Lecture Notes on Asymptotic Expansion

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Contents

Chapter 1

Asymptotic Expansions

1.1 Asymptotic Estimates

Let M be a set of real or complex numbers with a limit point a. Let $f, g : M \to \mathbb{R}$ (or $f, g : M \to \mathbb{C}$) be some functions on M.

Definition 1 *The following are* ***asymptotic estimates***

i) $f(x) \sim g(x) \quad (x \to a,\ x \in M)$

$$\text{if} \quad \lim_{x \to a,\ x \in M} \frac{f(x)}{g(x)} = 1 \tag{1.1}$$

ii) $f(x) = o(g(x)) \quad (x \to a,\ x \in M)$

$$\text{if} \quad \lim_{x \to a,\ x \in M} \frac{f(x)}{g(x)} = 0 \tag{1.2}$$

iii) $f(x) = O(g(x)) \quad (x \in M)$

$$\text{if} \quad \exists C : \quad |f(x)| \leq C|g(x)| \quad \forall x \in M \tag{1.3}$$

iv) $f(x) = O(g(x)) \quad (x \to a, x \in M)$

if $\exists C$ and a neighborhood U of a such that :

$$|f(x)| \leq C|g(x)| \quad \forall x \in M \cap U \tag{1.4}$$

1.1.1 Examples

1.
$$\ln x = o(x^{-\alpha}) \quad (x \to 0^+),\ \alpha > 0\ . \tag{1.5}$$

2.
$$\ln x = o(x^{\alpha}) \quad (x \to \infty),\ \alpha > 0\ . \tag{1.6}$$

3.
$$\sin z \sim z \quad (z \to 0)\ . \tag{1.7}$$

4.
$$\sin x = O(1) \quad (x \in \mathbb{R})\ . \tag{1.8}$$

5.
$$n! \sim \sqrt{2\pi n} e^{-n} n^n \quad (n \to \infty)\ . \tag{1.9}$$

Remarks. The relation $f(x) = o(g(x))$ means that $f(x)$ is *infinitesimal* with respect to $g(x)$ as $x \to a$. Similarly, $f(x) = O(g(x))$ means that $f(x)$ is *bounded* with respect to $g(x)$ as $x \to a$. In particular, $f(x) = o(1),\ (x \to a)$ means that $f(x)$ is *infinitesimal* as $x \to a$ and $f(x) = O(1),\ (x \to a)$ means that $f(x)$ is *bounded* as $x \to a$.

1.1.2 Properties of asymptotic estimates

There holds (as $x \to a$, $x \in M$):

$$o(f(x)) + o(f(x)) = o(f(x)) \tag{1.10}$$
$$o(f(x))o(g(x)) = o(f(x)g(x)) \tag{1.11}$$
$$o(o(f(x))) = o(f(x)) \tag{1.12}$$
$$O(f(x)) + O(f(x)) = O(f(x)) \tag{1.13}$$
$$O(f(x))O(g(x)) = O(f(x)g(x)) \tag{1.14}$$
$$O(O(f(x))) = O(f(x)) \tag{1.15}$$
$$o(f(x)) + O(f(x)) = O(f(x)) \tag{1.16}$$
$$o(f(x))O(g(x)) = o(f(x)g(x)) \tag{1.17}$$
$$O(o(f(x))) = o(f(x)) \tag{1.18}$$
$$o(O(f(x))) = o(f(x)) \tag{1.19}$$

1.2 Asymptotic Sequences

Definition 2 *Let $\varphi_n : M \to \mathbb{R}$, $n \in \mathbb{N}$, and a be a limit point of M. Let $\varphi_n(x) \neq 0$ in a neighborhood U_n of a. The the sequence $\{\varphi_n\}$ is called **asymptotic sequence** at $x \to a,\ \ x \in M$ if $\forall n \in \mathbb{N}$*

$$\varphi_{n+1}(x) = o(\varphi_n(x)) \quad (x \to a,\ x \in M) \tag{1.20}$$

1.2.1 Examples

1. Power asymptotic sequences

   (a)
$$\{(x-a)^n\}, \quad x \to a\ . \tag{1.21}$$

   (b)
$$\{x^{-n}\}, \quad x \to \infty\ . \tag{1.22}$$

2. Let $\{\lambda_n\}$ be a decreasing sequence of real numbers, i.e. $\lambda_n < \lambda_{n+1}$, and let $0 < \varepsilon \leq \pi/2$. Then the sequence

$$\{e^{\lambda_n z}\}, \quad z \to \infty, \quad |\arg z| \leq \frac{\pi}{2} - \varepsilon \tag{1.23}$$

   is an asymptotic sequence.

1.2.2 Properties of asymptotic sequences

1. Any subsequence of an asymptotic sequence is an asymptotic sequence.

2. Let $f(x) \neq 0$ for $x \in M$ in some neighborhood of a and $\{\varphi_n\}$ be an asymptotic sequence at $x \to a$, $x \in M$. Then the sequence $\{f(x)\varphi_n(x)\}$ is an asymptotic sequence as $x \to a$, $x \in M$.

3. Let $\{\varphi_n(x)\}$, $\{\psi_n(x)\}$ be asymptotic sequences as $x \to a,\ x \in M$. Then the sequence $\{\varphi_n(x)\psi_n(x)\}$ is an asymptotic sequence as $x \to a, x \in M$.

1.3 Asymptotic Series

Let $f : M \to \mathbb{R}$ and a be a limit point of M.

Definition 3 *Let $\{\varphi_n\}$ be an asymptotic sequence as $x \to a$, $x \in M$. We say that the function f is* ***expanded in an asymptotic series***

$$f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x), \quad (x \to a,\ x \in M), \tag{1.24}$$

where a_n are constants, if $\forall N \geq 0$

$$R_N(x) \equiv f(x) - \sum_{n=0}^{N} a_n \varphi_n(x) = o(\varphi_N(x)), \quad (x \to a,\ x \in M)\ . \tag{1.25}$$

This series is called ***asymptotic expansion*** *of the function f with respect to the asympotic sequence $\{\varphi_n\}$. $R_N(x)$ is called the* ***rest term*** *of the asymptotic series.*

Remarks

1. The condition $R_N(x) = o(\varphi_N(x))$ means, in particular, that

$$\lim_{x \to a} R_N(x) = 0 \quad \text{for any fixed } N \tag{1.26}$$

2. Asymptotic series could diverge. This happens if

$$\lim_{N \to \infty} R_N(x) \neq 0 \quad \text{for some fixed } x \tag{1.27}$$

3. There are three possibilities:

   (a) asymptotic series converges to $f(x)$;

   (b) asymptotic series converges to a function $g(x) \neq f(x)$;

   (c) asymptotic series diverges.

Theorem 1 *Asymptotic expansion of a function with respect to an asymptotic sequence is unique.*

Remark. Two different functions can have the same asymptotic expansion. For example, $f(x) = e^x$ and $g(x) = e^x + e^{-1/x}$ have the same asymptotic expansion with respect to the asymptotic sequence $\{x^n\}$:

$$e^x \sim e^x + e^{-1/x} \sim \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \to 0+ \tag{1.28}$$

Theorem 2 *Asymptotic series can be added and multiplied by numbers, but, cannot be multipied by asymptotic series.*

Theorem 3 *One can multiply and divide power asymptotic series.*

Definition 4 *Let $f : M \times S \to \mathbb{R}$ be a function of two variables and a be a limit point of M and $\{\varphi_n\}$ be an asymptotic sequence as $x \to a$. Let for any fixed $y \in S$ the function f is expanded in an asymptotic series*

$$f(x, y) \sim \sum_{n=0}^{\infty} a_n(y)\varphi_n(x), \quad (x \to a,\ x \in M) . \tag{1.29}$$

This asymptotic expansion is called ***uniform*** *with respect to the parameter $y \in S$, if the relation*

$$R_N(x, y) \equiv f(x, y) - \sum_{n=0}^{N} a_n(y)\varphi_n(x) = o(\varphi_N(x)), \quad (x \to a,\ x \in M) \tag{1.30}$$

is valid uniformly with respect to $y \in S$.

Theorem 4 *A uniform asymptotic expansion can be integrated with respect to the parameter term by term.*

Remark. One cannot, in general, differentiate asymptotic series, neither with respect to x nor with respect to a parameter.

1.4 Asymptotics of Integrals: Weak Singularities

Let us consider the integrals of the form

$$F(\varepsilon) = \int_0^a f(x, \varepsilon)\, dx \tag{1.31}$$

where $a > 0$ and $\varepsilon > 0$ is a *small positive parameter.* Here $f \in C^\infty([0,a] \times (0,\varepsilon_0])$ is a smooth function for $0 \le x \le a,\ \ 0 < \varepsilon \le \varepsilon_0$, with some ε_0. Then the integral converges for $\varepsilon > 0$. Let f have a singularity when $\varepsilon = 0$, i.e. $g(x) = f(x,0)$ has a singularity at some $0 \le x \le a$. If this singularity is of power or logarithmic type then we say that the integral $F(\varepsilon)$ has a **weak singularity**.

This definition is obviously extended for unbounded intervals. Let

$$F(\varepsilon) = \int_a^\infty f(x,\varepsilon)\,dx \tag{1.32}$$

where $a > 0$ and $\varepsilon > 0$ is a small parameter. Here $f \in C^\infty([a,\infty) \times (0,\varepsilon_0])$ is a smooth function for $a \le x \le \infty,\ \ 0 < \varepsilon \le \varepsilon_0$, with some ε_0. Let the integral converge for $\varepsilon > 0$ and diverge for $\varepsilon = 0$. If the function $g(x) = f(x,0)$ is of power or logarithmic order at $x \to \infty$, then we say that the integral $F(\varepsilon)$ has a **weak singularity**.

1.4.1 Power Singularity on a Bounded Interval

Let $a, \alpha, \beta \in \mathbb{R}$, $a, \beta > 0$, be some real numbers and $\varepsilon > 0$ be a small positive prameter. Let $\varphi \in C^\infty[0,a]$ be a smooth function on $[0,a]$. We will study the asymptotics as $\varepsilon \to 0+$ of the integrals of the form

$$F(\varepsilon) = \int_0^a t^{\beta-1}(t+\varepsilon)^\alpha \varphi(t)\,dt\ . \tag{1.33}$$

Remarks. The function F is holomorphic in complex plane ε with a cut along the negative half-axis. At the point $\varepsilon = 0$ this function has a singularity (if $\alpha > 0$ is not integer). The type of this singularity is determined by the behavior of the function φ at small $t \ge 0$.

