alpha \wedge \mathcal{A}\,.$$

- Finally, we consider matrix-valued p-forms valued in $V \otimes V^*$ and extend the operator $\mathcal{D}$ to such forms

$$\mathcal{D} : \Lambda_p \otimes V \otimes V^* \to \Lambda_{p+1} \otimes V \otimes V^*$$

by

$$(\mathcal{D}\alpha)^i{}_j = d\alpha^i{}_j + \mathcal{A}^i{}_k \wedge \alpha^k_j - (-1)^p \alpha^i_k \wedge \mathcal{A}^k{}_j$$

or, in matrix form,

$$\mathcal{D}\alpha = d\alpha + \mathcal{A} \wedge \alpha - (-1)^p \alpha \wedge \mathcal{A} .$$

- Now, let $\sigma = (\sigma^i)$, $\mathcal{F} = (\mathcal{F}^i{}_j)$ and $\alpha = (\alpha^i)$ be an arbitrary vector-valued 1-form. Then

$$\begin{aligned} \mathcal{D}\sigma &= 0 \\ \mathcal{D}^2\alpha &= \mathcal{F} \wedge \alpha \\ \mathcal{D}\mathcal{F} &= 0 . \end{aligned}$$

- **Problem**. Let the dimension $n = 2k$ of the manifold M be even. We define the following $2l$-forms

$$\Omega_{(l)} = \operatorname{tr} \underbrace{\mathcal{F} \wedge \cdots \wedge \mathcal{F}}_{l}$$

and the n-form

$$\Omega = \varepsilon^{i_1 \ldots i_{2k}} \mathcal{F}_{i_1 i_2} \wedge \cdots \wedge \mathcal{F}_{i_{2k-1} i_{2k}}$$

1. Prove that these forms are independent of the orthonormal basis and are closed, that is,

$$d\Omega_{(l)} = d\Omega = 0 .$$

2. Find the expressions in local coordinates for these forms.

- These forms define so called **characteristic classes**, which are closed invariant forms whose integrals over the manifold do not depend on the metric and, therefore, are topological invariants of the manifold.

Chapter 7

Homology Theory

7.1 Algebraic Preliminaries

7.1.1 Groups

- A **group** is a set G with a binary operation, $\cdot$, called the group multiplication, that is,
  1. associative,
  2. has an identity element,
  3. every element has an inverse.
- A group is **Abelian** if the group operation is commutative.
- For an Abelian group the group operation is called **addition** and is denoted by $+$. The identity element is called **zero** and denoted by 0. The inverse element of an element $g \in G$ is denoted by $(-g)$.
- Let G and E be Abelian groups. A map $F : G \to E$ is called a **homomorphism** if for any $g, g' \in G$,

$$F(g +_G g') = F(g) +_E F(g'),$$

  where $+_G$ and $+_E$ are the group operations in G and E respectively.
- In particular,

$$F(0_G) = 0_E, \qquad \text{and} \qquad F(-g) = -F(g).$$

- Let $F : G \to E$ be a homomorphism of an Abelian group G into an Abelian group E. The set of elements of G mapped to the identity element of E is denoted by
$$\operatorname{Ker} F = \{g \in G \mid F(g) = 0_E\}\,,$$
where 0_E is the identity element of E, and called the **kernel** of the homomorphism F.

- The **image** of the homomorphism $F : G \to E$ is the set
$$\operatorname{Im} F = \{h \in E \mid h = F(g) \text{ for some } g \in G\}\,.$$

- A homomorphism $F : G \to E$ of a group G into a group E is called an **isomorphism** if it is a bijection.

- A subset H of a group G is called a **subgroup** of G if it contains the identity element and is closed under the group operation.

- For any homomorphism $F : G \to E$ the kernel $\operatorname{Ker} F$ is a subgroup of G.

- Let G be an Abelian group and H be a subgroup of G. We say that two elements of G are **equivalent** if they differ by an element of H.

- Let $g \in G$ be an element of G. Then the set of all elements of G equivalent to g, denoted by $[g] = g + H$, is an **equivalence class** of g called a **coset**.

- The set of cosets is denoted by G/H.

- An element g used to describe a coset $[g]$ is called a **representative**.

- The set of cosets G/H is an Abelian group, called the **quotient group**, with the addition defined by
$$[g] + [g'] = [g + g']\,.$$

- The projection map $\pi : G \to G/H$ defined by
$$\pi(g) = [g]$$
is a homomorphism.

- **Theorem 7.1.1 Fundamental Theorem of Homomorphisms.** *Let G and F be groups. Let $F : G \to F$ be a homomorphism. Then*

$$G/\mathrm{Ker}\, F \cong \mathrm{Im}\, F\,.$$

Proof:

1. Since

$$F(g + \mathrm{Ker}\, F) = F(g)\,.$$

■

- **Theorem 7.1.2** *Let G and E be Abelian groups, and $H \subset G$ and $N \subset E$ be their subgroups so that G/H and E/N are the quotient groups. Let $F : G \to E$ be a homomorphism such that the image of the subgroup H of G is the subgroup N of E, that is,*

$$F(H) = N\,.$$

Then the homomorphism F induces a homomorphism of the quotient groups

$$F_* : G/H \to E/N\,.$$

Proof:

1. Let

$$\pi : E \to E/N$$

be the projection homomorphism defined by, for any $x \in E$

$$\pi(x) = [x] = x + N\,.$$

2. Then

$$F_* = \pi \circ F : G/H \to E/N$$

is a homomorphism.

■

- A **field** is a set K with two binary operations, addition, $+$, and multiplication, $\cdot$, that satisfy the following conditions:

1. both addition and multiplication are associative,
2. both addition and multiplication are commutative,

3. both addition and multiplication have identity elements, the additive identity 0 and the multiplicative identity 1, such that $0 \neq 1$,
4. the multiplication is distributive with respect to addition,
5. every element has an additive inverse,
6. every nonzero element has a multiplicative inverse.

- In particular, any field is an Abelian group with respect to addition.

- A **vector space** over a field K consists of a set E, whose elements are called **vectors**, and the field K, whose elements are called **scalars**, with two operations: **vector addition**, $+ : E \times E \to E$, and **multiplication by scalars**, $\cdot : K \times E \to E$, that satisfy the following conditions:

  1. the vector addition is associative and commutative,
  2. there is an additive identity, called the **zero vector**,
  3. every vector has an additive inverse, called the **opposite vector**,
  4. for any $\mathbf{v} \in E$, $a, b \in K$,
     (a) $a(b\mathbf{v}) = (ab)\mathbf{v}$,
     (b) $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$,
     (c) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$,
     (d) $1\,\mathbf{v} = \mathbf{v}$.

- Let E be a vector space and F be a vector subspace of E. Then E/F is a vector space.

- Let E be an inner product vector space and F be its subspace. Then

$$E/F = F^{\perp}$$

is the orthogonal complement of F in E, and

$$\pi : E \to F^{\perp}$$

is the orthogonal projection to $F^{\perp}$.

7.1.2 Finitely Generated and Free Abelian Groups

- Let G be an Abelian group. Let $g_1, \dots, g_r \in G$ be some elements of G and

$$H = \left\{ \sum_{k=1}^{r} n_k g_k \mid g_k \in G, n_k \in \mathbb{Z} \right\}$$

  be a set of linear combinations of g_k.

- Then H is a subgroup of G. The elements g_k are called the **generators** of H and H is said to be **generated** by g_k.

- If a group G is generated by finitely many elements of G, then G is called a **finitely generated group.**

- The elements $g_1, \dots, g_r$ are **linearly independent** if for any integer coefficients $n_1, \dots, n_r$ the linear combination

$$\sum_{k=1}^{r} n_k g_k \neq 0$$

  is not equal to zero.

- A finitely generated group G is called a **free Abelian group of rank** r if it is generated by r linearly independent elements.

7.1.3 Cyclic Groups

- An Abelian group generated by one element is called a **cyclic group**.

- Infinite cyclic groups.

- Finite cyclic group.

