Lecture Notes
Methods of Mathematical Physics
MATH 536

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Textbook: L. Debnath and P. Mikusinski, (Academic Press, 1999)

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Chapter 1

Integral Operators

1.1 Existense Theorems

1.1.1 Integral Equations

- Let g be a given function, $K(x,t)$ be a given function of two variables, and f be an unknown function. The **Volterra equation of the first kind** reads

$$g(x) = \int_a^x dt\, K(x,t) f(t)\,.$$

- **Volterra equation of the second kind** is

$$g(x) = f(x) + \int_a^x dt\, K(x,t) f(t)\,.$$

- The **Fredholm equation of the first kind** reads

$$g(x) = \int_a^b dt\, K(x,t) f(t)\,.$$

- **Fredholm equation of the second kind** is

$$g(x) = f(x) + \int_a^b dt\, K(x,t) f(t)\,.$$

1.1.2 Neumann Series and Fredholm Alternative

- Let $T : H \to H$ be a mapping on a vector space H. The **fixed point set** of T is the set of vectors $x \in H$ such that

$$T(x) = x_0 .$$

- If T is a linear operator, then its fixed point set is equal to the kernel of the operator $(I - T)$.

- The mapping T is a **contraction** if there is $0 < \alpha < 1$ such that for any $x, y \in H$

$$\| T(x) - T(y) \| \leq \alpha \| x - y \| .$$

- **Theorem 1.1.1 (Contraction Mapping Theorem)** *Let H be a Banach space and S be a closed subspace of H. Let $T : S \to S$ be a contraction mapping. Let x be an arbitrary vector in S and*

$$x_0 = \lim_{n\to\infty} T^n x .$$

Then x_0 belongs to S and

$$T(x_0) = x_0 .$$

Moreover, this is the unique solution of the equaion $T(x) = x$, that is the unique fixed point of the mapping T.

Proof: Banach Fixed Point Theorem.

- **Theorem 1.1.2** *Let H be a Banach space and T be a continuous mapping such that for some $m \in \mathbb{N}$, T^m is a contraction. Let $x \in H$ and*

$$x_0 = \lim_{n\to\infty} T^n x .$$

Then x_0 is the unique fixed point of T.

Proof: Easy.

∎

Theorem 1.1.3 *Let H be a Banach space and A be a non-zero bounded linear operator on H. Let $g \in H$ and $\alpha \in \mathbb{C}$ be a complex number such that*

$$|\alpha| < \frac{1}{\| A \|} .$$

- *Let T be a mapping on H defined by*

$$T(f) = \alpha A f + g .$$

Then T has a unique fixed point, that is the equation $T(f) = f$ has a unique solution.

Proof: Easy.

∎

- Note that

$$T^n(f) = \alpha^n A^n f + \sum_{k=0}^{n} \alpha^n A^n g .$$

- The fixed point

$$f = \sum_{k=0}^{\infty} \alpha^n A^n g ,$$

is the solution of the equation

$$(I - \alpha A) f = g .$$

obtained by the geometric series

$$(I - \alpha A)^{-1} = \sum_{n=0}^{\infty} \alpha^n A^n ,$$

called the **Neumann series**.

Theorem 1.1.4 (Neumann Series) *Let A be a bounded linear operator in a Banach space H. Then the resolvent $R(\lambda) = (A-\lambda I)^{-1}$ is a bounded operator for any $\lambda \in \mathbb{C}$ such that*

$$|\lambda| > \| A \|,$$

that is, outside the circle or radius $\| A \|$. Moreover,

$$R(\lambda) = -\sum_{n=0}^{\infty} \lambda^{-(n+1)} A^n$$

and

$$\| R(\lambda) \| \le \frac{1}{|\lambda| - \| A \|}.$$

Proof:

1. We have

$$\sum_{n=0}^{\infty} |\lambda^{-(n+1)}| \, \| A^n \| < \infty .$$

2. We check that

$$(A - \lambda I) \sum_{n=0}^{\infty} \lambda^{-(n+1)} A^n = \left(\sum_{n=0}^{\infty} \lambda^{-(n+1)} A^n \right) (A - \lambda I) = -I .$$

3. Finally,

$$\| R(\lambda) \| \le \sum_{n=0}^{\infty} |\lambda^{-(n+1)}| \, \| A^n \| = \frac{1}{|\lambda| - \| A \|} .$$