Standard Integral. One needs the following result. Let α and β be two complex numbers such that $\operatorname{Re}\beta > 0$, $\operatorname{Re}\alpha < 0$ and $\operatorname{Re}(\beta+\alpha) < 0$. Then

$$\int_0^\infty t^{\beta-1}(t+1)^\alpha\,dt = \frac{\Gamma(\beta)\Gamma(-\alpha-\beta)}{\Gamma(-\alpha)}, \qquad (0 < \operatorname{Re}\beta < -\operatorname{Re}\alpha)\,. \tag{1.34}$$

Theorem 5 *Let $\varphi \in C^\infty[0,a]$. Let $r, \delta > 0$ and $S_\delta = \{\varepsilon \in \mathbb{C} | 0 < |\varepsilon| \le r, |\arg\varepsilon| \le \pi - \delta\}$ be a sector in the complex plane of ϵ.*

1. *If $\alpha+\beta$ is not integer, then*

$$F(\varepsilon) \sim \sum_{n=0}^{\infty} \frac{\Gamma(\beta+n)\Gamma(-\alpha-\beta-n)}{\Gamma(-\alpha)} \frac{\varphi^{(n)}(0)}{n!} \varepsilon^{\alpha+\beta+n} + \sum_{n=0}^{\infty} a_n \varepsilon^n$$

$$(\varepsilon \to 0, \qquad \varepsilon \in S_\delta) \tag{1.35}$$

2. *If $\alpha+\beta = N$ is an integer, then*

$$F(\varepsilon) \sim -\sum_{n \geq \max\{0,-N\}}^{\infty} \frac{\Gamma(N+n)}{\Gamma(\alpha)\Gamma(N+n-\alpha)} \frac{\varphi^{(n)}(0)}{n!} \varepsilon^{n+N} \ln\varepsilon + \sum_{n=0}^{\infty} b_n \varepsilon^n$$

$$(\varepsilon \to 0, \qquad \varepsilon \in S_\delta) \tag{1.36}$$

The coefficients a_n and b_n depend on the values $\varphi(t)$ for $0 \leq t \leq a$. The branch for the functions ε^γ and $\ln\varepsilon$ is choosen in such a way that $\varepsilon^\gamma > 0$ and $\ln\varepsilon$ is real for $\varepsilon > 0$.

Examples. In all examples $\varphi \in C^\infty([0,a])$ is a smooth function bounded with all its derivatives.

1. Let $0 < a < 1$ and

$$F(\varepsilon) = \int_0^a \frac{\varphi(t)}{t+\varepsilon}\, dt\ . \tag{1.37}$$

Then

$$F(\varepsilon) = -\varphi(0)\ln\varepsilon + O(1), \quad (\varepsilon \to 0^+) \tag{1.38}$$

2. Let $0 < a < 1$ and

$$F(\varepsilon) = \int_0^a \frac{\varphi(t)}{t^2+\varepsilon^2}\, dt\ . \tag{1.39}$$

By using

$$\frac{1}{t^2+\varepsilon^2} = \frac{1}{2i\varepsilon}\left(\frac{1}{t-i\varepsilon} - \frac{1}{t+i\varepsilon}\right) \tag{1.40}$$

we obtain from the previous example

$$F(\varepsilon) = \varphi(0)\frac{\pi}{2\varepsilon} + O(1) \quad (\varepsilon \to 0^+) \tag{1.41}$$

3. Let $\alpha > 1/2$ and
$$F(\varepsilon) = \int_0^a \frac{\varphi(t)}{(t^2+\varepsilon^2)^\alpha}\, dt\,. \tag{1.42}$$
Then
$$F(\varepsilon) = \varphi(0)\frac{\sqrt{\pi}}{2}\frac{\Gamma(\alpha-1/2)}{\Gamma(\alpha)}\varepsilon^{1-2\alpha}+O(\varepsilon^{3-2\alpha})+O(1), \quad (\varepsilon\to 0^+) \tag{1.43}$$

1.4.2 Power singularity on Unbounded Interval

Standard Integral. To compute the following asymptotics one needs the following standard integral. Let $\operatorname{Re}\beta > 0$ and $\operatorname{Re}\alpha > -1$. Then
$$\int_0^\infty t^\alpha e^{-t^\beta} dt = \frac{1}{\beta}\Gamma\left(\frac{\alpha+1}{\beta}\right), \qquad (\operatorname{Re}\beta > 0,\ \operatorname{Re}\alpha > -1)\,. \tag{1.44}$$

Examples. Let $\varphi \in C^\infty([a,\infty))$ be a smooth function on $[a,\infty)$ that has asymptotic expansion as $x \to \infty$
$$\varphi(x) \sim \sum_{k=0}^\infty a_k x^{-k}\,. \tag{1.45}$$

1. Let $a, \beta > 0$, and
$$F(\varepsilon) = \int_a^\infty \varphi(x) x^\alpha e^{-\varepsilon x^\beta}\, dx\,. \tag{1.46}$$
If $\alpha < -1$, then the integral is not singular as $\varepsilon \to 0^+$. Its asymptotic expansion can be obtained either by integration by parts or by a change of variables.

So, let now $\alpha + 1 > 0$ and let $N = [\alpha+1] \geq 0$ be the integer part of $\alpha + 1$. Let us single out the first $N+1$ terms of the asymptotic expansion in φ, i.e.
$$\varphi(x) = \sum_{k=0}^N a_k x^{-k} + R_N(x)\,. \tag{1.47}$$
Since $R_N(x) = O(x^{-(N+1)})$ as $x \to \infty$, we have
$$F(\varepsilon) = \sum_{k=0}^N a_k \int_a^\infty x^{\alpha-k} e^{-\varepsilon x^\beta}\, dx + O(1)\,. \tag{1.48}$$

Now by changing the variables and extending the interval to $[0, \infty)$ we obtain

$$
\begin{aligned}
F(\varepsilon) &= \sum_{k=0}^{N} \frac{a_k}{\beta} \varepsilon^{-\frac{\alpha-k+1}{\beta}} \left[\Gamma\left(\frac{\alpha-k+1}{\beta}\right) + O(1) \right] + O(1) \\
&= \frac{a_0}{\beta} \varepsilon^{-\frac{\alpha+1}{\beta}} \Gamma\left(\frac{\alpha+1}{\beta}\right) + O(\varepsilon^{-\frac{\alpha}{\beta}}) \qquad (1.49)
\end{aligned}
$$

If $\alpha = -1$, then

$$F(\varepsilon) = -\frac{a_0}{\beta} \ln \varepsilon + O(1), \quad (\varepsilon \to 0^+) \; . \qquad (1.50)$$

2. Let $a > 0$ and let

$$P(x) = x^n + \cdots + a_1 x \qquad n \geq 1 \; . \qquad (1.51)$$

Consider the integral

$$F(\varepsilon) = \int_a^\infty \varphi(x) x^\alpha e^{-\varepsilon P(x)} \, dx \; . \qquad (1.52)$$

If $\alpha > -1$, then the main term of the asymptotics is

$$F(\varepsilon) = \frac{a_0}{n} \Gamma\left(\frac{\alpha+1}{n}\right) \varepsilon^{-\frac{\alpha+1}{n}} + O(\varepsilon^{-\frac{\alpha}{n}}), \quad (\varepsilon \to 0^+) \qquad (1.53)$$

If $\alpha = -1$, then

$$F(\varepsilon) = -\frac{a_0}{n} \ln \varepsilon + O(1), \qquad (\varepsilon \to 0^+) \qquad (1.54)$$

If $\alpha < -1$, then the integral is not singular as $\varepsilon \to 0^+$. By integration by parts the integral can be reduced to the cases considered above.

Chapter 2

Laplace Method

2.1 Laplace Integrals in One Dimension

Let $M = [a, b]$ be a closed bounded interval, $S : M \to \mathbb{R}$ be a real valued function, $\varphi : M \to \mathbb{C}$ a complex valued function and λ be a *large positive parameter*. Consider the integrals of the form

$$F(\lambda) = \int_a^b \varphi(x) \exp[\lambda S(x)]\, dx\,. \tag{2.1}$$

Such integrals are called **Laplace integrals**. We will study the asymptotics of the Laplace integrals as $\lambda \to \infty$.

Lemma 1 *Let* $\sup_{a<x<b} S(x) = L < \infty$ *and the integral (2.1) converges absolutely for some* $\lambda_0 > 0$*. Then*

1. $$|F(\lambda)| \le C|e^{\lambda L}| \quad (\operatorname{Re}\lambda \ge \lambda_0)\,. \tag{2.2}$$

2. *if* $f, S \in C(a, b)$*, then* $F(\lambda)$ *is holomorphic in the halfplane* $\operatorname{Re}\lambda > \lambda_0$.

2.1.1 Watson Lemma

Lemma 2 (Watson) *Let* $0 < a < \infty$, $\alpha > 0, \beta > 0$ *and let* S_ε *be the sector* $S_\varepsilon = \{\lambda \in \mathbb{C} \mid |\arg\lambda| \le \pi/2 - \varepsilon\}$ *in the complex plane* λ*. Let* $\varphi \in C^\infty([0, a])$ *and let*

$$\Phi(\lambda) = \int_0^a \varphi(x) x^{\beta-1} \exp(-\lambda x^\alpha)\, dx \tag{2.3}$$

Then, there is an asymptotic expansion as $\lambda \to \infty$, $\lambda \in S_\varepsilon$,

$$\Phi(\lambda) \sim \frac{1}{\alpha} \sum_{k=0}^{\infty} \lambda^{-(\beta+k)/\alpha} \Gamma\left(\frac{\beta+k}{\alpha}\right) \frac{\varphi^{(k)}(0)}{k!} \tag{2.4}$$

Laplace Transform. Let $\varphi \in C^\infty(\mathbb{R}_+)$ be a smooth function on the positive real axis such that its Laplace transform

$$\mathcal{L}(\varphi)(\lambda) = \int_0^\infty \varphi(x) e^{-\lambda x}\, dx \tag{2.5}$$

converges absolutely for some λ_0. Then

$$\mathcal{L}(\varphi)(\lambda) \sim \sum_{k=0}^{\infty} \lambda^{-k} \varphi^{(k)}(0) \qquad (|\lambda| \to \infty,\ \lambda \in S_\varepsilon) \tag{2.6}$$

2.1.2 Interior Nondegenerate Maximum Point

Let now S and φ be smooth functions and the function S have a maximum at an interior point x_0 of the interval $[a, b]$, i.e. $a < x_0 < b$. Then $S'(x_0) = 0$. Assume, for simplicity, that $S''(x_0) \neq 0$. Then $S''(x_0) < 0$. In other words, in a neighborhood of x_0 the function S has the following Taylor expansion

$$S(x) = S(x_0) + S''(x_0)\frac{(x-x_0)^2}{2} + O((x-x_0)^3)\ . \tag{2.7}$$

Such a point is called **nondegenerate critical point**.