- **Theorem 7.1.3 Fundamental Theorem of Finitely Generated Abelian Groups.** *Let G be a finitely generated Abelian group with m generators. Then G is isomorphic to the direct sum of cyclic groups,*

$$G \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{r} \oplus \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_p} ,$$

  where $m = r + p$. The number r is called the **rank** *of G.*

  Proof:

1.

■

7.2 Singular Chains

- The **standard Euclidean p-simplex** in $\mathbb{R}^p$ is the convex set $\Delta_p \subset \mathbb{R}^p$ generated by an ordered $(p+1)$-tuple $(P_0, P_1, \ldots, P_p)$ of points in $\mathbb{R}^p$

  $$P_0 = (0, \ldots, 0), \qquad P_i = (0, \ldots, 0, 1, 0, \ldots, 0), \qquad i = 1, \ldots, p\,,$$

  all of whose components are 0 except for the i-th component, which is equal to 1.

- We use the following notation

  $$\Delta_p = (P_0, P_1, \ldots, P_p)\,.$$

- Let M be an n-dimensional manifold. A **singular p-simplex** in M is a differentiable map

  $$\sigma_p : \Delta_p \to M\,.$$

- By slightly abusing notation we denote the image $\sigma_p(\Delta_p)$ of Δ_p in M under the map σ_p just by σ_p.

- Let α be a p-form on M and σ_p be a p-simplex in M. We define the integral of α over σ_p by

  $$\int_{\sigma_p} \alpha = \int_{\Delta_p} \sigma_p^* \alpha\,.$$

- The **k-th face** of a standard p-simplex $\Delta_p = (P_0, P_1, \ldots, P_p)$ (a face opposite to to the vertex P_k) is the convex set in $\mathbb{R}^p$ generated by an ordered p-tuple of points in $\mathbb{R}^p$

  $$\Delta^{(k)}_{p-1} = (P_0, \ldots, \hat{P}_k, \ldots, P_p)$$

  where the k-th point P_k is omitted.

- Since this points lie in $\mathbb{R}^p$ and $\mathbb{R}^{p-1}$ a face is not a standard $(p-1)$-simplex. It can be rather described as a singular simplex in $\mathbb{R}^p$ defined by the unique affine map

  $$f_k : \Delta_{p-1} \to \Delta_p$$

  whose image in $\mathbb{R}^p$ is exactly $\Delta^{(k)}_p$.

- Such a map is uniquely defined as follows. Let $\Delta_{p-1} = (Q_0, \dots, Q_{p-1})$, where $Q_i \in \mathbb{R}^{p-1}$, be the standard $(p-1)$-simplex in $\mathbb{R}^{p-1}$ and $\Delta_p = (P_0, \dots, P_p)$, where $P_i \in \mathbb{R}^p$, be the standard p-simplex in $\mathbb{R}^p$. Then
$$f_k(Q_i) = P_i \quad \text{for} \quad i = 0, \dots, k-1$$
and
$$f_k(Q_i) = P_{i+1} \quad \text{for} \quad i = k, \dots, p-1 \,.$$
- If Q is a point in $\mathbb{R}^{p-1}$ with coordinates $(x^i) = (x^1, \dots, x^{p-1})$, then $P = f_k(Q)$ is the point in $\mathbb{R}^p$ with coordinates $(y^\mu) = (y^1, \dots, y^p)$, where
$$y^\mu = A^\mu_{(k)j} x^j + b^\mu_{(k)} \,.$$
The above requirements fix the matrix $A_{(k)}$ and the vector $b_{(k)}$ uniquely.
- **Problem**. Find the maps f_k.
- Let M and V be manifolds and $\varphi : V \to M$ be a differentiable map. Let $\sigma_p : \Delta_p \to V$ be a singular p-simplex in V.
- Then the composition map
$$\varphi \circ \sigma_p : \Delta_p \to M$$
defines a singular p-simplex in M.
- Thus, the composition
$$\sigma_p \circ f_k : \Delta_{p-1} \to M$$
defines a singular $(p-1)$-simplex in M, which is the k-th face of the singular p-simplex σ_p in M.
- The **boundary** $\partial\Delta_p$ of the standard p-simplex Δ_p is defined as follows. For $p > 0$, we define
$$\partial(P_0, P_1, \dots, P_p) = \sum_{k=0}^{p} (-1)^k (P_0, \dots, \hat{P}_k, \dots, P_p) \,,$$
that is, the boundary is the formal sum
$$\partial\Delta_p = \sum_{k=0}^{p} (-1)^k \Delta^{(k)}_{p-1} \,.$$
For $p = 0$, we define
$$\partial\Delta_0 = 0 \,.$$

- **Examples.**
- The boundary of a standard simplex is not a simplex, but an **integer** $(p-1)$**-chain**.
- Let G be an Abelian group. A **singular** p**-chain on** M **with coefficients in** G is a finite formal sum
$$c_p = \sum_{k=1}^{r} g_k \sigma_p^k$$
of singular simplexes $\sigma_p^k : \Delta_p \to M$ with coefficients g_k which are elements of the group G.
- **Examples.**
- Let $S_p(M)$ be the set of all singular p-simplexes in M. Then, a p-chain is a function
$$c_p : S_p(M) \to G\,,$$
such that *its value is not equal to zero only for finitely many simplexes*. These simplexes are exactly σ_p^k listed in the formal sum, and the values of the function c_p are exactly the coefficients of the formal sum, that is
$$c_p(\sigma_p^k) = g_k\,.$$
- Alternatively, a p-chain can be thought of as a finite subset of $S_p(M) \times G$, that is, a finite set of ordered pairs
$$c_p = \{(\sigma_p^k, g_k)\}_{k=1}^r$$
- The notation of a p-chain as a sum, or as a function, is useful since we can define the addition of p-chains simply as addition of corresponding functions. If two p-chains c_p and c'_p have the same p-simplex σ_p in both of them, then in the sum $c_p + c'_p$ we add the corresponding group elements g and g'. That is, if
$$c_p = g\sigma_p + \dots, \qquad \text{and} \qquad c'_p = g'\sigma_p + \dots,$$
then
$$c_p + c'_p = (g + g')\sigma_p + \dots.$$

- We denote the identity element of G by 0 and define the **zero p-chain** simply by
$$0_p = 0\,.$$
- Then the set of all singular p-chains on M with coefficients in G forms an Abelian group, called the **singular p-chain group** of M with coefficients in G and denoted by $C_p(M;G)$.
- A chain with integer coefficients, when $G = \mathbb{Z}$, is called an **integer chain**.
- The standard p-simplex Δ_p is an integer p-chain in $\mathbb{R}^p$ with only one term: $c_p = 1 \cdot \Delta_p$. That is, it is an element of $C_p(\mathbb{R}^p;\mathbb{Z})$.
- The boundary of the standard p-simplex in $\mathbb{R}^p$,
$$\partial\Delta_p = \sum_{k=0}^{p}(-1)^k\Delta^{(k)}_{p-1}$$
is an integer $(p-1)$-chain in $\mathbb{R}^p$, that is, an element of $C_{p-1}(\mathbb{R}^p;\mathbb{Z})$.
- Let M and V be closed manifolds. Let $F : M \to V$ be a map of M into V and $\sigma_p : \Delta_p \to M$ be a singular p-simplex in M. Then the composition $F \circ \sigma_p : \Delta_p \to V$ is a singular p-simplex in V. We denote it by
$$F_*\sigma_p = F \circ \sigma_p\,.$$
- The **induced chain homomorphism**
$$F_* : C_p(M;G) \to C_p(V;G)$$
is defined by: for any $g_k \in G$ and $\sigma^k_p \in S_p(M)$,
$$F_*\left(\sum_{k=1}^{r} g_k\sigma^k_p\right) = \sum_{k=1}^{r} g_k F_*\sigma^k_p$$
- Let $F : M \to V$ and $E : V \to W$ be two maps of manifolds and $F_* : C_p(M;G) \to C_p(V;G)$ and $E_* : C_p(V;G) \to C_p(W;G)$ be the corresponding induced chain homomorphisms. Then
$$(E \circ F)_* = E_* \circ F_*\,.$$

- Let $\sigma_p : \Delta_p \to M$ be a singular p-simplex in M. Then its boundary $\partial\sigma_p$ is the integer $(p-1)$-chain in M defined by

$$\partial\sigma_p = \sigma_{p*}(\partial\Delta_p)\,.$$

- In more detail

$$\begin{aligned}\partial\sigma_p &= \sigma_{p*}(\partial\Delta_p)\\ &= \sum_{k=0}^{p}(-1)^k\sigma_{p*}(\Delta^{(k)}_{p-1}) = \sum_{k=0}^{p}(-1)^k(\sigma_p\circ f_k)\,.\end{aligned}$$

  Recall that $\sigma_{p*}(\Delta^{(k)}_{p-1}) = (\sigma_p\circ f_k)$ is the k-th face of the singular p-simplex σ_p.