■

Corollary 1.1.1 *1. Let A be a bounded linear operator in a Banach space H, $y \in H$ and $\alpha \in \mathbb{C}$ be a complex number such that*

$$|\alpha| < \frac{1}{\| A \|} .$$

- *Then the equation*

$$x - \alpha A x = y$$

has a unique solution given by

$$x = \sum_{n=0}^{\infty} \alpha^n A^n y .$$

Theorem 1.1.5 (Fredholm Alternative) *Let H be a Hilbert space and A be a self-adjoint compact operator on H. Then the following are equivalent statements:*

- 1. *The non-homogeneous equation*

$$f - Af = g$$

has a unique solution for every $g \in H$.

2. *The homogeneous equation*

$$f - Af = 0$$

has only the trivial solution.

Moreover, if the non-homogeneous equation $(\mathbb{I} - A)u = g$ has a solution u, then every solution v of the homogeneous equation, $(\mathbb{I} - A)v = 0$, is orthogonal to g.

Proof:

1. Let $\lambda_n \in \mathbb{C}$ and $v_n \in H$ be the spectral resolution of A.
2. Let

$$g = \sum_{n=1}^{\infty} c_n v_n \qquad \text{and} \qquad f = \sum_{n=1}^{\infty} a_n v_n .$$

3. Then for any $n \in \mathbb{N}$, if $\lambda_n \neq 1$, then

$$a_n = \frac{c_n}{1 - \lambda_n}\,.$$

4. So, if the non-homogeneous equation has a solution then it is unique and has the form

$$f = \sum_{n=1}^{\infty} \frac{c_n}{1 - \lambda_n} v_n\,.$$

5. The convergence of the series follows from the fact that $\lambda_n \to 0$.
6. If the homogeneous equation has a non-trivial solution, then the non-homogeneous equation has infinitely many solutions.
7. The orthogonality of the solutions follows from the equations.

■

1.1.3 Homework

- Exercises: 5.12[1,2,3]

1.2 Fredholm Integral Equations

1.2.1 Hilbert-Schmidt and Trace -Class Operators

- Let H be a Hilbert space and (e_n) be an orthonormal basis in H. An operator A on H is called a **trace-class operator** if the series

$$\operatorname{Tr} A = \sum_{n=1}^{\infty} (Ae_n, e_n) < \infty$$

  converges.

- For every trace-class operator A the operator A^* is trace-class and

$$\operatorname{Tr} A^* = \overline{\operatorname{Tr} A}\,.$$

- If A is a compact self-adjoint operator with the eigenvalues λ_n, then A is trace-class if and only if

$$\sum_{n=1}^{\infty} |\lambda_n| < \infty\,,$$

  where each eigenvalue is taken with its multiplicity, and

$$\operatorname{Tr} A = \sum_{n=1}^{\infty} \lambda_n\,.$$

  This is also true for non-self-adjoint operators.

- It A is a trace-class operator and B is a bounded operator, then AB and BA are trace-class operators and

$$\operatorname{Tr}(AB) = \operatorname{Tr}(BA)\,.$$

- For a bounded operator A the operator $|A|$ is defined by

$$|A| = \sqrt{A^* A}\,.$$

- An operator A is trace-class if and only if $|A|$ is trace-class.

- The trace-norm of a trace-class operator A is

$$\|\, A \,\|_{\mathrm{tr}} = \operatorname{Tr} |A|\,.$$

- For any trace-class operator A

$$\| A^* \|_{\rm tr} = \| A \|_{\rm tr}$$

$$\| A \| \le \| A \|_{\rm tr}$$

and

$$|{\rm Tr}\, A| \le \| A \|_{\rm tr}$$

- Let A be a bounded operator on a Hilbert space H and A^* be its adjoint. The **Hilbert-Schmidt norm** of A is defined by

$$\| A \|_{HS} = ({\rm Tr}\, AA^*)^{1/2} .$$

- An operator A is called a **Hilbert-Schmidt operator** if its Hilbert-Schmidt norm is finite.

- There holds

$$\| A^* \|_{HS} = \| A \|_{HS} \quad .$$

- The operator norm of the operator A is related to the Hilbert-Schmidt norm by

$$\| A \| \le \| A \|_{HS} \quad .$$

- An operator A is Hilbert-Schmidt if and only if $|A|$ is Hilbert-Schmidt.