Then, as $\lambda \to \infty$ the main contribution to the integral comes from a small neighborhood of x_0. In this neighborhood the function φ is almost constant and can be replaced by its value at x_0. The terms of order $(x - x_0)^3$ can be neglected in the exponent and the remaining integral can be extended to the whole real line. By using the **standard Gaussian integral**

$$\int_{-\infty}^{\infty} \exp\left(-\frac{\alpha}{2}\, y^2\right) dy = \sqrt{\frac{2\pi}{\alpha}}, \qquad (\operatorname{Re}\alpha > 0)\,, \tag{2.8}$$

one obtains finally the main term of the asymptotics

$$F(\lambda) \sim \lambda^{-1/2} \sqrt{\frac{2\pi}{-S''(x_0)}}\, \varphi(x_0) e^{\lambda S(x_0)}, \qquad (\lambda \to \infty) \tag{2.9}$$

One can prove the general theorem.

Theorem 6 *Let $M = [a, b]$ and $\varphi, S \in C^\infty(M)$, S has a maximum only at one point x_0, $a < x_0 < b$ and $S''(x_0) \neq 0$. Then as $\lambda \to \infty, \lambda \in S_\varepsilon$ there is asymptotic expansion*

$$F(\lambda) \sim e^{\lambda S(x_0)} \sum_{k=0}^{\infty} c_k \lambda^{-1/2-k} . \tag{2.10}$$

The coefficients c_k are expressed in terms of derivatives of φ and S at x_0.

The theorem can be proved as follows. First, we change the integration variable

$$x = x_0 + \lambda^{-1/2} y . \tag{2.11}$$

So, y is the scaled fluctuation from the maximum point x_0. The interval of integration should be changed accordingly, so that the maximum point is now $y = 0$. Then, we expand both functions S and φ in Taylor series at x_0 getting

$$\lambda S(x_0 + \lambda^{-1/2} y) = \lambda S(x_0) + \frac{1}{2} S''(x_0) y^2 + \sum_{n=3}^{\infty} \frac{S^{(n)}(x_0)}{n!} y^n \lambda^{-(n-2)/2} , \tag{2.12}$$

$$\varphi(x_0 + \lambda^{-1/2} y) = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(x_0)}{n!} y^n \lambda^{-n/2} . \tag{2.13}$$

Since the quadratic terms are of order $O(1)$ we leave it in the exponent and expand the exponent of the rest in a power series. Next, we extend the integration interval to the whole real line and compute the standard Gaussian integrals of the form

$$\int_{-\infty}^{\infty} \exp\left(-\frac{\alpha}{2} y^2\right) y^{2k+1} \, dy = 0 , \tag{2.14}$$

$$\int_{-\infty}^{\infty} \exp\left(-\frac{\alpha}{2} y^2\right) y^{2k} \, dy = \Gamma\left(k + \frac{1}{2}\right) \left(\frac{\alpha}{2}\right)^{-k-1} , \tag{2.15}$$

where k is a nonnegative integer and α has a positive real part, $\mathrm{Re}\,\alpha > 0$. Finally, we get a power series in inverse powers of λ. The coefficients c_k of the asymptotic expansion are polynomials in the higher derivatives $S^{(k)}(x_0)$, $k \geq 3$, and derivatives $\varphi^{(l)}(x_0)$, $l \geq 0$, and involve inverse powers of $S''(x_0)$.

Stirling Formula

$$\Gamma(x+1) = \sqrt{2\pi} x^{x+1/2} e^{-x} [1 + O(x^{-1})], \qquad (x \to \infty) \tag{2.16}$$

is obtained by applying the Laplace method to the integral

$$\Gamma(x+1) = x^{x+1} \int_0^\infty \exp[x(\ln t - t)]\, dt \tag{2.17}$$

Stieltjes Transform Let $\varphi : \mathbb{R}_+ \to \mathbb{C}$ have finite moments

$$m_n(\varphi) = \int_0^\infty t^n \varphi(t)\, dt < \infty \qquad \forall n \in \mathbb{N}. \tag{2.18}$$

Then the Stieltjes transform of φ

$$\mathcal{S}(\varphi)(x) = \int_0^\infty \frac{\varphi(t)}{t+x}\, dt \tag{2.19}$$

has asymptotic expansion as $x \to \infty$

$$\mathcal{S}(\varphi)(x) \sim \sum_{k=0}^\infty (-1)^k m_k(\varphi) x^{-1-k} \tag{2.20}$$

2.1.3 Boundary Maximum Point

Let the function S have a maximum at a boundary point $x_0 = a$. Let, for simplicity, $S'(a) \neq 0$, i.e. $S'(a) < 0$, and $\varphi(a) \neq 0$. Then, as $\lambda \to \infty$, the main contribution to the integral comes from the interval $[a, a+\epsilon]$, where

$$S(x) = S(a) + (x-a)S'(a) + O((x-a)^2)\,. \tag{2.21}$$

Now, by replacing the function φ by its value at a and neglecting nonlinear terms in $S(x)$, we get

$$F(\lambda) \sim \lambda^{-1} \frac{\varphi(a)}{-S'(a)} e^{\lambda S(a)}, \quad (\lambda \to \infty) \tag{2.22}$$

In this way one can prove the following theorem.

Theorem 7 *Let $M = [a,b]$, $\varphi, S \in C^\infty(M)$, S has a maximum only at the point $x = a$ and $S'(a) \neq 0$. Then, as $\lambda \to \infty$, $\lambda \in S_\varepsilon$, there is asymptotic expansion*

$$F(\lambda) \sim e^{\lambda S(a)} \sum_{k=0}^\infty c_k \lambda^{-1-k}\,. \tag{2.23}$$

The coefficients c_k are expressed in terms of derivatives of φ and S at $x = a$.

Error Function The asymptotic expansion of the (complementary) error function as $x \to \infty$ has the form

$$\operatorname{Erfc} x = \int_x^\infty e^{-t^2}\,dt \sim \frac{e^{-x^2}}{2x}\sum_{k=0}^\infty (-1)^k \frac{(2k-1)!!}{2^k} x^{-2k} \qquad (x \to \infty)\,. \tag{2.24}$$

Incomplete Gamma Function The incomplete gamma-function

$$\gamma(a,x) = \int_0^x t^{a-1} e^{-t}\,dt, \qquad (0 < a < \infty,\ x > 0) \tag{2.25}$$

has the following asympotic expansion as $x \to \infty$

$$\gamma(a,x) = \Gamma(a) + e^{-x} x^{a-1} \sum_{k=0}^\infty \frac{\Gamma(a)}{\Gamma(a-k)} x^{-k} \tag{2.26}$$

2.2 Background from Analysis

2.2.1 Definitions

1. Let x^j, $j = 1, \dots, n$ be real numbers and let x be the n-tuple $x = (x^1, \dots x^n)$. The set of all n-tuples of real numbers is denoted by $\mathbb{R}^n$. A *connected open subset* Ω of $\mathbb{R}^n$ is called a **domain**. The set $\partial\Omega$ of boundary points of Ω is called the **boundary** of Ω. The union $\Omega \cup \partial\Omega$ is called the **closure** of Ω and is denoted by $\overline{\Omega}$.

2. Let $x, \xi \in \mathbb{R}^n$. Then the **scalar product** of x and ξ is defined by $x \cdot \xi = x^1\xi^1 + \cdots + x^n\xi^n$.

3. On $\mathbb{R}^n$ there is standard Lebesgue measure $dx = dx^1 \cdots dx^n$.

4. Let α_j, $j = 1, \dots, n$, be non-negative integers, $\alpha_j \geq 0$. The n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ is called a **multi-index**. Further, let

$$|\alpha| = \alpha_1 + \cdots + \alpha_n \tag{2.27}$$

$$\alpha! = \alpha_1! \cdots \alpha_n! \tag{2.28}$$

$$\partial^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial(x^1)^{\alpha_1} \cdots \partial(x^n)^{\alpha_n}} \tag{2.29}$$

$$D^\alpha f(x) = \left(\frac{1}{i}\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{1}{i}\frac{\partial}{\partial x^n}\right)^{\alpha_n} f(x) \tag{2.30}$$

Then the Taylor expansion of a smooth function φ at x_0 can be written in the form

$$\varphi(x) = \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!}[\partial^\alpha \varphi(x_0)]\,(x - x_0)^\alpha\,. \tag{2.31}$$

5. The boundary $\partial\Omega$ is said to be **smooth**, denoted $\partial\Omega \in C^\infty$, if in a neighborhood of any boundary point $x_0 \in \partial\Omega$ it can be locally defined by an equation $x_j = \varphi(x')$ with a smooth function φ.

6. The set of all continuous functions on $\overline{\Omega}$ is denoted by $C(\overline{\Omega})$.

7. The set of all functions with continuous partial derivatives up to order k on Ω is denoted by $C^k(\Omega)$.

8. The set of all functions with continuous partial derivatives up to order k on $\overline{\Omega}$ is denoted by $C^k(\overline{\Omega})$.

9. The set of all functions with continuous partial derivatives up to order k on Ω that vanish in a neighborhood of the boundary $\partial\Omega$ is denoted by $C^k_0(\overline{\Omega})$.

10. The closure of the set where a function is not equal to zero is called the **support** of the function, denoted by

$$\operatorname{supp} f = \overline{\{x \in \Omega \mid f(x) \neq 0\}}\,. \tag{2.32}$$

11. A mapping $\varphi : \Omega \to \Omega$, is said to be of class C^k if $\varphi \in C^k(\Omega)$.

12. A one-to-one mapping $\varphi : \Omega \to \Omega$ of Ω onto Ω is called **diffeomorphism** of class C^k if $\varphi \in C^k(\Omega)$ and $\varphi^{-1} \in C^k(\Omega)$.

13. Let φ^j, $j = 1, \dots, k$ be some scalar functions on $\mathbb{R}^n$ and φ be a k-tuple $\varphi = (\varphi^1, \dots, \varphi^k)$. In other words, $\varphi : \mathbb{R}^n \to \mathbb{R}^k$. The matrix

$$\partial_x \varphi(x) = \left(\frac{\partial \varphi^i(x)}{\partial x^j}\right), \quad i = 1, \dots, k; \quad j = 1, \dots, n \tag{2.33}$$

is called the **Jacobi matrix**.