- That is, *the boundary of the image of* Δ_p *is the image of the boundary of* Δ_p.

- The boundary of any singular p-chain with coefficients in G is defined by, for any $g_k\in G$, $\sigma^k_p\in S_p(M)$,

$$\partial\left(\sum_{k=1}^{r} g_k\sigma^k_p\right) = \sum_{k=1}^{r} g_k\partial\sigma^k_p\,.$$

- This leads to the **boundary homomorphism**

$$\partial : C_p(M;G)\to C_{p-1}(M;G)\,.$$

- Let $F : M\to V$ be a map, σ_p be a singular p-simplex in M and $F_*\sigma_p$ be the induced singular p-simplex in V. Then

$$\begin{aligned}\partial(F_*\sigma_p) &= \partial(F\circ\sigma_p)\\ &= (F\circ\sigma_p)_*(\partial\Delta_p)\\ &= (F_*\circ\sigma_{p*})(\partial\Delta_p)\\ &= F_*[\sigma_{p*}(\partial\Delta_p)]\\ &= F_*(\partial\sigma_p)\end{aligned}$$

- More generally, let

$$c_p = \sum_{k=1}^{r} g_k\sigma^k_p$$

  be a p-chain on M and F_*c_p be the induced p-chain on V.

- Then
$$\partial(F_* c_p) = F_*(\partial c_p)\,.$$
- Therefore,
$$\partial \circ F_* = F_* \circ \partial\,,$$
in other words, *the boundary of an image is the image of the boundary*.
- Thus, we obtain a **commutative diagram**
$$\begin{array}{ccc} & F_* & \\ C_p(M;G) & \to & C_p(V;G) \\ \partial\downarrow & & \partial\downarrow \\ C_{p-1}(M;G) & \to & C_{p-1}(V;G) \\ & F_* & \end{array}$$
which is a fancy way to say that for any p-chain $c_p \in C_p(M;G)$ on M we have
$$F_*(\partial c_p) = \partial(F_* c_p)\,.$$
- **Theorem 7.2.1** *The boundary of a boundary is zero, that is,*
$$\partial^2 = 0\,.$$

Proof:

1. For a standrad p-simplex Δ_p we have
$$\begin{aligned} \partial\partial\Delta_p &= \sum_{k=0}^{p}(-1)^k \partial\Delta_p^{(k)} \\ &= \sum_{k=0}^{p}(-1)^k \partial(P_0,\dots,\hat{P}_k,\dots,P_p) \\ &= \sum_{k=0}^{p}(-1)^k \sum_{j=0}^{k-1} \partial(P_0,\dots,\hat{P}_j,\dots,\hat{P}_k,\dots,P_p) \\ &+ \sum_{k=0}^{p}(-1)^k \sum_{j=k+1}^{p} \partial(P_0,\dots,\hat{P}_k,\dots,\hat{P}_j,\dots,P_p) \\ &= 0 \end{aligned}$$
because of the pairwise cancellation.

2. Then, for a singular p-simplex σ_p

$$\partial\partial\sigma_p = \partial[\sigma_{p*}(\partial\Delta_p)] = \sigma_{p*}\partial(\partial\Delta_p) = \sigma_*(0) = 0$$

∎

7.2.1 Examples

- **Cylinder**.
- **Möbius Band**.

7.3 Singular Homology Groups

7.3.1 Cycles, Boundaries and Homology Groups

- We can define the singular p-chains with coefficients in a field K.

- Furthermore, we can define the multiplication of p-chains by elements of the field K, called the **scalars** by, for any $a, b_i \in K$,

$$a\left(\sum_{i=1}^{r} b_i\sigma^i_p\right) = \sum_{i=1}^{r} ab_i\sigma^i_p \,.$$

- The chain groups $C_p(M, K)$ with coefficients in a field K become infinite-dimensional vector spaces.

- In this case the boundary homomorphism becomes a linear transformation (operator) in a vector space.

- Let M be a manifold and G an Abelian group. A singular p-chain z_p in M whose boundary is 0 is called a singular p**-cycle**.

- The set of all p-cycles in M

$$Z_p(M; G) = \{z_p \in C_p(M; G) \mid \partial z_p = 0\}$$

  is a subgroup of the chain group $C_p(M; G)$ called the p**-cycle group**.

- Obviously the p-cycle group is the kernel of the boundary homomorphism

$$Z_p(M; G) = \mathrm{Ker}\, \partial_p \,,$$

  where

$$\partial_p : C_p(M; G) \to C_{p-1}(M; G) \,.$$

- In the case, when $G = K$ is a field, then $Z_p(M; K)$ is a vector subspace of the vector space $C_p(M; K)$.

- A singular p-chain b_p in M that is the boundary of a singular $(p + 1)$-chain is called a p**-boundary**.

- The set of all p-boundaries in M

$$B_p(M;G) = \{b_p \in C_p(M;G) \mid b_p = \partial c_{p+1} \text{ for some } c_{p+1} \in C_{p+1}(M;G)\}$$

  is a subgroup of the chain group $C_p(M;G)$ called the p-**boundary group**.

- Obviously the p-boundary group is the image of the boundary homomorphism

$$B_p(M;G) = \operatorname{Im} \partial_{p+1} \,.$$

- Since every p-boundary is a p-cycle (because of $\partial^2 = 0$) the group $B_p(M;G)$ is a subgroup of $Z_p(M;G)$.

- In the case, when $G = K$ is a field, then $B_p(M;K)$ is a vector subspace of the vector space $Z_p(M;K)$.

- Let M be a manifold and G be an Abelain group. We say that two p-cycles are **homologous** if they differ by a boundary.

- The set of equivalence classes of p-cycles homologous to each other, that is, the quotient group

$$H_p(M;G) = Z_p(M;G)/B_p(M;G)\,,$$

  is called the p-th **homology group**.

- In the case when the coefficient group G is a field $G = K$, all the groups, Z_p, B_p and H_p are vector spaces.

7.3.2 Simplicial Homology

- The important fact is that if M is a compact manifold, then the vector space $H_p(M;K)$ is *finite dimensional*. (This can be proved but we will not do that).

- Let M be a compact n-dimensional manifold. Then there is a **triangulation** of M by finitely many n-simplexes diffeomorphic to the standard n-simplex Δ_n.

- Thus, M is a union of finitely many n-simplexes, which are either disjoint of intersect along common r-simplexes with $r = 0, 1, \ldots, n-1$.

- The set of all these simplexes form a finite **simplicial complex** with some coefficient group G.
- All the simplicial chain groups $\bar{C}_p(M;G)$, $\bar{Z}_p(M;G)$ and $\bar{B}_p(M;G)$ are finitely generated Abelian groups.
- Therefore, the homology group $\bar{H}_p(M;G)$ is a finitely generated group.
- Since any simplicial cycle can be described as a singular cycle, there are homomorphisms
$$\bar{Z}_p \to Z_p, \qquad \bar{B}_p \to B_p$$
and the induced homomorphism
$$\bar{H}_p \to H_p\,.$$
- **Theorem 7.3.1** *Let M be a compact manifold and G be an Abelian group. Then the singular homology groups are isomorphic to the simplicial homology groups*
$$\bar{H}_p(M;G) = H_p(M;G)$$
and are finitely generated Abelian groups.