- The products AB and BA of two Hilbert-Schmidt operators A and B are Hilbert-Schmidt operators and

$${\rm Tr}\,(AB) = {\rm Tr}\,(BA) .$$

- Every trace-class opertator is Hilbert-Schmidt.

- For any trace-class operator A

$$\| A \| \le \| A \|_{HS} \le \| A \|_{\rm tr} \quad .$$

- Let $H = L^2(M, \mu)$ be a Hilbert space where M is a space with a measure μ. The operator A in H is Hilbert-Schmidt if and only if there exists a function $K \in L^2(M \times M, \mu \times \mu)$ such that the operator A is defined by

$$(Af)(x) = \int_M d\mu(y)\; K(x, y) f(y) .$$

The function K, called the **Schwartz kernel** of the operator A, is defined uniquely up to a its values on a set of $(\mu \times \mu)$ measure zero. The Hilbert-Schmidt norm of the operator A is

$$\| A \|_{HS} = \left(\int_M d\mu(x) \int_M d\mu(y)\, |K(x,y)|^2 \right)^{1/2} .$$

1.2.2 Linear Fredholm Equations

- **Definition 1.2.1** *Let $I = [a,b]$, K be a function on $I \times I$. The operator A on $L^2(I)$ defined by*

  $$(Af)(x) = \int_a^b dy\, K(x,y) f(y)$$

  is called **Fredholm integral operator** *with the kernel K.*

- **Lemma 1.2.1** *Let $I = [a,b]$, K be a continuous function on $I \times I$, such that*

  $$k = \left(\int_a^b dx \int_a^b dy\, |K(x,y)|^2 \right)^{1/2} < \infty .$$

  Then A is Hilbert-Schmidt operator with the norm

  $$\| A \|_{HS} = k .$$

Proof:

1. We have (Schwartz inequality)

$$|(Af)(x)|^2 \le \int_a^b dy |K(x,y)|^2 \, \| f \|^2$$

2. Thus

$$\| Af \|^2 \le k^2 \| f \|^2 < \infty .$$

3. So, A is a bounded operator on $L^2(I)$.

■

- **Theorem 1.2.1** *Let $I = [a, b]$, K be a continuous function on $I \times I$, and A be the Fredholm integral operator with the kernel K. Let*

$$k = \| A \|_{HS}$$

and let $\alpha \in \mathbb{C}$ be a complex number such that

$$|\alpha| k < 1 .$$

Then for any $g \in L^2(I)$ there is a unique $f \in L^2(I)$ such that

$$(I - \alpha A) f = g .$$

Proof:

1. Let T be an operator on $L^2(I)$ defined by

$$T(f) = \alpha A f + g .$$

2. Then T is a contraction if $|\alpha| k < 1$ and the equation $T f = f$ has a unique solution.

■

- **Example**.

1.2.3 Nonlinear Fredholm Equations

Theorem 1.2.2 *Let $I = [a, b]$ and K be a function on $I \times I \times \mathbb{C}$ and A be a mapping on $L^2(I)$ defined by*

$$A(f)(x) = \int_a^b dy\, K(x, y, f(y))\,.$$

Suppose there exists a function N on $I \times I$ and constants M and k such that

•

1. *for all $f \in L^2(I)$*
$$\| A(f) \| \leq M \| f \|,$$
2. *for all $x, y \in I$, $z_1, z_2 \in \mathbb{C}$*
$$|K(x, y, z_1) - K(x, y, z_2)| \leq N(x, y)|z_1 - z_2|$$
3. $$\int_a^b dx \int_a^b dy |N(x, y)|^2 = k^2 < \infty\,.$$

Let $g \in L^2(I)$ and $\alpha \in \mathbb{C}$ be such that

$$|\alpha| k < 1\,.$$

Then there is a unique $f \in L^2(I)$ such that

$$f = \alpha A(f) + g\,.$$

Proof:

1. Let T be an operator on $L^2(I)$ defined by
$$T(f) = \alpha A(f) + g\,.$$
2. Then T is a contraction if $|\alpha| k < 1$ and the equation $Tf = f$ has a unique solution.