Theorem 8 (Inverse Function Theorem) *Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is of class C^k, $k \geq 1$, in a neighborhood of a point x_0 and $\det \partial_x \varphi(x_0) \neq 0$. Then φ is local diffeomorphism of class C^k in a neighborhood of the point x_0.*

Theorem 9 (Implicit Function Theorem) *Let Ω be a domain in $\mathbb{R}^{2n}$, let $F : \Omega \to \mathbb{R}^n$ be a mapping of class $C^k(\Omega)$ and let $(x_0, y_0) \in \Omega$ be a point in Ω such that*

$$F(x_0, y_0) = 0, \qquad \det \partial_y F(x_0, y_0) \neq 0 \,. \tag{2.34}$$

Then in a neighborhood of the point x_0 there is a mapping $y = f(x)$ of class C^k such that $y_0 = f(x_0)$ and

$$F(x, f(x)) \equiv 0\,. \tag{2.35}$$

2.2.2 Morse Lemma

Let $S : \Omega \to \mathbb{R}$ be a real valued function of class C^k on a domain Ω in $\mathbb{R}^n$ with $k \geq 2$. Let

$$\partial_x^2 S(x) = \left(\frac{\partial^2 S(x)}{\partial x^i \partial x^j} \right), \qquad i, j = 1, \dots, n \,. \tag{2.36}$$

Definition 5 *1. The point x_0 is called a **critical point** of the function S if $\partial S(x_0) = 0$*

*2. A critical point x_0 is called **non-degenerate** if $\det \partial_x^2 S(x_0) \neq 0$.*

*3. The determinant $\det \partial_x^2 S(x_0)$ is called the **Hessian** of the function S at the point x_0.*

Lemma 3 (Morse) *Let $S : \mathbb{R}^n \to \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ be a non-degenerate critical point of the function S. Let $S \in C^\infty$ in a neighborhood of the point x_0 and let $\mu_j \neq 0$, $j = 1, \dots, n$ be the eigenvalues of the matrix $\partial_x^2 S(x_0)$. Then there are neighborhoods U and V of the points x_0 and 0 and a smooth local diffeomorphism $\varphi : V \to U$ of class C^∞ such that $\det \partial_y \varphi(0) = 1$ and*

$$S(\varphi(y)) = S(x_0) + \frac{1}{2} \sum_{j=1}^{n} \mu_j (y^j)^2 \,. \tag{2.37}$$

Remark. Nondegenerate critical points are isolated.

2.2.3 Gaussian Integrals

Proposition 1 *Let $A = (a_{ij}$ be a complex symmetric nondegenerate $n \times n$ matrix with the eigenvalues $\mu_j(A), j = 1, \dots, n,$. Let $\mathrm{Re}\, A \geq 0$, which means that $x \cdot \mathrm{Re}\, A\, x \geq 0 \quad \forall x \in \mathbb{R}^n, x \neq 0$, or $\mathrm{Re}\, \mu_j(A) \geq 0$, $j = 1, \dots, n$. Then for $\lambda > 0$, $\xi \in \mathbb{R}^n$ there holds*

$$\int_{\mathbb{R}^n} \exp\left(-\frac{\lambda}{2}\, x \cdot A\, x - i\xi \cdot x\right) dx$$
$$= \left(\frac{2\pi}{\lambda}\right)^{n/2} (\det A)^{-1/2} \exp\left(-\frac{1}{2\lambda}\, \xi \cdot A^{-1}\, \xi\right) . \qquad (2.38)$$

The branch of $\sqrt{\det A}$ is choosen as follows

$$(\det A)^{-1/2} = |\det A|^{-1/2} \exp\left(-i\, \mathrm{Ind}\, A\right) , \qquad (2.39)$$

where

$$\mathrm{Ind}\, A = \frac{1}{2} \sum_{j=1}^{n} \arg \mu_j(A), \qquad |\arg \mu_j(A)| \leq \frac{\pi}{2} . \qquad (2.40)$$

By expanding both sides of this equation in Taylor series in ξ we obtain the following result.

Corollary 1

$$\int_{\mathbb{R}^n} \exp\left(-\frac{\lambda}{2}\, x \cdot A\, x\right) x^{i_1} \cdots x^{i_{2k+1}} dx = 0 \qquad (2.41)$$

$$\int_{\mathbb{R}^n} \exp\left(-\frac{\lambda}{2}\, x \cdot A\, x\right) x^{i_1} \cdots x^{i_{2k}}\, dx$$
$$= \left(\frac{2\pi}{\lambda}\right)^{n/2} (\det A)^{-1/2} (2\lambda)^{-k} \frac{(2k)!}{k!}\, G^{(i_1 i_2} \cdots G^{i_{2k-1} i_{2k})} . \qquad (2.42)$$

Here k is a non-negative integer, $G = A^{-1}$, and the round brackets denote complete symmetrization over all indices included.

An important particular case of the previous formula is when the matrix A is real.

Proposition 2 *Let A be a real symmetric nondegenerate $n\times n$ matrix. Let $\nu_+(A)$ and $\nu_-(A)$ be the numbers of positive and negative eigenvalues of A and*

$$\operatorname{sgn} A = \nu_+ - \nu_- \tag{2.43}$$

be the signature of the matrix A. Then for $\lambda>0,\ \xi\in\mathbb{R}^n$ there holds

$$\int_{\mathbb{R}^n} \exp\left(i\frac{\lambda}{2}\, x\cdot A\, x - i\xi\cdot x\right) dx$$
$$= \left(\frac{2\pi}{\lambda}\right)^{n/2} |\det A|^{-1/2} \exp\left(-\frac{i}{2\lambda}\,\xi\cdot A^{-1}\,\xi + \frac{i\pi}{4}\operatorname{sgn}(A)\right). \tag{2.44}$$

2.3 Laplace Integrals in Many Dimensions

2.3.1 Interior Maximum Point

Let Ω be a bounded domain in $\mathbb{R}^n$, $S:\Omega\to\mathbb{R}$, $f:\Omega\to\mathbb{C}$ are some functions on Ω and $\lambda>0$ be a large positive parameter. We will study the asymptotics as $\lambda\to\infty$ of the multidimensional Laplace integrals

$$F(\lambda) = \int_\Omega f(x)\exp[\lambda S(x)]\,dx\,. \tag{2.45}$$

Let S and f be smooth functions and the function S have a maximum only at one interior nondegenerate critical point $x_0\in\Omega$. Then $\partial_x S(x_0)=0$ and $[\det\partial_x^2 S(x_0)]<0$. Then in a neighborhood of x_0 the function S has the following Taylor expansion

$$S(x) = S(x_0) + \frac{1}{2}(x-x_0)\cdot[\partial_x^2 S(x_0)](x-x_0) + O((x-x_0)^3)\,. \tag{2.46}$$

One could also use the Morse Lemma to replace the function S by a quadratic form. Then as $\lambda\to\infty$ the main contribution to the integral comes from a small neghborhood of x_0. In this neighborhood the terms of the third order in the Taylor expansion of S can be neglected. Also, since the function f is continuous at x_0, it can be replaced by its value at x_0. Then the region of integration can be extended to the whole $\mathbb{R}^n$. By using the formula for the standard Gaussian integral one gets then the leading asymptotics of the integral $F(\lambda)$ as $\lambda\to\infty$

$$F(\lambda)\sim\exp[\lambda S(x_0)]\left(\frac{2\pi}{\lambda}\right)^{n/2}\left[-\det\partial_x^2 S(x_0)\right]^{-1/2} f(x_0)\,. \tag{2.47}$$

One can prove the general theorem.

Theorem 10 *Let $f, S \in C^\infty(\Omega)$ and let x_0 be a nondegenerate critical point of the function S where it has the only maximum in Ω. Let $0 < \varepsilon < \pi/2$. Then there is asymptotic expansion as $\lambda \to \infty$ in the sector $S_\varepsilon = \{\lambda \in \mathbb{C} \mid |\arg\lambda| \le \pi/2 - \varepsilon\}$*

$$F(\lambda) \sim \exp[\lambda S(x_0)]\lambda^{-n/2}\sum_{k=0}^{\infty} a_k \lambda^{-k}\,. \tag{2.48}$$

The coefficients a_k are expressed in terms of the derivatives of the functions f and S at the point x_0.

The idea of the proof is the same as in the one-dimensional case and goes as follows. First, we change the integration variables

$$x^i = x_0^i + \lambda^{-1/2}y^i\,. \tag{2.49}$$

So, y is the scaled fluctuation from the maximum point x_0. The interval of integration should be changed accordingly, so that the maximum point is now $y = 0$. Then, we expand both functions S and φ in Taylor series at x_0 getting

$$\lambda S(x_0 + \lambda^{-1/2}y) = \lambda S(x_0) + \frac{1}{2}y\cdot[\partial_x^2 S(x_0)]y + \sum_{|\alpha|=3}^{\infty}\frac{\lambda^{-(|\alpha|-2)/2}}{\alpha!}[\partial^\alpha S(x_0)]\,y^\alpha\,, \tag{2.50}$$

$$\varphi(x_0 + \lambda^{-1/2}y) = \sum_{|\alpha|=0}^{\infty}\frac{\lambda^{-|\alpha|/2}}{\alpha!}\,\partial^\alpha\varphi(x_0)\,y^\alpha\,. \tag{2.51}$$

Since the quadratic terms are of order $O(1)$ we leave them in the exponent and expand the exponent of the rest in a power series. Next, we extend the integration domain to the whole $\mathbb{R}^n$ and compute the standard Gaussian integrals. Finally, we get a power series in inverse powers of λ. The coefficients a_k of the asymptotic expansion are polynomials in the higher derivatives $\partial^\alpha S(x_0)$, $|\alpha| \ge 3$, and derivatives $\partial^\alpha\varphi(x_0)$, $|\alpha| \ge 0$, and involve inverse matrices $G = [\partial_x^2 S(x_0)]^{-1}$.

Remark. If x_0 is a **degenerate** maximum point of the function S, then the asymptotic expansion as $\lambda \to \infty$ has the form

$$F(\lambda) \sim \exp[\lambda S(x_0)]\lambda^{-n/2} \sum_{k=0}^{\infty} \sum_{l=0}^{N} a_{kl} \lambda^{-r_k} (\ln \lambda)^l \,, \tag{2.52}$$

where N is some positive integer and $\{r_k\}$, $r_k \geq n/2$, $k \in \mathbb{N}$, is a increasing sequence of nonnegative rational numbers.

The coefficients a_k (and a_{kl}) of the asymptotic expansion of the integral $F(\lambda)$ are **invariants** under smooth local diffeomorphisms in a neighborhood of x_0 and play very important role in various applications.

2.3.2 Boundary Maximum Point

Let now S has maximum at the boundary point $x_0 \in \partial\Omega$. We assume, for simplicity, that both the boundary and the function S are smooth, i.e. $S \in C^\infty$ and $\partial\Omega \in C^\infty$.

Since the boundary is smooth we can smoothly parametrize it in a neighborhood of x_0 by $(n-1)$ parameters $\xi = (\xi^a)$, $(a = 1, \dots, n-1)$. Let the the parametric equations of the boundary be

$$x^i = x^i(\xi), \qquad i = 1, \dots, n \,. \tag{2.53}$$

Then

$$T_a = (T^i_a) = \left(\frac{\partial x^i}{\partial \xi^a}\right) \tag{2.54}$$

are tangent vectors to the boundary.

Let $r = r(x)$ be the *normal distance* to the boundary. Then the equation of the boundary can be written as

$$r(x) = 0 \,. \tag{2.55}$$

and for $x \in \Omega$ we have $r > 0$. Obviously, $r(x(\xi)) \equiv 0$. From this equation we obtain that the vector

$$N = (N_i) = \left(\frac{\partial r}{\partial x^i}\right) \tag{2.56}$$

is orthogonal to all tangent vectors and is therefore normal to the boundary. It can be certainly normalized, since it is nowhere zero. We choose it to be the *inward* normal.