Proof: Nontrivial.

■
- **Corollary 7.3.1** *Let M be a compact manifold and K be field. Then the homology groups $H_p(M;K)$ are finite-dimensional vector spaces.*

Proof: Follows from above theorem.

■

7.3.3 Betti Numbers and Topological Invariants

- Let M be a compact manifold. The dimensions of the real homology groups $H_p(M;\mathbb{R})$ are called the **Betti numbers**, that is,
$$B_p(M) = \dim H_p(M;\mathbb{R})\,.$$
- The Betti number B_p is the maximal number of linearly independent p-cycles modulo a boundary.

- Let M and V be manifolds, G be a n Abelain group, $F : M \to V$ be a map and $F_* : C_p(M;G) \to C_p(V;G)$ be the induced homomorphism of chain groups.

- Since the induced homomorphism F_* commutes with the boundary homomorphism ∂, the groups $Z_p(M;G)$ and $B_p(M;G)$ are closed under F_*.

- Therefore, the homomorphism F_* naturally acts on the homology groups

$$F_* : H_p(M;G) \to H_p(V;G)\,.$$

- If $F : M \to V$ is a homeomorphism, then there is the inverse homeomorphism $F^{-1} : V \to M$ and the inverse induced homomorphism

$$F_*^{-1} : H_p(V;G) \to H_p(M;G)\,.$$

- In this case, the induced homomorphism F_* is an isomorphism.

- **Theorem 7.3.2** *Let M and V be compact homeomorphic manifolds and G be an Abelian group. Then their homology groups are isomorphic, that is, for any p*

$$H_p(M;K) \cong H_p(V;G)\,.$$

Proof: Follows from above.

■

- Thus, homology groups are **topological invariants**.

- **Corollary 7.3.2** *Let M and V be compact manifolds. If there is an Abelian group G and an integer p such that their homology groups are not isomorphic, that is,*

$$H_p(M;K) \ncong H_p(V;G)\,,$$

then the manifolds M and V are not homeomorphic to each other.

- **Remark**. The converse is not true.

7.3.4 Some Theorems from Algebraic Topology

- A manifold M is **path-connected** (or just **connected**) if every two points in M can be connected by a piecewise-smooth curve.
- Let M be a manifold and G be an Abelian group.
- A point p in a manifold M is a 0-chain. Since $\partial p = 0$, then each point is a 0-cycle (by definition).
- A smooth map $C : [0,1] \to M$ defines a singular 1-simplex and
$$\partial C = C(1) - C(0)$$
is a 0-chain.
- More generally, for any $g \in G$, then gC is a singular 1-simplex and
$$\partial(gC) = gC(1) - gC(0)\,.$$
- A piecewise-smooth map $C : [0,1] \to M$ defines a 1-chain (as a formal sum
$$c_1 = \sum_{k=1}^{r} gC_k$$
of smooth pieces with the same coefficient) and
$$\partial c_1 = gC(1) - gC(0)$$
is a 0-chain.
- **Theorem 7.3.3** *Let M be a compact connected manifold and G be an Abelian group. Then*
$$H_0(M;G) = Gp = \{gp \mid g \in G, p \in M\}\ .$$

Proof:

1. In a connected manifold any two 0-simplexes with the same coefficient are homologous.
2. Moreover, for any point $p \in M$, $g \neq 0$, and any element $g \in G$, there is no 1-chain c_1 such that $\partial c_1 = gp$.

3. Therefore, a multiple gp of a single point is not a boundary for any $g \neq 0$.
4. Thus, any point $p \in M$ is a 0-cycle that is not a boundary.
5. Moreover, for any $g \in G$ and any $p \in M$ the 0-chain gp is a 0-cycle that is not a boundary.

■

- In particular,
$$H_0(M;\mathbb{Z}) = \{0, \pm p, \pm 2p, \dots\}$$
and
$$H_0(M;\mathbb{R}) = \mathbb{R}$$
is a one-dimensional vector space.

- **Corollary 7.3.3** *Let M be a compact connected manifold. Then the zero Betti number is equal to*
$$B_0(M) = 1\,.$$

Proof: Follows from above.

■

- **Theorem 7.3.4** *Let M be a compact manifold consisting of k connected pieces $M_1, \dots M_k$. Then*
$$H_0(M;\mathbb{R}) = \mathbb{R}p_1 + \cdots + \mathbb{R}p_k\,,$$
where $p_i \in M_i$, $i = 1, \dots, k$, meaning
$$\mathbb{R}p_1 + \cdots + \mathbb{R}p_k = \left\{ \sum_{i=1}^{k} a_i p_i \;\middle|\; a_i \in \mathbb{R}, p_i \in M_i, i = 1, \dots, k \right\}\,.$$

Proof:

■

- In this case
$$H_0(M;\mathbb{R}) = \mathbb{R}^k$$
is a k-dimensional vector space.

- **Corollary 7.3.4** *Let M be a compact manifold consisting of k connected pieces. Then the zero Betti number is equal to*

$$B_0(M) = k\,.$$

Proof: Follows from above.

■

- Let V be a p-dimensional oriented closed (compact without boundary) manifold.

- Then a triangulation of V defines an integer p-cycle, denoted by $[V]$ so that

$$\partial[V] = 0\,.$$

- This does not work for non-orientable closed manifolds. A triangulation of a non-orientable closed manifold V gives an integer p-chain, which is not a p-cycle since

$$\partial[V] \neq 0\,.$$

- **Example**. Klein bottle.

- By changing the coefficient group G sometimes one can get a p-cycle even for non-orientable manifolds.

- **Theorem 7.3.5** *Let M be an n-dimensional compact manifold and V be a closed oriented p-dimensional submanifold of M. Let G be an Abelian group and g be an element of G. Consider a triangulation of V in p-triangles and assign to each p-triangle in this triangulation of V the same coefficient $g \in G$. Then $g[V]$ defines a p-cycle z_p in $H_p(M;G)$.*

Proof: Without proof.

■

- Remark. A p-cycle is a generalization of the concept of a closed oriented submanifold.

- In the case of real homology groups, when $G = \mathbb{R}$, the following theorem is true.

• **Theorem 7.3.6** *Let M be an n-dimensional compact manifold. Every real p-cycle z_p in $H_p(M;\mathbb{R})$ is homologous to a finite formal sum*

$$z_p \sim \sum_{k=1}^{r} a_k V_k$$

of closed oriented p-dimensional submanifolds V_k of M with real coefficients a_k.

Proof: Nontrivial.

■

• **Theorem 7.3.7** *Let M be an n-dimensional manifold and G be an Abelian group. Let z_p and z'_p be two cycles in $H_p(M;G)$ that can be deformed into each other. Then they are homologous to each other*

$$z_p \sim z'_p \,.$$

Proof:

1. Since the deformation defines a deformation chain c_{p+1} such that

$$\partial c_{p+1} = z'_p - z_p \,.$$

■

• **Proposition 7.3.1** *Let M be an n-dimensional closed manifold and G be an Abelian group. Then for $p > n$ the singular homology groups $H_p(M;G)$ are trivial*

$$H_p(M;G) = 0 \,.$$

Proof:

1. Singular homology groups are isomorphic to the simplicial homology groups.
2. Since there are no simplicial complexes of dimension $p > n$ then all simplicial homology groups are trivial for $p > n$.

■

7.3.5 Examples.

- **Sphere** S^n.
- From the facts that S^n is connected, orientable and closed it follows that

$$
\begin{aligned}
&H_0(S^n; G) = H_n(S^n; G) = G\,,\\
&H_p(S^n; G) = 0\,, \qquad \text{for } p \neq 0, n\,,\\
&B_0(S^n) = B_n(S^n) = 1\,,\\
&B_p(S^n) = 0\,, \qquad \text{for } p \neq 0, n\,.
\end{aligned}
$$

- **Torus** T^2.

$$
\begin{aligned}
&H_0(T^2; G) = H_2(T^2; G) = G\,,\\
&H_1(T^2; G) = GA + GB\,,\\
&B_0(T^2) = 1, \qquad B_1(T^2) = 2, \qquad B_2(T^2) = 1,
\end{aligned}
$$

  where A and B are the basic 1-cycles.