■

1.2.4 Method of Successive Approximations

- Let A be a linear integral operator

$$(Af)(x) = \int_a^b dy K(x,y) f(y)\,.$$

- Then A^n is also an integral operator

$$(A^n f)(x) = \int_a^b dy\; K_n(x,y) f(y)\,,$$

  where the kernel $K_n(x,y)$ is defined by the recursion

$$K_n(x,y) = \int_a^b dz\; K(x,z) K_{n-1}(z,y)$$

  and is equal to

$$K_n(x,y) = \int_a^b dz_1 \cdots \int_a^b dz_{n-1}\; K(x,z_{n-1}) \cdots K(z_1,y)\,.$$

- The resolvent $R_\lambda = (A - \lambda I)^{-1}$ is an operator

$$[R_\lambda f](x) = f(x) - \frac{1}{\lambda} \int_a^b dy\; \Gamma_{\lambda^{-1}}(x,y) f(y)\,,$$

  where

$$\Gamma_\alpha(x,y) = \sum_{n=1}^{\infty} \alpha^n K_n(x,y)\,.$$

- The solution of the Fredholm integral equation

$$f(x) = \alpha \int_a^b dy\; K(x,y) f(y) + g(x)$$

  is

$$f(x) = g(x) + \int_a^b dy\; \Gamma_\alpha(x,y) g(y)\,.$$

- **Example.**

1.2.5 Separable Kernel

- Let M_k and N_k, where $k = 1, \dots, n$, be some functions from $L^2([a, b])$. The kernel K of an integral operator A is **separable** (or **degenerate**) if it has the form

$$K(x, y) = \sum_{k=1}^{n} M_k(x) N_k(y) .$$

- Let $g \in L^2([a, b])$. The Fredholm equation of the second kind

$$f(x) = \alpha \int_a^b dy\, K(x, y) f(y) + g(x)$$

  for a separable kernel is a **finite-dimensional** problem.

- Let A be a $n \times n$ matrix, and B and C be n column vectors defined by

$$c_k = (N_k, \overline{f}), \qquad b_k = (N_k, \overline{g}), \qquad a_{mk} = (N_m, \overline{M}_k)$$

  Then the integral equation is equivalent to the matrix equations

$$(I - \alpha A)C = B .$$

  The solution of this equation is

$$C = (I - \alpha A)^{-1} B ,$$

  if $\det(I - \alpha A) \neq 0$. Otherwise, there are infinitely many solutions.

- The solution of the integral equation is

$$f = g + \alpha \sum_{k=1}^{n} c_k M_k .$$

- **Example.**

1.2.6 Convolution Kernel

- Let K be integrable functions on $\mathbb{R}$ from $L^1(\mathbb{R})$. An integral operator A on $L^1(\mathbb{R})$ with the kernel of the form

$$\tilde{K}(x,y) = K(x-y)$$

is called the **convolution operator** and the kernel $K(x-y)$ is called the **convolution kernel**.

- Let $g \in L^1(\mathbb{R})$. Fredholm equation with convolution kernel is the equation

$$f(x) = \alpha \int_{-\infty}^{\infty} dy\, K(x-y) f(y) + g(x)\,.$$

or

$$f = \alpha\sqrt{2\pi} K * f + g\,.$$

where $*$ is the convolution.

- Applying the Fourier transform we get

$$\hat{f} = \alpha (2\pi)^{1/2} \hat{K} \hat{f} + \hat{g}\,.$$

The formal solution is given by the inverse Fourier transform

$$f = \mathcal{F}\left(\frac{\hat{g}}{1-\alpha\sqrt{2\pi}\hat{K}}\right)$$

or

$$f(x) = \int_{-\infty}^{\infty} d\omega\, e^{i\omega x}\left(\frac{\hat{g}(\omega)}{1-\alpha\sqrt{2\pi}\hat{K}(\omega)}\right)$$

1.2.7 Hilbert Transform

- We consider smooth functions with compact support on $\mathbb{R}$ from $C_0^\infty(\mathbb{R})$.

- The Cauchy principal value integral is defined by

$$P\int_{-\infty}^{\infty} dy f(y) = \lim_{\varepsilon\to 0^+}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right) dy\; f(y)\,.$$

- The **Hilbert transform** is the operator $\mathcal{H}$ on this space by

$$(\mathcal{H}f)(x) = P\int_{-\infty}^{\infty} dy \frac{f(y)}{x-y}\,.$$

- **Proposition**. The Hilbert transform is an anti-involution

$$\mathcal{H}^2 = -\mathrm{Id}\,.$$

  That is the inverse Hilbert tranform is given by

$$\mathcal{H}^{-1} = -\mathcal{H}\,.$$

  Proof: Use Fourier transform.