The normal and tangential derivatives are defined as usual

$$\partial_r = \sum_{i=1}^{n} \frac{\partial x^i}{\partial r}\frac{\partial}{\partial x^i}, \qquad \frac{\partial}{\partial \xi^a} = \sum_{i=1}^{n} \frac{\partial x^i}{\partial \xi^a}\frac{\partial}{\partial x^i} \tag{2.57}$$

The point x_0 is *not*, in general, a critical point of S, since the normal derivative of S at x_0 does not have to be equal to zero.

Definition 6 *The point x_0 is said to be a* ***nondegenerate boundary maximum point*** *of S if*

$$\partial_r S(x_0) \neq 0 \tag{2.58}$$

and the $(n-1)\times(n-1)$ matrix $\partial_\xi^2 S(x(\xi))$ is negative definite.

In a neighborhood of a nondegenerate boundary maximum point the function S has the following Taylor expansion

$$\begin{aligned} S(x) \;=\; & S(x_0) + [\partial_r S(x_0)]r + \frac{1}{2}[\partial_r^2 S(x_0)]r^2 \\ & + [\partial_r\partial_\xi S(x_0)]\cdot(\xi-\xi_0)r + \frac{1}{2}(\xi-\xi_0)\cdot[\partial_\xi^2 S(x_0)](\xi-\xi_0) \\ & + \cdots\,, \end{aligned} \tag{2.59}$$

up to third order terms in r and $(\xi - \xi_0)$.

Now we replace the integral $F(\lambda)$ by an integral over a small neighborhood of x_0. We change the variables of integration from x^i, $i = 1, \dots, n$, to (ξ^a, r), $a = 1, \dots, n-1$, and neglect the terms of third order in the Taylor series. We also replace the function f by its value at the point x_0. In the remaining integral we extend the integration to the whole space $\mathbb{R}_+ \times \mathbb{R}^{n-1}$, i.e. we integrate over r from 0 to ∞ and integrate over the whole tangent plane at x_0. These integrals are standard Gaussian integrals and we obtain the leading asymptotics as $\lambda \to \infty$

$$\begin{aligned} F(\lambda) \;\sim\; & -\lambda^{-(n+1)/2}(2\pi)^{(n-1)/2}\exp[\lambda S(x_0)] \\ & \times[\partial_r S(x_0)]^{-1}[-\det\partial_\xi^2 S(x_0)]^{-1/2} J(x_0) f(x_0) \end{aligned} \tag{2.60}$$

where $J(x_0)$ is the Jacobian of change of variables.

The general form of the asymptotic expansion is given by the following theorem.

Theorem 11 *Let $f, S \in C^\infty(\Omega)$ and let S have a maximum only at a non-degenerate boundary maximum point $x_0 \in \partial\Omega$. Then as $\lambda \to \infty$, $\lambda \in S_\varepsilon$,*

$$F(\lambda) \sim \lambda^{-(n+1)/2} \exp[\lambda S(x_0)] \sum_{k=0}^{\infty} a_k \lambda^{-k} \tag{2.61}$$

2.3.3 Integral Operators with Singular Kernels

Let Ω be a bounded domain in $\mathbb{R}^n$ including the origin, $0 \in \Omega$. Let S be a real valued non-positive function on Ω of class C^2 that has maximum equal to zero, $S(0) = 0$, only at a nondegenerate maximum critical point $x_0 = 0$. Let $K_\lambda : C^\infty(\Omega) \to C^\infty(\Omega)$ be a linear integral operator defined by

$$(K_\lambda f)(x) = \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_\Omega \exp[\lambda S(x-y)] f(y)\, dy\,. \tag{2.62}$$

Let M be a compact subset of Ω. Then

$$\lim_{\lambda\to\infty} (K_\lambda f)(x) = [-\det \partial_x^2 S(0)]^{-1/2} f(x) \tag{2.63}$$

uniformly for $x \in M$.

Formally

$$\left(\frac{\lambda}{2\pi}\right)^{n/2} \exp[\lambda S(x-y)] \longrightarrow [-\det \partial_x^2 S(0)]^{-1/2}\, \delta(x-y) \tag{2.64}$$

Chapter 3

Stationary Phase Method

3.1 Stationary Phase Method in One Dimension

3.1.1 Fourier Integrals

Let $M = [a, b]$ be a closed bounded interval, $S : M \to \mathbb{R}$ be a real valued nonconstant function, $f : M \to \mathbb{C}$ be a complex valued nonzero function and λ be a large positive parameter. Consider the integrals of the form

$$F(\lambda) = \int_a^b f(x)\, \exp[i\lambda S(x)]\, dx\,. \tag{3.1}$$

The function S is called **phase function** and such integrals are called **Fourier Integrals**. We will study the asymptotics of such integrals.

As $\lambda \to \infty$ the integral $F(\lambda)$ is small due to rapid oscillations of $\exp(i\lambda S)$.

Lemma 4 (Riemann-Lebesgue) *Let f be an integrable function on the real line, i.e. $f \in L^1(\mathbb{R})$. Then*

$$\int_{\mathbb{R}} f(x) e^{i\lambda x}\, dx = o(1), \qquad (\lambda \to \infty)\,. \tag{3.2}$$

Definition 7 *1. A point x_0 is called the **regular point** of the Fourier integral $F(\lambda)$ if the functions f and S are smooth in a neighborhood of x_0 and $S'(x_0) \neq 0$.*

2. *A point x_0 is called the* ***critical point*** *of the integral $F(\lambda)$ if it is not a regular point.*

3. *A critical point x_0 is called* ***isolated critical point*** *if there is a neighborhood of x_0 that does not contain any other critical points.*

4. *An interior isolated critical point is called* ***stationary point****.*

5. *The integral over a neighborhood of an isolated critical point that does not contain other critical points will be called the* ***contribution of the critical point*** *to the integral.*

Clearly the main contribution comes from the critical points since close to these points the oscillations slow down. As always, we will assume that functions S and f are smooth, i.e. of class $C^\infty(M)$. Otherwise, the **singularities** of the functions S and f and their derivatives would contribute significantly to $F(\lambda)$.

3.1.2 Localization Principle

Lemma 5 *Let $S \in C^\infty(\mathbb{R})$ be smooth function and $f \in C_0^\infty(\mathbb{R})$ be a smooth function of compact support. Then as $\lambda \to \infty$*

$$\int_{\mathbb{R}} f(x) \exp[i\lambda S(x)]\, dx = O(\lambda^{-\infty}) \tag{3.3}$$

Remarks.

1. Since the function f has compact support, the integral is, in fact, over a finite interval.

2. This is the main technical lemma for deriving the (power) asympotics of the Fourier integrals. It means that such integrals can be neglected in a power asymptotic expansion.

3. The Fourier integrals are in general much more subtle object than the Laplace integrals. Instead of exponentially decreasing integrand one has a rapidly oscillating one. This requires much finer estimates and also much stronger conditions on the phase function S and the integrand f.

Theorem 12 *Let the Fourier integral $F(\lambda)$ have finite number of isolated critical points. Then as $\lambda \to \infty$ the integral $F(\lambda)$ is equal to the sum of the contributions of all critical points up to $O(\lambda^{-\infty})$.*

Thus, the problem reduces to computing the asymptotics of the contributions of critical points. In a neighborhood of a critical point we can replace the functions S and f by more simple functions and then compute some standard integrals.

3.1.3 Boundary Points

If the phase function does not have any stationary points, then by integration by parts one can easily obtain the asymptotic expansion.

Theorem 13 *Let $S'(x) \neq 0 \quad \forall x \in M$. Then as $\lambda \to \infty$*

$$F(\lambda) \sim \sum_{k=0}^{\infty} (i\lambda)^{-k-1} \left(\frac{1}{-S'(x)} \frac{\partial}{\partial x} \right)^k \left(\frac{f(x)}{S'(x)} \right) e^{i\lambda S(x)} \Bigg|_a^b . \tag{3.4}$$

The leading asymptotics is

$$F(\lambda) = (i\lambda)^{-1} \{ f(b) \exp[i\lambda S(b)] - f(a) \exp[i\lambda S(a)] \} + O(\lambda^{-2}) \tag{3.5}$$

The same technique, i.e. integration by parts, applies to the integrals over an unbounded interval, say,

$$F(\lambda) = \int_0^{\infty} f(x) \, \exp[i\lambda S(x)] \, dx \, . \tag{3.6}$$

with some additional conditions that guarantee the converges at ∞ as well as to the integrals of the form

$$F(x) = \int_x^{\infty} f(t) \, \exp[iS(t)] \, dt \tag{3.7}$$

as $x \to \infty$.

Examples

1. Let $\alpha > 0$. Then as $\lambda \to \infty$

   (a)
$$\int_0^\infty \frac{e^{i\lambda x}}{(1+x)^\alpha}\,dx = i\lambda^{-1} + O(\lambda^{-2})\,. \tag{3.8}$$

   (b)
$$\int_0^\infty \frac{\sin(\lambda x)}{(1+x)^\alpha}\,dx = \lambda^{-1} + O(\lambda^{-2})\,. \tag{3.9}$$

   (c)
$$\int_0^\infty \frac{\cos(\lambda x)}{(1+x)^\alpha}\,dx = \alpha\lambda^{-2} + O(\lambda^{-3})\,. \tag{3.10}$$

2. Let $\alpha > 0$. Then as $x \to \infty$

$$F(x) = \int_x^\infty t^{-\alpha}\, e^{it}\, dt \sim ie^{ix}x^{-\alpha}\sum_{k=0}^\infty \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)}(ix)^{-k}\,. \tag{3.11}$$

   In particular, the Frenel integral has the asymptotic expansion as $x \to \infty$

$$\Phi(x) = \int_x^\infty t^{-\alpha}\, e^{it}\, dt \sim \frac{i}{2\sqrt{\pi}}e^{ix^2}x^{-1/2}\sum_{k=0}^\infty (-1)^k\Gamma\left(k+\frac{1}{2}\right)x^{-k}\,. \tag{3.12}$$

3.1.4 Standard Integrals

Consider the integral

$$\Phi(\lambda) = \int_0^a f(x)x^{\beta-1}e^{i\lambda x^\alpha} \tag{3.13}$$

Lemma 6 (Erdéyi) *Let $\alpha \geq 1$, $\beta > 0$ and f is a smooth function on a closed bounded interval $[0,a]$, $f \in C^\infty([0,a])$, that vanish at $x = a$ with all its derivatives. Then as $\lambda \to \infty$*

$$\Phi(\lambda) = \int_0^a f(x)x^{\beta-1}e^{i\lambda x^\alpha} \sim \sum_{k=0}^\infty a_k\lambda^{-(\beta+k)/\alpha}\,, \tag{3.14}$$

where

$$a_k = \frac{f^{(k)}(0)}{k!}\frac{1}{\alpha}\Gamma\left(\frac{\beta+k}{\alpha}\right)\exp\left[i\pi\frac{\beta+k}{\alpha}\right] \tag{3.15}$$

This lemma plays he same role in the stationary phase method as Watson lemma in the Laplace method.