- **Klein Bottle** K^2.
- Since K^2 is connected closed non-orientable it follows that

$$
\begin{aligned}
&H_0(K^2; \mathbb{Z}) = \mathbb{Z},\\
&H_2(K^2, \mathbb{Z}) = 0\,,\\
&H_1(K^2; \mathbb{Z}) = \mathbb{Z}A + \mathbb{Z}_2 B\,,
\end{aligned}
$$

  where A and B are the basic 1-cycles, and

$$
\begin{aligned}
&H_0(K^2; \mathbb{R}) = \mathbb{R},\\
&H_2(K^2, \mathbb{R}) = 0\,,\\
&H_1(K^2; \mathbb{R}) = \mathbb{R}A\,,\\
&B_0(K^2) = 1, \qquad B_1(K^2) = 1, \qquad B_2(K^2) = 0\,.
\end{aligned}
$$

- **Real Projective Plane** $\mathbb{R}P^2$.
- $\mathbb{R}P^2$ is connected closed non-orientable.

$$
\begin{aligned}
&H_0(\mathbb{R}P^2; \mathbb{Z}) = \mathbb{Z},\\
&H_2(\mathbb{R}P^2, \mathbb{Z}) = 0\,,\\
&H_1(\mathbb{R}P^2; \mathbb{Z}) = \mathbb{Z}_2 A\,,
\end{aligned}
$$

where A is the basic 1-cycle, and

$$\begin{aligned}
&H_0(\mathbb{R}P^2;\mathbb{R}) = \mathbb{R},\\
&H_2(\mathbb{R}P^2,\mathbb{R}) = 0\,,\\
&H_1(\mathbb{R}P^2;\mathbb{R}) = 0\,,\\
&B_0(\mathbb{R}P^2) = 1, \qquad B_1(\mathbb{R}P^2) = 0, \qquad B_2(\mathbb{R}P^2) = 0\,.
\end{aligned}$$

- **Torus** T^3.
- T^3 is a connected closed orientable manifold.

$$\begin{aligned}
&H_0(T^3;\mathbb{Z}) = H_3(T^3,\mathbb{Z}) = \mathbb{Z}\,,\\
&H_1(T^3;\mathbb{Z}) = \mathbb{Z}A + \mathbb{Z}B + \mathbb{Z}C\,,\\
&H_2(T^3;\mathbb{Z}) = \mathbb{Z}D + \mathbb{Z}E + \mathbb{Z}F\,,
\end{aligned}$$

where A, B and C are basic 1-cycles, and D, E and F are basic 2-cyles,

$$B_0(T^3) = 1, \qquad B_1(T^3) = B_2(T^3) = 3, \qquad B_2(T^3) = 1\,.$$

- **Real Projective Space** $\mathbb{R}P^3$.
- $\mathbb{R}P^3$ is connected closed orientable.

$$\begin{aligned}
&H_0(\mathbb{R}P^3;\mathbb{R}) = H_3(\mathbb{R}P^3,\mathbb{R}) = \mathbb{R}\,,\\
&H_1(\mathbb{R}P^3;\mathbb{R}) = H_2(\mathbb{R}P^3;\mathbb{R}) = 0\,,\\
&B_0(\mathbb{R}P^3) = B_3(\mathbb{R}P^3) = 1, \qquad B_1(\mathbb{R}P^3) = B_2(\mathbb{R}P^3) = 0\,.
\end{aligned}$$

7.4 de Rham Cohomology Groups

- Let M be a manifold and $G = \mathbb{R}$ be the coefficient group.
- Then $C_p(M;\mathbb{R})$, $Z_p(M;\mathbb{R})$, $B_p(M;\mathbb{R})$ and $H_p(M;\mathbb{R})$ are vector spaces.
- For simplicity we will denote them in this section simply by $C_p(M)$, $Z_p(M)$, $B_p(M)$ and $H_p(M)$.
- Let $C^p(M) = C^\infty(\Lambda_p(M))$ be the space of smooth p-forms on M.
- We will call the closed p-form **p-cocyles** and the space

$$Z^p(M) = \{\alpha_p \in C^p(M) \mid d\alpha_p = 0\}$$

  of all closed p-forms on M, the **cocycle group**.
- The exact p-forms on M are called the **p-coboundaries** and the space

$$B^p(M) = \{\alpha_p \in Z^p(M) \mid \alpha_p = d\beta_{p+1} \text{ for some } \beta_{p+1} \in C^{p+1}(M)\}$$

  of all exact p-forms on M is called the **coboundary group.**
- Both $Z^p(M)$ and $B^p(M)$ are vector spaces with real coefficients.
- Recall that the exterior derivative is a map

$$d_p : C^p(M) \to C^{p+1}(M)$$

  such that

$$\operatorname{Ker} d_p = Z^p(M)$$

  and

$$\operatorname{Im} d_{p-1} = B^p(M)\,.$$

- Two closed forms are said to be equivalent (or **cohomologous**) if they differ by an exact form.
- The collection of all equivalence classes of closed forms is the quotient vector space

$$H^p(M) = Z^p(M)/B^p(M)$$

  called the p-th **de Rham cohomology group**.

- de Rham cohomology groups are vector spaces.
- Let
$$c_p = \sum_{k=1}^{r} a_k \sigma_p^k$$
be a real p-chain in M, and α be a p-form on M.
- We define the integral of α over c_p by
$$\left\langle \alpha, c_p \right\rangle = \int_{c_p} \alpha = \sum_{k=1}^{r} a_k \int_{\sigma_p^k} \alpha \,.$$
- Thus every p-form on M defines a linear functional on $C_p(M)$
$$\alpha : C_p(M) \to \mathbb{R}\,,$$
by
$$c_p \overset{\alpha}{\longmapsto} \left\langle \alpha, c_p \right\rangle \,.$$
- The space of all p-forms can be naturally identified with the dual space $C_p^*(M)$
$$C^p(M) \cong C_p^*(M)\,.$$
- Furthermore, by Stokes theorem we have for a $(p-1)$-form
$$\left\langle d\alpha, c_p \right\rangle = \left\langle \alpha, \partial c_p \right\rangle \,.$$
- Thus, for every p-cycle z_p, that is, if $\partial z_p = 0$,
$$\left\langle d\alpha, z_p \right\rangle = 0\,,$$
and for every closed form α, that is, if $d\alpha = 0$,
$$\left\langle \alpha, \partial c_p \right\rangle = 0\,.$$
- More generally, let $\alpha_p \in Z^p(M)$ be a closed p-form, $\beta_{p+1} \in C^{p+1}(M)$ be a $(p+1)$-form, $z_p \in Z_p(M)$ be a p-cycle and $c_{p+1} \in C_{p+1}(M)$ be a $(p+1)$-chain. Then
$$\left\langle \alpha + d\beta, z_p + \partial c_p \right\rangle = \left\langle \alpha, z_p \right\rangle \,.$$

- Therefore, for every equivalence class $[\alpha_p] \in H^p(M)$ of closed forms we can define a linear functional $H_p(M) \to \mathbb{R}$ on the space of homology groups by, for any $[z_p] \in H_p(M)$,

$$\Big\langle [\alpha_p], [z_p] \Big\rangle = \Big\langle \alpha_p, z_p \Big\rangle \,.$$

  This is well defined since the right hand side does not depend on the choice of representatives in the equivalence classes.

- This naturally identifies the space of cycles with the space of cocycles

$$Z^p(M) \cong Z^*_p(M)\,.$$

- For a closed p-form α_p and a p-cycle z_p the value of the functional

$$\Big\langle \alpha, z_p \Big\rangle = \int_{z_p} \alpha_p$$

  is called the **period** of the form α_p on the cycle z_p.