1.2.8 Homework

- Exercises: 5.12[4,6,7,8,11,12,13,14,15,32]. In each exercise that has many subproblems, a,b,c, . . . , do only one problem of your choice.

1.3 Volterra Integral Equations

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Definition 1.3.1 *Let $I = [a, b]$, K be a function on $I \times I$. The operator A on $L^2(I)$ defined by*

$$(Af)(x) = \int_a^x dy\, K(x, y) f(y)$$

is called **Volterra integral operator** *with the kernel K.*

1.3.1 Volterra Equations of the Second Kind

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Theorem 1.3.1 *Let $I = [a, b]$, K be a function on $I \times I$ with the Hilbert-Schmidt norm k, and A be the Volterra integral operator with the kernel K. Then for any $g \in L^2(I)$ and for any $\alpha \in \mathbb{C}$ there exists a unique $f \in L^2(I)$ such that*

$$f = \alpha A f + g\,.$$

The solution f of this equation has the form

$$f = g + \sum_{n=1}^{\infty} \alpha^n A^n g\,,$$

where A^n is the Volterra integral operator with the kernel

$$K_n(x, y) = \int_a^x dz_{n-1} \cdots \int_a^{z_2} dz_1 K(x, z_{n-1}) \cdots K(z_1, y)\,,$$

Proof:

1. Let T be a mapping on $L^2(I)$ defined by
$$T(f) = \alpha A f + g\,.$$
2. Then
$$T^n(f) = \alpha^n A^n f + \sum_{k=0}^{n-1} \alpha^k A^k g\,.$$
3. Then for any $n \geq 2$ we have
$$\| T^n(f_1) - T^n(f_2) \|^2 \leq \frac{|\alpha|^{2n} k^n}{(n-1)!} \| f_1 - f_2 \|^2\ .$$

4. There exists an $m \in \mathbb{N}$ such that T^m is a contraction.
5. Thus $T(f) = f$ has a unique solution given by

$$f = \lim_{n\to\infty} T^n(f)\,.$$

■

- **Corollary 1.3.1 (Homogeneous Volterra Equation.)** *Let $I = [a, b]$, K be a function on $I \times I$ with the finite Hilbert-Schmidt norm k and A be the Volterra intergal operator on $L^2(I)$ with the kernel K. Then for any $\alpha \in \mathbb{C}$ the equation*

$$f = \alpha A f$$

has only the trivial solution $f = 0$. Thus, the operator A does not have any eigenvalues, the resolvent R_α is a bounded operator for any $\alpha \in \mathbb{C}$, and the spectrum $\sigma(A)$ is empty.

Proof: Follows from the previous theorem.

■

1.3.2 Volterra Equations of the First Kind

- Let $I = [a, b]$ and $g \in L^2(I)$, K be a function on $I \times I$, and A be the Volterra integral operator on $L^2(I)$ with the kernel K and a finite Hilbert-Schmidt norm. The **Volterra equation of the first kind** has the form

$$Af = g\,.$$

- Suppose that g and K are differentiable and $K(x, x) \neq 0$. Then

$$K(x, x) f(x) + \int_a^x dy\, \partial_x K(x, y) f(y) = g'(x)\,,$$

and, therefore, we get a Volterra equation of the second kind

$$f(x) + \int_a^x dy\, N(x, y) f(y) = h(x)\,,$$

where

$$N(x, y) = \frac{\partial_x K(x, y)}{K(x, x)}$$

and

$$h(x) = \frac{g'(x)}{K(x,x)}\,.$$

Let B be the Volterra integral operator with the kernel N. Then we have the equation

$$f = Bf + h\,.$$

Then, if B is Hilbert-Schmidt operator and $h \in L^2(I)$, then this equation has a unique solution.

1.3.3 Abel's Integral Equation

- Let $0 \le \alpha < 1$, and let A_α be the Volterra integral operator with the kernel $(x-y)^{-\alpha}$ on the interval $[0,1]$.