3.1.5 Stationary Point

Theorem 14 *Let $M = [a,b]$ be a closed bounded interval, $S \in C^\infty(M)$ be a smooth real valued nonconstant function, $f \in C_0^\infty(M)$ be a complex valued function with compact support in M. Let S have a single isolated nondegenerate critical point x_0 in M, i.e. $S'(x_0) = 0$ and $S''(x_0) \neq 0$. Then as $\lambda \to \infty$ there is asymptotic expansion of the Fourier integral*

$$\begin{aligned} F(\lambda) &= \int_a^b f(x)\exp[i\lambda S(x)]\,dx \\ &\sim \exp\left[i\lambda S(x_0) + [\operatorname{sgn} S''(x_0)]\,i\,\frac{\pi}{4}\right]\lambda^{-1/2}\sum_{k=0}^\infty \lambda^{-k}\,. \end{aligned} \tag{3.16}$$

The coefficients a_k are determined in terms of the derivatives of the functions S and f at x_0.

The leading asymptotics as $\lambda \to \infty$ is

$$F(\lambda) = \sqrt{\frac{2\pi}{|S''(x_0)|}}\lambda^{-1/2}\exp\left[i\lambda S(x_0) + [\operatorname{sgn} S''(x_0)]\,i\,\frac{\pi}{4}\right]\left(f(x_0) + O(\lambda^{-1})\right)\,. \tag{3.17}$$

To prove this theorem one does a change of variables in a sufficiently small neghborhood of x_0 and applies the Erdélyi lemma.

3.1.6 Principal Values of Integrals

Let f be a smooth function and consider the integral

$$\int_a^b \frac{f(x)}{x}\,dx \tag{3.18}$$

This integral diverges, in general, at $x = 0$. One can regularize it by cutting out a *symmetric* neighborhood of the singular point

$$I(\varepsilon) = \int_a^{-\varepsilon} \frac{f(x)}{x}\,dx + \int_\varepsilon^b \frac{f(x)}{x}\,dx\,. \tag{3.19}$$

Definition 8 *If the limit of $I(\varepsilon)$ as $\varepsilon \to 0+$ exists, then it is called the **principal value** of the integral I*

$$\mathcal{P}\int_a^b \frac{f(x)}{x}\,dx = \lim_{\varepsilon\to 0^+}\left(\int_a^{-\varepsilon} \frac{f(x)}{x}\,dx + \int_\varepsilon^b \frac{f(x)}{x}\,dx\right)\,. \tag{3.20}$$

In this section we consider the asymptotics of the integrals of the form

$$F(\lambda) = \mathcal{P}\int_{\mathbb{R}} e^{\pm i\lambda S(x)}\, f(x)\,\frac{dx}{x} \tag{3.21}$$

as $\lambda \to \infty$.

Lemma 7 *Let $f \in C_0^\infty(\mathbb{R})$ be a smooth function of compact support. Then as $\lambda \to \infty$*

$$\mathcal{P}\int_{\mathbb{R}} e^{\pm i\lambda x}\, f(x)\,\frac{dx}{x} = \pm i\pi f(0) + O(\lambda^{-\infty})\,. \tag{3.22}$$

Theorem 15 *Let $f \in C_0^\infty(\mathbb{R})$ be a smooth function of compact support, $S \in C^\infty(\mathbb{R})$ be a real valued smooth function and $S'(0) \neq 0$. Then as $\lambda \to \infty$*

$$F(\lambda) = \mathcal{P}\int_{\mathbb{R}} e^{\pm i\lambda S(x)}\, f(x)\,\frac{dx}{x} = [\operatorname{sgn} S'(0)]\, i\pi f(0)\exp[i\lambda S(0)] + O(\lambda^{-\infty})\,. \tag{3.23}$$

Theorem 16 *Let $f \in C_0^\infty(\mathbb{R})$ be a smooth function of compact support, $S \in C^\infty(\mathbb{R})$ be a real valued smooth function. Let $x = 0$ be the only statinary point of the function S on* $\operatorname{supp} f$*, and let it be nondegenerate, i.e. $S'(0) = 0$ and $S''(0) \neq 0$. Then as $\lambda \to \infty$ there is asymptotic expansion*

$$\begin{aligned} F(\lambda) &= \mathcal{P}\int_{\mathbb{R}} e^{\pm i\lambda S(x)}\, f(x)\,\frac{dx}{x} \\ &\sim \exp[i\lambda S(0)]\lambda^{-1/2}\sum_{k=0}^\infty a_k\lambda^{-k}\,. \end{aligned} \tag{3.24}$$

The leading asymptotics has the form

$$F(\lambda) = \exp\left[i\lambda S(0) + [\operatorname{sgn} S''(0)]\, i\,\frac{\pi}{4}\right]\sqrt{\frac{2\pi}{|S''(0)|}}$$

$$\times\lambda^{-1/2}\left[-\frac{S'''(0)}{6S''(0)}f(0) + f'(0) + O(\lambda^{-1})\right]. \qquad (3.25)$$

3.2 Stationary Phase Method in Many Dimensions

Let Ω be a domain in $\mathbb{R}^n$ and $f \in C_0^\infty(\Omega)$ be a smooth function of compact support, $S \in C^\infty(\Omega)$ be a real valued smooth function. In this section we study the asympotics as $\lambda \to \infty$ of the multi-dimensional Fourier integrals

$$F(\lambda) = \int_\Omega f(x)\, \exp[i\lambda S(x)]\, dx\,. \qquad (3.26)$$

3.2.1 Nondegenerate Stationary Point

Localization Principle

Lemma 8 *Let Ω be a domain in $\mathbb{R}^n$ and $f \in C_0^\infty(\Omega)$ be a smooth function of compact support, $S \in C^\infty(\Omega)$ be a real valued smooth function without stationary points in* $\operatorname{supp} f$*, i.e. $\partial_x S(x) \neq 0$ for $x \in \operatorname{supp} f$. Then as $\lambda \to \infty$*

$$F(\lambda) = O(\lambda^{-\infty}) \qquad (3.27)$$

This lemma is proved by integration by parts.

Definition 9 1. *The set $\mathcal{S}(\mathbb{R}^n)$ of all smooth functions on $\mathbb{R}^n$ that decrease at $|x| \to \infty$ together with all derivatives faster than any power of $|x|$ is called the **Schwartz space**.*

2. *For any integrable function $f \in L^1(\mathbb{R}^n)$ the **Fourier transform** is defined by*

$$\mathcal{F}(f)(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \exp(i\,x\cdot\xi)\, f(x)\, dx \qquad (3.28)$$

Proposition 3 *1. Fourier transform is a one-to-one onto map (bijection) $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, i.e. if $f \in \mathcal{S}(\mathbb{R}^n)$, then $\mathcal{F}(f) \in \mathcal{S}(\mathbb{R}^n)$.*

2. The inverse Fourier transform is

$$\mathcal{F}^{-1}(f)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp(-i\, x \cdot \xi)\, f(\xi)\, d\xi \tag{3.29}$$

Theorem 17 *Let Ω be a finite domain in $\mathbb{R}^n$, $f \in C_0^\infty(\Omega)$ be a smooth function with compact support in Ω and $S \in C^\infty(\Omega)$ be a real valued smooth function. Let S have a single stationary point x_0 in Ω and let it be non-degenerate. Then as $\lambda \to \infty$ there is asymptotic expansion*

$$F(\lambda) \sim \lambda^{-n/2} \exp[i\lambda S(x_0)] \sum_{k=0}^{\infty} a_k \lambda^{-k}\,. \tag{3.30}$$

The coefficients a_k are determined in terms of derivatives of the functions f and S at x_0.

The leading asymptotics is

$$\begin{aligned} F(\lambda) \quad = \quad & \left(\frac{2\pi}{\lambda}\right)^{n/2} \exp\left[i\lambda S(x_0) + [\operatorname{sgn} \partial_x^2 S(x_0)]\, i\, \frac{\pi}{4}\right] \\ & \times \left|\det \partial_x^2 S(x_0)\right|^{-1/2} \left[f(x_0) + O(\lambda^{-1})\right]\,. \end{aligned} \tag{3.31}$$

Recall that $\operatorname{sgn} A = \nu_+(A) - \nu_-(A)$ denotes the signature of a real symmetric nondegenerate matrix A, where $\nu_\pm(A)$ are the number of positive and negative eigenvalues of A.

3.2.2 Integral Operators with Singular Kernels

Let Ω be a bounded domain in $\mathbb{R}^n$ including the origin, $0 \in \Omega$. Let $f \in C_0^\infty(\Omega)$ be a smooth function with compact support and $S \in C^\infty(\mathbb{R}^n)$ be a real valued non-positive smooth function. Let S have a single stationary point $x = 0$, and let it to a nondegenerate, i.e. let $S(0) = S'(0) = 0$ and $\partial_x^2 S(0) \neq 0$. Let $K_\lambda : C_0^\infty(\Omega) \to C^\infty(\Omega)$ be a linear integral operator defined by

$$(K_\lambda f)(x) = \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_\Omega \exp[i\lambda S(x-y)] f(y)\, dy\,. \tag{3.32}$$

Then as $\lambda \to \infty$

$$(K_\lambda f)(x) = \exp\left[[\operatorname{sgn} \partial_x^2 S(0)]\, i\, \frac{\pi}{4}\right] \left|\det \partial_x^2 S(0)\right|^{-1/2} \left[f(x) + O(\lambda^{-1})\right] \quad (3.33)$$

uniformly for $x \in \Omega$. On the other hand, if $x \notin \Omega$, then as $\lambda \to \infty$

$$(K_\lambda f)(x) = O(\lambda^{-\infty})\,. \quad (3.34)$$

Chapter 4

Saddle Point Method

4.1 Saddle Point Method for Laplace Integrals

Let γ be a contour in the complex plane and the functions f and S are holomorphic in a neighborhood of this contour. In this section we will study the asymptotics as $\lambda \to \infty$ of the Laplace integrals

$$F(\lambda) = \int_\gamma f(z) \exp[\lambda S(z)] \, dz \,. \tag{4.1}$$

4.1.1 Heuristic Ideas of the Saddle Point Method

The idea of the **saddle point method** (or **method of steepest descent**) is to deform the contour in such a way that the main contribution to the integral comes from a neighborhood of a single point. This is possible since the functions f and S are holormorphic.