- We conjecture that

$$H^p(M) \cong H^*_p(M)\,.$$

- **Proposition 7.4.1** . *Let M be a closed manifold. Then for any linear functional $\varphi : H_p(M) \to \mathbb{R}$ on homology groups there is a closed p-form α_p such that*

$$\varphi(z_p) = \Big\langle \alpha_p, z_p \Big\rangle \,.$$

Proof : Difficult.

■

- **Corollary 7.4.1** . *Let M be a closed manifold. Let $k = B_p(M)$ be the p-th Betti number. Let $z_p^{(1)}, \dots, z_p^{(k)}$, be a basis of p-cycles in the homology groups $H_p(M)$ and $\pi_1, \dots, \pi_k$ be arbitrary real numbers. Then there is a closed p-form α_p such that*

$$\Big\langle \alpha_p, z_p^{(i)} \Big\rangle = \pi_i, \qquad i = 1, 2, \dots, k\,.$$

Proof :

■

- **Proposition 7.4.2** . *Let M be a closed manifold. Let $\alpha_p \in Z^p(M)$ be a closed p-form on M such that for any p-cycle $z_p \in Z_p(M)$*

$$\left\langle \alpha_p, z_p \right\rangle = 0\,.$$

Then the p-form α_p is exact.

Proof : Difficult.

■

- **Theorem 7.4.1 de Rham Theorem**. *Let M be a closed manifold. Then the map*

$$H^p(M) \to H^*_p(M)\,,$$

that associates to each equivalence class $[\alpha_p]$ of closed p-forms a linear functional $H_p(M) \to \mathbb{R}$ on the homology group $H_p(M)$ defined by

$$[z_p] \overset{[\alpha_p]}{\longmapsto} \left\langle \alpha_p, z_p \right\rangle\,,$$

is an isomorphism.

Proof :

■

7.4.1 Examples

- **Torus** T^2.
- **Closed Surfaces** in $\mathbb{R}^2$.

7.5 Harmonic Forms

- Let (M, g) be a closed oriented n-dimensional Riemannian manifold with a Riemannian metric g and the Riemannian volume n-form vol.

- Let $\Lambda_p(M)$ be the bundle of p-forms.

- Recall that there is a natural fiber inner product on Λ_p defined by

$$\langle \alpha, \beta \rangle = \frac{1}{p!} g^{i_1 j_1} \cdots g^{i_p j_p} \alpha_{i_1 \dots i_p} \beta_{j_1 \dots j_p} ,$$

  and the corresponding fiber norm

$$||\alpha|| = \sqrt{\langle \alpha, \alpha \rangle} .$$

- Also, there is a duality between p-forms and $(n-p)$-forms defined by the **Hodge star operator**

$$* : \Lambda_p \to \Lambda_{n-p} ,$$

  that maps any p-form α to a $(n-p)$-form $*\alpha$ **dual** to α defined as follows.

- For each p-form α the form $*\alpha$ is the unique $(n-p)$-form such that for any p-form β

$$\beta \wedge *\alpha = \langle \beta, \alpha \rangle \, \text{vol} .$$

- In components, this means that

$$\begin{aligned} (*\alpha)_{i_{p+1} \dots i_n} &= \frac{1}{p!} \varepsilon_{i_1 \dots i_p i_{p+1} \dots i_n} \sqrt{|g|} g^{i_1 j_1} \cdots g^{i_p j_p} \alpha_{j_1 \dots j_p} \\ &= \frac{1}{p!} \frac{1}{\sqrt{|g|}} g_{i_{p+1} j_{p+1}} \cdots g_{i_n j_n} \varepsilon^{j_1 \dots j_p j_{p+1} \dots j_n} \alpha_{j_1 \dots j_p} . \end{aligned}$$

- Recall that the Hodge star maps forms to pseudo-forms and vice-versa.

- Recall also that for any p-form α,

$$*^2 \alpha = (-1)^{p(n-p)} \alpha ,$$

  meaning that

$$*^{-1} = (-1)^{p(n-p)} * .$$

- The **coderivative** is a linear map

$$\delta : \Lambda_p \to \Lambda_{p-1}$$

defined by

$$\delta = *^{-1} d * = (-1)^{(n-p+1)(p-1)} * d * \ .$$

- The exterior derivative and the coderivative satisfy the important conditions

$$d^2 = \delta^2 = 0 \, .$$

- **Problem**. Show that in local coordinates the coderivative of a p-form α is the $(p-1)$-form $\delta\alpha$ with components

$$(\delta\alpha)_{i_1 \dots i_{p-1}} \quad = \quad g_{i_1 j_2} \cdots g_{i_{p-1} j_p} \frac{1}{\sqrt{|g|}} \partial_j \left(\sqrt{|g|} g^{jk_1} g^{j_2 k_2} \cdots g^{j_p k_p} \alpha_{k_1 k_2 \dots k_p} \right)$$

- Now we define the L^2**-inner product** of p-forms by

$$(\alpha, \beta)_{L^2} = \int_M \alpha \wedge *\beta = \int_M \langle \alpha, \beta \rangle \, \text{vol} \ ,$$

and the L^2-norm

$$||\alpha||_{L^2} = \sqrt{(\alpha, \alpha)_{L^2}} \, .$$

- This makes the space $C^\infty(\Lambda_p(M))$ of smooth p-forms an inner-product vector space.

- The completion of $C^\infty(\Lambda_p(M))$ in the L^2-norm gives the **Hilbert space** $L^2(\Lambda_p(M))$ of square-integrable p-forms.

- Let $A : H \to H$ be an operator on a Hilbert space H. The **adjoint of the operator** A with respect to the inner product of the space H is the operator $A^* : H \to H$ defined by, for any $\varphi, \psi \in H$,

$$(A\varphi, \psi) = (\varphi, A^* \psi) \, .$$

- **Theorem 7.5.1** *Let M be a closed orientable Riemannian manifold. Then the adjoint of the exterior derivative is the negative coderivative*

$$d^* = -\delta\,.$$

That is, for any p-form α and any $(p+1)$-form β,

$$(d\alpha, \beta) = -(\alpha, \delta\beta)\,.$$

Proof:

1.

∎

- **Theorem 7.5.2** *Let M be a compact orientable Riemannian manifold with boundary. Then for any p-form α and any $(p+1)$-form β,*

$$(d\alpha, \beta) + (\alpha, \delta\beta) = \int_{\partial M} \alpha \wedge *\beta\,.$$

Proof:

1. Direct calculation.

∎

- Notice that in case of a manifold with boundary the coderivative is the negative adjoint of the exterior derivative only on the forms that satisfy one of the two types of boundary conditions:

$$\alpha|_{\partial M} = 0 \qquad \text{or} \qquad *\,\alpha|_{\partial M} = 0\,.$$

- Let g be the Riemannian metric and ∇ be the corresponding Levi-Civita connection. It defines a natural connection on the bundle of p-forms. The **Laplacian** on p-forms is the operator

$$\Delta : C^\infty(\Lambda_p(M)) \to C^\infty(\Lambda_p(M))$$

defined by

$$\Delta = g^{ij}\nabla_i\nabla_j\,.$$

- The **Hodge Laplacian** on p-forms is the operator

$$L : C^\infty(\Lambda_p(M)) \to C^\infty(\Lambda_p(M))$$

defined by

$$L = d\delta + \delta d = (d+\delta)^2 .$$

- The operators d and δ commute with the Hodge Laplacian, i.e.

$$dL = Ld , \qquad \delta L = L\delta .$$

- **Theorem 7.5.3** *For any p there holds*

$$L = \Delta + W ,$$

where $W : \Lambda_p \to \Lambda_p$ is an endomorphism on the bundle of p-forms called the **Weitzenböck endomorphism**.

***Proof*:**

1.

∎

- Weitzenböck endomorphism is a linear combination of Riemann curvature tensor, that is, when acting on p-forms W has the form

$$W^{i_1\dots i_p}{}_{j_1\dots j_p} = F^{mni_1\dots i_p}{}_{klj_1\dots j_p} R^{kl}{}_{mn} ,$$

where $F^{mni_1\dots i_p}{}_{klj_1\dots j_p}$ is constructed only from the Kronecker symbol δ^i_j and the metric g_{ij} and g^{ij}.