- Let g be a continuous function on $[0,1]$ such that $g(0) = 0$. **Abel's equation** is the equation of the form

$$\int_0^x dy\, \frac{f(y)}{(x-y)^\alpha} = g(x)\,.$$

or

$$A_\alpha f = g\,.$$

- For $\alpha = 0$ this equation is

$$(A_0 f)(x) = \int_0^x dy\, f(y) = g(x)\,.$$

So, if g is differentiable, then the solution of this equation is

$$f = g'\,.$$

Thus, the operator A_0 is invertible and the inverse

$$A_0^{-1} = \partial_x$$

is the differential operator.

- The composition of two operators A_α and A_β is the operator

$$A_\alpha A_\beta = \frac{\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(2-\alpha-\beta)} A_{\alpha+\beta-1}$$

- Let $\beta = 1 - \alpha$. Then

  $$A_\alpha A_{1-\alpha} = \frac{\pi}{\sin(\pi\alpha)} A_0 \,.$$

  Thus

  $$A_\alpha^{-1} = \frac{\sin(\pi\alpha)}{\pi} A_0^{-1} A_{1-\alpha} \,.$$

- Therefore, the solution of the Abel's equation is

  $$f = \frac{\sin(\pi\alpha)}{\pi} A_0^{-1} A_{1-\alpha} g$$

  or

  $$f(x) = \frac{\sin(\pi\alpha)}{\pi} \partial_x \int_0^x dy\, \frac{g(y)}{(x-y)^{1-\alpha}} \,.$$

- Let h be a positive differentiable strictly increasing function on $[0,1]$ and g is a differentiable function on $[0,1]$. Then the solution of the integral equation

  $$\int_0^x dy\, \frac{f(y)}{[h(x) - h(y)]^\alpha} = g(x)$$

  is

  $$f(x) = \frac{\sin(\pi\alpha)}{\pi} \partial_x \int_0^x dy\, \frac{h'(y)}{[h(x) - h(y)]^{1-\alpha}} g(y) \,.$$

1.3.4 Riemann-Liouville Transform

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Let E be the space of smooth functions on $[0,\infty)$ with compact support whose all derivatives vanish at 0. $f^{(k)}(0) = 0$. Let R be the integral operator defined by

$$(Rf)(x) = \int_0^x dy\, f(y) \,.$$

We have

$$\partial_x (Rf)(x) = f(x) \,, \qquad (R\partial f)(x) = f(x) \,.$$

Therefore

$$R\partial = \partial R = \mathrm{Id} \,,$$

so that

$$R = \partial^{-1} \,.$$

For any $n \geq 1$ R^n

$$(R^n f)(x) = \int_0^x dy_{n-1} \cdots \int_0^{y_2} dy_1 \; f(y_1)$$

is equal to

$$(R^n f)(x) = \frac{1}{(n-1)!} \int_0^x dy \; (x-y)^{n-1} f(y) \, .$$

Now, we generalize this for any complex α with $\operatorname{Re} \alpha > 0$

$$(R_\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x dy \; (x-y)^{\alpha-1} f(y) \, .$$

One can show that

$$R_\alpha R_\beta = R_{\alpha+\beta}$$

and

$$R_0 = \mathrm{Id} \; .$$

So, R_α is a complex power of the operator R (or ∂)

$$R_\alpha = R^\alpha = (\partial)^{-\alpha} \, .$$

By analytic continuation we obtain an operator for any α. In particular,

$$R_n = \partial^n \, .$$

Similar transforms, say on smooth functions of compact support, can be defined as follows

$$(B_\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x dy \; (x-y)^{\alpha-1} f(y) \, ,$$

$$(W_\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty dy \; (y-x)^{\alpha-1} f(y) \, ,$$

$$(C_\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^0 dy \; (y-x)^{\alpha-1} f(y) \, ,$$

which are related to each other.

1.3.5 Homework

- Exercises: 5.12[9,16]

Chapter 2

Ordinary Differential Operators

2.1 Ordinary Differential Operators

- Let $I = (a, b)$, where a can be $-\infty$ and b can be $+\infty$. Let $a_2 \in C^2(I)$ be a positive twice-differentiable function, $a_1 \in C^1(I)$ be a differentiable function and $a_0 \in C(I)$ be a continuous function.