First of all, let us find a bound for $|F(\lambda)|$. For a contour $\gamma = \gamma_0$ of finite length $l(\gamma_0)$ we have, obviously,

$$|F(\lambda)| \le l(\gamma_0) \max_{z\in\gamma_0} f(z) \exp[\lambda \operatorname{Re} S(z)] \,. \tag{4.2}$$

Now, let Γ be the set of all contours obtained by smooth deformations of the contour γ_0 keeping the endpoints fixed. Then such an estimate is valid for any contour $\gamma \in \Gamma$, hence,

$$|F(\lambda)| \le \inf_{\gamma\in\Gamma} \left\{ l(\gamma) \max_{z\in\gamma} f(z) \exp[\lambda \operatorname{Re} S(z)] \right\} \,. \tag{4.3}$$

Since we are interested in the limit $\lambda \to \infty$, we expect that the length of the contour does not affect the accuracy of the estimate. Also, intuitively it is clear that the behavior of the function S is much more important that that of the function f (since S is in the exponent and its variations are scaled significantly by the large parameter λ). Thus we expect an estimate of the form

$$|F(\lambda)| \leq C(\gamma, f) \inf_{\gamma\in\Gamma} \left\{ \max_{z\in\gamma} \exp[\lambda \mathrm{Re}\, S(z)] \right\}, \tag{4.4}$$

where $C(\gamma, f)$ is a constant that depends on the contour γ and the function f but does not depend on λ. So, we are looking for a point on a given contour γ where the maximum of $\mathrm{Re}\, S(z)$ is attained. Then, we look for a contour γ^* where the minimum of this maximum is attained, i.e. we assume that there exists a contour γ_* where

$$\min_{\gamma\in\Gamma} \max_{z\in\gamma} \mathrm{Re}\, S(z) \tag{4.5}$$

is attained. Such a contour will be called a **minimax** contour.

Let $z_0 \in \gamma_*$ be the only point on the contour γ_* where the maximum of $\mathrm{Re}\, S(z)$ is attained. Then, we have an estimate

$$|F(\lambda)| \leq C(\gamma_*, f) \exp[\lambda\, \mathrm{Re}\, S(z_0)] \,. \tag{4.6}$$

By deforming the contour of integration to γ_* we obtain

$$F(\lambda) = \int_{\gamma_*} f(z)\, \exp[\lambda\, S(z)]\, dz \,. \tag{4.7}$$

The asymptotics of this integral can be computed by Laplace method.

1. **Boundary Point.** Let z_0 be an endpoint of γ_*, say, the initial point. Suppose that $S'(z_0) \neq 0$. Then one can replace the integral $F(\lambda)$ by an integral over a small arc with the initial point z_0. Finally, integrating by parts gives the leading asymptotics

$$F(\lambda) = \frac{1}{-S'(z_0)} \exp[\lambda\, S(z_0)]\, \lambda^{-1} \left[f(z_0) + O(\lambda^{-1}) \right] \,. \tag{4.8}$$

2. **Interior Point.** Let z_0 be an interior point of the contour γ_*. From the minimax property of the contour γ_* it follows that the point z_0 is

the **saddle point** of the function $\mathrm{Re}\, S(z)$. Let $z = x + iy$. Since the saddle point is a stationary point, then

$$\frac{\partial}{\partial x}\mathrm{Re}\, S(z_0) = \frac{\partial}{\partial y}\mathrm{Re}\, S(z_0) = 0\,. \tag{4.9}$$

Then from Cauchy-Riemann conditions it follows that $S'(z_0) = 0$.

Definition 10 *1. A point $z_0 \in \mathbb{C}$ is called a* ***saddle point*** *of the complex valued function $S : \mathbb{C} \to \mathbb{C}$ if $S'(z_0) = 0$.*

2. A saddle point z_0 is said to be of ***order*** *n if*

$$S'(z_0) = \cdots = S^{(n)}(z_0) = 0, \qquad S^{(n+1)}(z_0) \neq 0\,. \tag{4.10}$$

3. A first order saddle point is called ***simple****, i.e. for a simple saddle point $S''(z_0) \neq 0$.*

4. The number $\mathrm{Re}\, S(z_0)$ is called the ***height*** *of the saddle point.*

To compute the asymptotics at an interior saddle point, we replace the contour γ_* by a small arc γ_*^0 containing the point z_0. Then we expand the function S in the Taylor series in the neighborhood of z_0 and neglect the terms of third order and higher, i.e. we replace S by

$$S(z) = S(z_0) + \frac{1}{2}S''(z_0)(z - z_0)^2 + O((z - z_0)^3)\,. \tag{4.11}$$

Finally, by changing the variables and evaluating the integral by Laplace method we obtain the asymptotics as $\lambda \to \infty$

$$F(\lambda) = \sqrt{\frac{2\pi}{-S''(z_0)}}\lambda^{-1/2} \exp[\lambda\, S(z_0)] \left[f(z_0) + O(\lambda^{-1})\right]\,. \tag{4.12}$$

The saddle point method consists of two parts: the *toplogical* part and the *analytical* part.

The topological part consists of the deformation of the contour to the minimax contour γ_* that is most suitable for asymptotical estimates. The analytical part contains then the evaluation of the asymptotics over the contour γ_*.

The analytical part is rather straightforward. Here one can apply the same methods and as in the Laplace method; in many cases one can even use the same formulas.

The topological part is usually much more complicated since it is a global problem. It could happen, for example, that a contour γ_* where the minimax $\min_{\gamma\in\Gamma}\max_{z\in\gamma}\operatorname{Re}S(z)$ is attained does not exist at all! Next, strictly speaking we need to look for a contour where $\min_{\gamma\in\Gamma}\max_{z\in\gamma}f(z)\exp[\operatorname{Re}S(z)]$ is attained what makes the problem even more complicated.

Thus, if one can find the minimax contour, then one can compute the asymptotics of $F(\lambda)$ as $\lambda\to\infty$. Unfortunately, there is no simple algorithm that would always enable one to find the minimax contour. Nevertheless, under certain conditions one can prove that such a contour exists and, in fact, find one. We will discuss this point later.

4.1.2 Level Curves of Harmonic Functions

Lemma 9 *Let $S:\mathbb{C}\to\mathbb{C}$ be holomorphic at z_0 and $S'(z_0)\neq 0$. Then in a small neighborhood of the point z_0 the arcs of the level curves*

$$\operatorname{Re}S(z)=\operatorname{Re}S(z_0),\qquad \operatorname{Im}S(z)=\operatorname{Im}S(z_0), \tag{4.13}$$

are analytic curves. These curves are orthogonal at z_0.

Let $\varphi(z)=S(z)-S(z_0)$. Since $S'(z_0)\neq 0$ the function $w=\varphi(z)$ is is a one-to-one holomorphic, in fact, conformal, mapping of a neghborhood of the point $z=z_0$ onto a neighborhood of the point $w=0$. The inverse function $z=\varphi^{-1}(w)$ is holomorphic in a neighborhood of the origin $w=0$. Let $w=u+iv$ and $\psi(u,v)\equiv\varphi^{-1}(w)$. The arc of the level curve $\operatorname{Re}S(z)=\operatorname{Re}S(z_0)$ is mapped onto an open interval on the imaginary axis. It is defined by $z=\psi(0,v)$ and is analytic. The same is true for the level curve $\operatorname{Im}S(z)=\operatorname{Im}S(z_0)$. It is defined by $z=\psi(u,0)$ and is analytic as well. The tangent vectors to the level curves at z_0 are determined by

$$\left.\frac{\partial\psi(u,0)}{\partial u}\right|_{u=0}=(\partial_w\varphi^{-1})(0)\,,\qquad \left.\frac{\partial\psi(0,v)}{\partial v}\right|_{v=0}=i\,(\partial_w\varphi^{-1})(0) \tag{4.14}$$

and are obviously orthogonal. One could also conlude this from the fact that the map is conformal and, therefore, preserves the angles.

Lemma 10 *Let z_0 be a simple saddle point of the function S. Then in a small neighborhood of the point z_0 the level curve $\operatorname{Re} S(z) = \operatorname{Re} S(z_0)$ consists of two analytic curves that intersect orthogonally at the point z_0 and separate the neighborhood of z_0 in four sectors. The signs of the function $\operatorname{Re}[S(z) - S(z_0)]$ in adjacent sectors are different.*

In a neighborhood of a simple saddle point the there is a one-to-one holomorphic function $z = \psi(w)$ such that $\psi(0) = z_0$, $\psi'(0) \neq 0$, and $S(\psi(w)) = S(z_0) + w^2$. In the complex plane of w the level curve $\operatorname{Re} S(z) = \operatorname{Re} S(z_0)$ takes the form $\operatorname{Re} w^2 = 0$. Its solution consists of two orthogonal lines $w_\pm = (1 \pm i)t$ with $-\varepsilon < t < \varepsilon$ that intersect at the point $w = 0$. The level curves $z_\pm = \psi(w_\pm)$ have listed properties.

More generally :

Lemma 11 *Let z_0 be a saddle point of the function S of order n. Then in a small neighborhood of the point z_0 the level curve $\operatorname{Re} S(z) = \operatorname{Re} S(z_0)$ consists of $(n+1)$ analytic curves that intersect at the point z_0 and separate the neighborhood of z_0 in $2(n+1)$ sectors with angles $\pi(n+1)$ at the vertex. The signs of the function $\operatorname{Re}[S(z) - S(z_0)]$ in adjacent sectors are different.*

Definition 11 *Let S be a complex valued function and γ be a simple curve with the initial point z_0. The curve γ is called* ***curve of steepest descent*** *of the function $\operatorname{Re} S$ if $\operatorname{Im} S(z) = \text{const}$ and $\operatorname{Re} S(z) < \operatorname{Re} S(z_0)$ for $z \in \gamma$, $z \neq z_0$. If $\operatorname{Im} S(z) = \text{const}$ and $\operatorname{Re} S(z) > \operatorname{Re} S(z_0)$ for $z \in \gamma$, $z \neq z_0$, then the curve γ is called* ***curve of steepest ascent*** *of the function $\operatorname{Re} S$.*

Lemma 12

1. *If z_0 is not a saddle point, then there is exactly one curve of steepest descent.*

2. *If z_0 is a simple saddle point, then there are 2 curves of steepest descent.*

3. *If z_0 is a saddle point of order n, then there are $(n+1)$ curves of steepest descent.*

4. *In a neighborhood of a saddle point z_0 in each sector in which $\operatorname{Re}[S(z) - S(z_0)] > 0$ there is exatly one curve of steepest descent.*

This is proved by a change of variables in a neighborhood of z_0.

Remarks. Let $S : D \to \mathbb{C}$ be a nonconstant holomorphic function in a domain D. Let $z = x+iy$ and $S(z) = u(x, y)+iv(x, y)$. Then both $u : D \to \mathbb{R}$ and $v : D \to \mathbb{R}$ are harmonic functions in D. Harmonic functions do not have maximum or minimum points in the interior of D. They are attained only at the boundary of the domain D. All critical points of harmonic functions, i.e. the points where $\partial u = \partial v = 0$ are saddle points. These are exactly the points where $S'(z) = 0$. That is why such points are called saddle points of the function S. In the simplest case $S = z^2$ the surface $u = x^2 - y^2$ is hyperbolic paraboloid (saddle).