- **Problem**. Obtain the expression for the Weitzenböck endomorphism for p-forms. *Hint: replace partial derivatives by covariant derivatives and use the definition of the curvature.*

- A p-form α is called **harmonic** if

$$L\alpha = 0 .$$

-

• **Theorem 7.5.4** *Let M be a closed Riemannian manifold. Then a p-form α is harmonic if and only if it is closed and coclosed, that is,*

$$d\alpha = \delta\alpha = 0\,.$$

***Proof*:**

1. Use
$$0 = (L\alpha, \alpha) = ||d\alpha||^2 + ||\delta\alpha||^2\,.$$

∎

• On a closed Riemannian manifold the Laplacian and the Hodge Laplacian are **self-adjoint elliptic** operators.

• **Theorem 7.5.5 Hodge Theorem.** *Let M be a closed Riemannian manifold. Then:*

1. *The vector space $\mathcal{H}^p$ of harmonic p-forms is finite-dimensional.*
2. *The equation*
$$L\alpha = \rho$$
has a solution if and only if ρ is orthogonal to $\mathcal{H}^p$.

***Proof*:**

1.

∎

• **Theorem 7.5.6** *Let M be a closed Riemannian manifold. Then any p-form β is a sum of an exact form $d\alpha$, a coexact form $\delta\gamma$ and a harmonic form h, that is,*

$$\beta = d\alpha + \delta\gamma + h\,.$$

In other words, there is an orthogonal decomposition (called the **Hodge decomposition***)*

$$\Lambda_p = \operatorname{Im} d_{p-1} + \operatorname{Im} \delta_{p+1} + \mathcal{H}^p\,.$$

***Proof*:**

1.

■

• **Corollary 7.5.1** *Let M be a closed Riemannian manifold. Then any closed p-form β is a sum of an exact form $d\alpha$ and a harmonic form h, that is,*

$$\beta = d\alpha + h\,.$$

Proof:

1.

■

• **Corollary 7.5.2** *Let M be a closed Riemannian manifold. Then each de Rham class of cohomologous closed p-forms has a harmonic representative.*

Let $k = B_p(M)$ be the p-th Betti number. Let $z_p^{(1)}, \dots, z_p^{(k)}$, be a basis of real p-cycles in the real homology groups $H_p(M)$ and $\pi_1, \dots, \pi_k$ be arbitrary real numbers. Then there is a unique harmonic p-form h_p such that

$$\left\langle h_p, z_p^{(i)} \right\rangle = \pi_i, \qquad i = 1, 2, \dots, k\,.$$

Proof:

1.

■

• The metric g is said to have **positive Ricci curvature** if its Ricci tensor is positive-definite.

• **Corollary 7.5.3 Bochner Theorem** *Let M be a closed Riemannian manifold with positive Ricci curvature. Then the first Betti number vanishes, i.e.*

$$B_1(M) = 0\,.$$

That is there are no harmonic 1-forms on M.

Proof:

1. Let h be a harmonic 1-form. Then

$$0 = \frac{1}{2}\int_M \Delta \langle h, h\rangle \,\mathrm{vol} \; = \int_M R^{ij} h_i h_j \mathrm{vol} \; + ||\nabla_j h_i||^2 \geq 0\,.$$

2. Thus $h = 0$.

∎

- **Remark**. The elements of the first homology group $H_1(M, G)$ are equivalence classes of 1-cycles.
- The 1-cycles are closed oriented curves (loops) on M.
- If a closed curve can be deformed to a point, then it is a boundary of a surface (a 2-simplex).
- That is, a closed curve that can be contracted to a point is a trivial 1-cycle.
-
- **Corollary 7.5.4** *For any simply connected manifold M and any Abelian group G,*

$$H_1(M, G) = 0 .$$

7.6 Relative Homology and Morse Theory

7.6.1 Relative Homology

- Let M be a compact Riemannian n-dimensional manifold with smooth boundary ∂M and
$$i : \partial M \to M$$
be the inclusion map.

- Let $\hat{x}^i$, $i = 1, \dots, (n-1)$, be the local coordinates on the boundary ∂M and x^μ, $\mu = 1, \dots, n$, be the local coordinates on M in a patch U close to the boundary. Then the inclusion map is defined locally by
$$x^\mu = x^\mu(\hat{x})\,.$$

- Close to the boundary there exists a system of local coordinates $(x^\mu) = (\hat{x}^i, r)$ so that the boundary is described by the equation
$$r = 0\,.$$
The coordinate r can be chosen to be the normal geodesic distance from the boundary so that the vector ∂_r is normal to the boundary.

- The inclusion map in this case is given by
$$x^i = \hat{x}^i, \qquad r = 0\,.$$
Therefore,
$$\frac{\partial x^k}{\partial \hat{x}^j} = \delta^k_j\,, \qquad \frac{\partial r}{\partial \hat{x}^j} = 0\,.$$

- A p-form α on M is called **normal** to ∂M if
$$i^*\alpha = 0\,,$$
that is,
$$\frac{\partial x^{\mu_1}}{\partial \hat{x}^{j_1}} \cdots \frac{\partial x^{\mu_p}}{\partial \hat{x}^{j_p}} \alpha_{\mu_1 \dots \mu_p}(x(\hat{x})) = 0\,.$$

- This just means that a normal p-form α vanishes on tangent vectors to the boundary.

- In other words, a p-form α is normal to the boundary if

$$\alpha(\mathbf{v}_1, \dots, \mathbf{v}_p)\Big|_{\partial M} = 0$$

for any tangent vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$.

- In the special coordinate system $(\hat{x}^i, r)$ this means that a p-form is normal if all purely tangential components vanish on the boundary

$$\alpha_{j_1 \dots j_p}|_{\partial M} = 0 \,.$$

- Obviously,

$$i^* dr = 0 \,.$$

Therefore, a p-form α is normal if there is a $(p-1)$-form β such that

$$\alpha = \beta \wedge dr \,.$$

- A p-form α is called **tangent** to the boundary if the $(n-p)$-form $*\alpha$ is normal, that is,

$$i^* * \alpha = 0 \,.$$

- In other words, a p-form α is tangent to the boundary if

$$\alpha(\mathbf{v}_1, \dots, \mathbf{v}_{p-1}, \mathbf{N})\Big|_{\partial M} = 0$$

for any tangent vectors $\mathbf{v}_1, \dots, \mathbf{v}_{p-1}$ and a normal vector $\mathbf{N}$, meaning that the interior product with a normal vector vanishes

$$i_{\mathbf{N}}\alpha\Big|_{\partial M} = 0 \,.$$

- In local coordinates, this condition becomes

$$\varepsilon_{\lambda_1 \dots \lambda_{n-p} \mu_1 \dots \mu_p} \frac{\partial x^{\lambda_1}}{\partial \hat{x}^{j_1}} \cdots \frac{\partial x^{\lambda_{n-p}}}{\partial \hat{x}^{j_{n-p}}} g^{\mu_1 \nu_1} \cdots g^{\mu_p \nu_p} \alpha_{\nu_1 \dots \nu_p}\Big|_{\partial M} = 0 \,.$$

- In the special coordinate system $(\hat{x}^i, r)$ this just means that the components with at least one index along the normal direction vanish

$$\alpha_{j_1 \dots j_{p-1} r}\Big|_{\partial M} = 0 \,.$$

- **Proposition 7.6.1** *Let M be a compact Riemannian manifold with smooth boundary. Let α and β be two p-forms on M, which are either both normal to the boundary or both tangent to the boundary. Then*

$$(d\alpha, \beta) = -(\alpha, \delta\beta)\,.$$

That is, on normal or tangent p-forms

$$d^* = -\delta\,.$$

***Proof*:** Obvious.

■

- **Theorem 7.6.1** *Let M be a compact Riemannian manifold with smooth boundary. Let $k = B_p(M)$ be the p-th Betti number. Let $z_p^{(1)}, \dots, z_p^{(k)}$, be a basis of real p-cycles in the real homology groups $H_p(M)$ and $\pi_1, \dots, \pi_k$ be arbitrary real numbers. Then there is a unique tangent harmonic p-form h_p such that*

$$dh = \delta h = 0$$

and

$$\left\langle h_p, z_p^{(i)} \right\rangle = \pi_i, \qquad i = 1, 2, \dots, k\,.$$

***Proof*:**

1.

■

- More generally,

Theorem 7.6.2 *Let M be a compact Riemannian manifold with smooth boundary. Let $k = B_p(M)$ be the p-th Betti number. Let $z_p^{(1)}, \dots, z_p^{(k)}$, be a basis of real p-cycles in the real homology groups $H_p(M)$ and $\pi_1, \dots, \pi_k$ be arbitrary real numbers. Let γ be a closed $(n-p)$-form on ∂M such that for every $(n-p)$-cycle y_{n-p} on ∂M that is a boundary of a $(n-p+1)$-chain c_{n-p+1} in M, that is,*

$$i^*\gamma = \partial c_{n-p+1}\,,$$

- *the value of the integral of γ over y_p vanishes,*

$$\left\langle \gamma, y_p \right\rangle_{\partial M} = \int_{\partial M} \gamma = 0\,.$$

Then there is a unique harmonic p-form h_p such that

$$dh = \delta h = 0\,,$$

$$i^* * h = *h|_{\partial M} = \gamma\,,$$

$$\left\langle h_p, z_p^{(i)} \right\rangle = \pi_i, \qquad i = 1, 2, \dots, k\,.$$

Proof:

1.

■

- Let M be a compact manifold with boundary. A **relative p-cycle** (mod ∂M) is a p-chain on M whose boundary lies on ∂M.

- A p-cycle with no boundary is called an **absolute p-cycle**.

- From the point of view of relative homology any p-chain that lies on ∂M is neglected.

- **Example.**

- Two relative p-cycles are **homologous** (mod ∂M) if they differ by a true boundary and a p-chain that lies on ∂M,

$$c'_p - c_p = \partial u_{p+1} + v_p\,, \qquad v \subset \partial M\,.$$

- A **relative boundary** (mod ∂M) is a sum of an absolute boundary and a chain that lies on ∂M.

- **Example.**

- The **relative homology group** is the quotient group of relative cycles modulo the relative boundaries

$$H_p(M, \partial M; G) = Z_p(M, \partial M; G)/B_p(M, \partial M; G)\,.$$

- **Theorem 7.6.3** *Let M be a compact Riemannian manifold with smooth boundary. Let $k = B_p(M)$ be the p-th Betti number. Let $z_p^{(1)}, \dots, z_p^{(k)}$, be a basis of real relative p-cycles of M (mod ∂M) in the real homology groups $H_p(M, \partial M; \mathbb{R})$, that is,*

$$H_p(M, \partial M; \mathbb{R}) = \sum_{i=1}^{k} \mathbb{R} c_i\,.$$

Let $\pi_1, \dots, \pi_k$ be arbitrary real numbers. Then there is a unique normal harmonic p-form h_p on M such that

$$dh = \delta h = 0\,,$$

and

$$\left\langle h_p, z_p^{(i)} \right\rangle = \pi_i, \qquad i = 1, 2, \dots, k\,.$$

Proof:

1.

■

7.6.2 Morse Theory

- Let M be a closed manifold and $f : M \to \mathbb{R}$ be a smooth function on M.

- **Example.**

- A point $x_0 \in M$ is called a **critical point** of the function f if

$$df\Big|_{x_0} = 0\,, \qquad \frac{\partial f}{\partial x^i}\bigg|_{x_0} = 0\,.$$

- A real number $a \in \mathbb{R}$ is called a **critical value** of f if the inverse image $f^{-1}(a) \subset M$ contains at least one critical point.
- A critical point is called **inessential** if it can be removed by a small deformation of the function f.
- A critical point x_0 is called **non-degenerate** if the **Hessian**

$$H_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}$$

is non-degenerate, that is,

$$\det H_{ij}\Big|_{x_0} \neq 0\,.$$

- The Hessian defines a linear operator on the tangent space

$$H : T_x M \to T_x M$$

with the matrix

$$H^i{}_j = g^{ik} H_{kj}\,.$$

- For a non-degenerate critical point the Hessian operator has non-zero real eigenvalues.
- The number of the negative eigenvalues (counted with multiplicities), that is the dimension of the subspace of the tangent space on which the Hessian is negative-definite, is called the **Morse index** of the critical point.
- **Lemma 7.6.1 Morse Lemma.** *Let M be a closed manifold and $f : M \to \mathbb{R}$ be a smooth real-valued function on M. Let x_0 be a nondegenerate critical point of f with Morse index p. Then in a neighborhood of x_0 there exist local coordinates $(x^1, \dots, x^p, y^1, \dots, y^{n-p})$ such that*

$$f(x,y) = f(x_0) - (x^1)^2 - \cdots - (x^p)^2 + (y^1)^2 + \cdots + (y^{n-p})^2\,.$$

Proof:

1.

■

- The number of critical points of index p is called the p-th **Morse type number** and denoted by M_p.

- Let t be a formal variable. The polynomial

$$M(t) = \sum_{p=0}^{n} M_p t^p$$

  is called the **Morse polynomial**.

- For each real number $a \in \mathbb{R}$ we define

$$M_a = \{x \in M \mid f(x) \le a\}$$

  and

$$M_a^- = \{x \in M \mid f(x) < a\} .$$

- A real number a is called a **homotopically critical value** of the function f if some relative homology group $H_p(M_a, M_a^-; G)$ is non-zero.

- It turns out that for non-degenerate critical points a value is homotopically critical if and only if it is critical.

- That is, for non-degenerate critical points the critical values of f are precisely the values at which new relative cycles appear.

- Let $B_p(M) = \dim H_p(M)$ be the Betti numbers. The polynomial

$$P(t) = \sum_{p=0}^{n} B_p t^p$$

  is called the **Poincaré polynomial**.

Theorem 7.6.4 Morse Theorem. *Let M be a closed manifold and $f; M \to \mathbb{R}$ be a smooth function. Suppose that the function f has only non-degenerate critical points. Let M_p be the Morse type numbers, B_p be the Betti numbers, $M(t)$ be the Morse polynomial and $P(t)$ be the Poincaré polynomial. Then there is a polynomial*

•

$$Q(t) = \sum_{p=0}^{n-1} q_p t^p$$

with non-negative coefficients, $q_p \geq 0$, such that

$$M(t) - P(t) = (1 + t)Q(t),$$

that is,

$$M_p - B_p = q_p + q_{p-1}.$$

Proof:

1.

■

Corollary 7.6.1 *There hold:*

1. **Weak Morse inequalities**

$$M_p \geq B_p \,,$$

2. *In particular, the total number of critical points is bounded below by the sum of all Betti numbers*

$$\sum_{p=0}^{n} M_p \geq \sum_{p=0}^{n} B_p \,.$$

•

3. **Strong Morse inequalities**

$$\begin{aligned} M_0 &\geq B_0 \\ M_1 - M_0 &\geq B_1 - B_0 \\ &\cdots \\ M_n - M_{n-1} + \cdots + (-1)^n M_0 &= B_n - B_{n-1} + \cdots + (-1)^n B_0 \,. \end{aligned}$$

4. *In particular,*

$$\sum_{p=0}^{n} (-1)^p M_p = \sum_{p=0}^{n} (-1)^p B_p$$

Proof:

1.

■

Chapter 8

8.1

-
-
- **Theorem 8.1.1**

  Proof :

  1.

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  1.

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Notation

Add all common notation

$f : X \to Y$	mapping (function) from X to Y
$f(X)$	range of f
χ_A	characteristic function of the set A
$\emptyset$	empty set
$\mathbb{N}$	set of natural numbers (positive integers)
$\mathbb{Z}$	set of integer numbers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of positive real numbers
$\mathbb{C}$	set of complex numbers