Definition 12 *Two contours γ_1 and γ_2 are called **equivalent** if*

$$\int_{\gamma_1} f(z) \exp[\lambda\, S(z)]\, dz = \int_{\gamma_2} f(z) \exp[\lambda\, S(z)]\, dz\,. \qquad (4.15)$$

Lemma 13 *Let S and f be holormorphic functions on a finite contour γ. Let the points where $\max_{z\in\gamma} \operatorname{Re} S(z)$ is attained are neither saddle points nor the endpoints of the contour γ. Then there is a contour γ' equivalent to the contour γ and such that*

$$\max_{z\in\gamma'} \operatorname{Re} S(z) < \max_{z\in\gamma} \operatorname{Re} S(z)\,. \qquad (4.16)$$

Theorem 18 *Let $F(\lambda)$ be a Laplace integral (4.1) If there exists a contour γ_* such that: i) it is equivalent to the contour γ and ii) the integral $F(\lambda)$ attains the minimax $\min_{\gamma\in\Gamma} \max_{z\in\gamma} \operatorname{Re} S(z)$ on it. Then among the points where $\max_{z\in\gamma_*} \operatorname{Re} S(z)$ is attained there are either endpoints of the contour or saddle points z_j such that in aneighborhood of z_j the contour γ_* goes through two different sectors where $\operatorname{Re} S(z) < \operatorname{Re} S(z_j)$.*

4.1.3 Analytic Part of Saddle Point Method

In this section we always assume that γ is a simple smooth (or piece-wise smooth) curve in the complex plane, which may be finite or infinite. The functions f and S are assumed to be holomorphic on γ. Also, we assume that the integral $F(\lambda)$ converges absolutely.

First of all, we have

Lemma 14 *If $\max_{z\in\gamma} \operatorname{Re} S(z) \geq C$, then*

$$F(\lambda) = O(e^{C}\lambda), \qquad (\lambda \geq 1)\,. \qquad (4.17)$$

Theorem 19 *Let z_0 be the initial endpoint of the curve γ. Let f and S are analytic at z_0, $\operatorname{Re} S(z_0) > \operatorname{Re} S(z)$ $\forall z \in \gamma$, and $S'(z_0) \neq 0$. Then, as $\lambda \to \infty$ there is asympotic expansion*

$$F(\lambda) \sim \exp[\lambda S(z_0)] \sum_{k=0}^{\infty} a_k \lambda^{-k-1} \tag{4.18}$$

where

$$a_k = -\left(-\frac{1}{S'(z)}\frac{\partial}{\partial z}\right)^k \left[\frac{f(z)}{S'(z)}\right]\Bigg|_{z=z_0} \tag{4.19}$$

The proof is by integration by parts.

Theorem 20 *Let z_0 be an interior point of the curve γ. Let f and S are analytic at z_0, $\operatorname{Re} S(z_0) > \operatorname{Re} S(z)$ $\forall z \in \gamma$. Let z_0 be a simple saddle point of S such that in a neighborhood of z_0 the contour γ goes through two different sectors where $\operatorname{Re} S(z) < \operatorname{Re} S(z_0)$. Then, as $\lambda \to \infty$ there is asympotic expansion*

$$F(\lambda) \sim \exp[\lambda S(z_0)] \sum_{k=0}^{\infty} a_k \lambda^{-k-1/2} . \tag{4.20}$$

The branch of the square root is choosen so that $\arg \sqrt{-S''(z_0}$ is equal to the angle between the positive direction of the tangent to the curve γ at the point z_0 and the positive direction of the real axis.

In a neighborhood of z_0 there is a mapping $z = \psi(w)$ such that

$$S(\psi(w)) = S(z_0) - \frac{w^2}{2} . \tag{4.21}$$

After this change of variables and deforming the contour to the steepest descent contour the integral becomes

$$F(\lambda) = \exp[\lambda S(z_0)] \int_{\varepsilon}^{\varepsilon} e^{-\lambda w^2/2} f(\psi(w))\psi'(w)\, dw + O(\lambda^{-\infty}) . \tag{4.22}$$

Since both f and ψ are holomorphic there is a Taylor expansion

$$f(\psi(w)\psi'(w) = \sum_{k=0}^{\infty} c_k w^k . \tag{4.23}$$

Then the coefficients a_k are easily computed in terms of c_k

$$a_k = 2^{k+1/2}\Gamma\left(k + \frac{1}{2}\right) c_{2k}\,. \tag{4.24}$$

Theorem 21 *Let γ be a finite contour and f and S be holomorphic in a neighborhood of γ. Let $\max_{z\in\gamma} \mathrm{Re} S(z)$ be attained at the points z_j and these points are either the endpoints of the curve or saddle points such that in a neighborhood of a saddle points the contour γ goes through two different sectors where $\mathrm{Re}\, S(z) < \mathrm{Re}\, S(z_j)$. Then as $\lambda \to \infty$ the integral $F(\lambda)$ is asymptotically equal to the sum of contributions of the points z_j.*

Remark. The integral over a small arc containing a saddle point is called the contribution of the saddle point to the integral.

Proposition 4 *Let f and S be holomorphic functions on γ and $\mathrm{Im}\, S(z) = \mathrm{const}$ on γ. If γ has finite number of saddle points, then as $|lambda| \to \infty$, $|\arg\lambda| \le \pi/2 - \varepsilon < \pi/2$, the asymptotic expansion of the integral $F(\lambda)$ is equal to the sum of contributions of the saddles points and the endpoints of the contour.*

4.1.4 Examples

1.

$$\int_{-\infty}^{\infty} e^{i\lambda x}(1+x^2)^{-\lambda}\, dx \sim \sqrt{\pi(1-c)}e^{-\lambda c}\lambda^{-1/2}(2c)^{-\lambda} \qquad (\lambda \to \infty) \tag{4.25}$$

where $c = \sqrt{2} - 1$. There are two saddle points $z_{1,2} = i(-1 \pm \sqrt{2})$. The asymptotics is determined by the contribution from the point z_1.

2.

$$\int_{i-\infty}^{i+\infty} e^{-z^2}(1+z)^{-n}\, dz \sim \sqrt{\frac{\pi}{2}}\, i^{-n}\, e^{(n-1)/2}\left(\frac{n}{2}\right)^{-n/2}, \qquad (n \to \infty)\,. \tag{4.26}$$

Here $n > 0$ is a positive integer.

3. Let $a > 0$. As $\lambda \to \infty$

$$\int_{ia-\infty}^{ia+\infty} \exp(-2\lambda z^2 - 4\lambda/z)\, dz \sim \pi^{1/6}(6\lambda)^{-1/2} \exp(3\lambda + i3\sqrt{3}\,\lambda)\,. \quad (4.27)$$

There are three saddle points, which are the roots of the equation $z^3 = 1$. The asymptotics is determined by the contribution from the point $z_1 = e^{i\,2\pi/3}$.

4.

$$\int_{-i\infty}^{i\infty} \exp(-\lambda z^2 + z^2 \ln z)\, dz \sim i\sqrt{\pi} \exp\left(-\frac{1}{2} e^{2\lambda-1}\right), \qquad (\lambda \to \infty)\,. \quad (4.28)$$

There are two saddle points $z_1 = 0$ and $z_2 = e^{\lambda - 1/2}$. The asymptotics is equal to the contribution from the saddle point z_2.

Notation

Logic

$A \Longrightarrow B$	A implies B
$A \Longleftarrow B$	A is implied by B
iff	if and only if
$A \Longleftrightarrow B$	A implies B and is implied by B
$\forall x \in X$	for all x in X
$\exists x \in X$	there exists an x in X such that

Sets and Functions (Mappings)

$x \in X$	x is an element of the set X
$x \notin X$	x is not in X
$\{x \in X \mid P(x)\}$	the set of elements x of the set X obeying the property $P(x)$
$A \subset X$	A is a subset of X
$X \setminus A$	complement of A in X
$\overline{A}$	closure of set A
$X \times Y$	Cartesian product of X and Y
$f : X \to Y$	mapping (function) from X to Y
$f(X)$	range of f

χ_A	characteristic function of the set A
$\emptyset$	empty set
$\mathbb{N}$	set of natural numbers (positive integers)
$\mathbb{Z}$	set of integer numbers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of positive real numbers
$\mathbb{C}$	set of complex numbers

Vector Spaces

$H \oplus G$	direct sum of H and G
H^*	dual space
$\mathbb{R}^n$	vector space of n-tuples of real numbers
$\mathbb{C}^n$	vector space of n-tuples of complex numbers
l^2	space of square summable sequences
l^p	space of sequences summable with p-th power

Normed Linear Spaces

$\|\|x\|\|$	norm of x
$x_n \longrightarrow x$	(strong) convergence
$x_n \xrightarrow{w} x$	weak convergence

Function Spaces

$\operatorname{supp} f$	support of f
$H \otimes G$	tensor product of H and G

$C_0(\mathbb{R}^n)$	space of continuous functions with bounded support in $\mathbb{R}^n$
$C(\Omega)$	space of continuous functions on Ω
$C^k(\Omega)$	space of k-times differentiable functions on Ω
$C^\infty(\Omega)$	space of smooth (infinitely diffrentiable) functions on Ω
$\mathcal{D}(\mathbb{R}^n)$	space of test functions (Schwartz class)
$L^1(\Omega)$	space of integrable functions on Ω
$L^2(\Omega)$	space of square integrable functions on Ω
$L^p(\Omega)$	space of functions integrable with p-th power on Ω
$H^m(\Omega)$	Sobolev spaces
$C_0(V, \mathbb{R}^n)$	space of continuous vector valued functions with bounded support in $\mathbb{R}^n$
$C^k(V, \Omega)$	space of k-times differentiable vector valued functions on Ω
$C^\infty(V, \Omega)$	space of smooth vector valued functions on Ω
$\mathcal{D}(V, \mathbb{R}^n)$	space of vector valued test functions (Schwartz class)
$L^1(V, \Omega)$	space of integrable vector valued functions functions on Ω
$L^2(V, \Omega)$	space of square integrable vector valued functions functions on Ω
$L^p(V, \Omega)$	space of vector valued functions functions integrable with p-th power on Ω
$H^m(V, \Omega)$	Sobolev spaces of vector valued functions

Linear Operators

D^α	differential operator

$L(H, G)$	space of bounded linear transformations from H to G
$H^* = L(H, \mathbb{C})$	space of bounded linear functionals (dual space)

Bibliography

[1] M. V. Fedoriuk, *Asymptotics: Integrals and Series*, (Moscow: Nauka, 1987)

[2] E. T. Copson, *Asymptotic Expansions*, (Cambridge, UK: Cambridge University Press, 1965)

[3] F. W. J. Olver, *Introduction to Asymptotics and Special Functions*, (New York: Academic Press, 1973)

[4] F. W. J. Olver, *Asymptotics and Special Functions*, (New York: Academic Press, 1974)

[5] A. Erdélyi, *Asymptotic Expansions*, (New York: Dover Publications, 1956)

[6] R. B. Dingle, *Asymptotic Expansions: Their Derivation and Interpretation*, (London: Academic Press, 1973)