  For simplicity we can just assume that all coefficients are smooth, that is are functions from $C^\infty(I)$ and

  $$a_2(x) \neq 0, \qquad \text{for any } x \in I\,.$$

  A **second-order ordinary differential operator** L is an operator acting on twice differentiable functions by

  $$L = a_2\partial_x^2 + a_1\partial_x + a_0\,.$$

  Again, we could restrict first only to smooth functions, that is, restrict first the domain of L to $C^\infty(I)$, or, even, to smooth functions of compact support $C_0^\infty(I)$.

- The **boundary data** of a function f are the values of the function f and its derivative f' at the points a and b.

- Let (α_{ij}) and (β_{ij}), where $i, j = 1, 2$, be two constant matrices and (b_i) be a constant 2-vector. Assume that the 4-vectors $(\alpha_{11}, \alpha_{12}, \beta_{11}, \beta_{12})$ and $(\alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22})$ are linearly independent.

- The **boundary conditions** are

$$B_1(f) = c_1, \qquad B_2(f) = c_2 .$$

- The **homogeneous boundary conditions** are

$$B_1(f) = 0, \qquad B_2(f) = 0 .$$

- The **boundary operators** are defined by

$$B_1(f) = \alpha_{11} f(a) + \alpha_{12} f'(a) + \beta_{11} f(b) + \beta_{12} f'(b),$$

$$B_2(f) = \alpha_{21} f(a) + \alpha_{22} f'(a) + \beta_{21} f(b) + \beta_{22} f'(b) .$$

- Let Γ be the **boundary data map** defined by

$$(\Gamma f)(a) = \begin{pmatrix} f(a) \\ f'(a) \end{pmatrix}, \qquad (\Gamma f)(b) = \begin{pmatrix} f(b) \\ f'(b) \end{pmatrix},$$

- Let $A(a)$ be the matrix (α_{ij}) and $A(b)$ be the matrix (β_{ij}). Then the boundary conditions are

$$(A\Gamma f)(a) + (A\Gamma f)(b) = C ,$$

  where C is the column vector $C = (c_i)$.

- The **separated (or local) boundary operators** have the form

$$B_1(f) = \alpha_{11} f(a) + \alpha_{12} f'(a),$$

$$B_2(f) = \beta_{21} f(b) + \beta_{22} f'(b) .$$

- The **periodic boundary conditions** have the form

$$B_1(f) = f(a) - f(b) = 0,$$

$$B_2(f) = f'(a) - f'(b) = 0 .$$

- **Definition 2.1.1** *Let $I = [a, b]$ and L be a differential operator of order n in $L^2(I)$. The domain $D(L)$ of the differential operator L is the set of all functions whose k-th derivative is square integrable, that is $Lf \in L^2(I)$, and which satisfy the homogeneous boundary conditions $B_i(f) = 0$, where $i = 1, \dots, n-1$.*

- **Definition 2.1.2** *Let $I = [a, b]$ and L be a differential operator of order n in $L^2(I)$. An operator L^* is the adjoint of the operator L if for any $f \in D(L)$ and any $g \in D(L^*)$*

  $$(Lf, g) = (f, L^*g)\,.$$

- Let $L^2(I)$ have the standard inner product with the weight function equal to 1. The **adjoint** of the operator

  $$L = a_2\partial_x^2 + a_1\partial_x + a_0\,.$$

  is given by

  $$\begin{aligned} L^* &= \partial_x^2 a_2 - \partial_x a_1 + a_0 \\ &= a_2\partial_x^2 + (2a_2' - a_1)\partial_x + (a_2'' - a_1' + a_0)\,. \end{aligned}$$

- Let f and g be differentiable on $[a, b]$. The **bilinear concomitant** of f and g is

  $$J(f, g) = a_2[(\partial_x f)g - f\partial_x g] + (a_1 - a_2')fg\,.$$

- For a $f \in D(L)$ and any $g \in L^2(I)$ we have

  $$(Lf, g) = (f, L^*g) + J(f, g)\big|_a^b\,.$$

- The domain $D(L^*)$ of the adjoint L^* of the operator L is the set of all twice-differentiable functions g in $L^2(I)$ such that $J(f, g)\big|_a^b = 0$ for any $f = D(L)$. That is the functions that satisfy the **adjoint homogeneous boundary conditions**

  $$B_1^*(g) = B_2^*(g) = 0\,.$$

- The operator

  $$L = a_2\partial_x^2 + a_1\partial_x + a_0\,.$$

  is **formally self-adjoint** if

  $$a_2' = a_1\,.$$

  It can be written in the form

  $$L = \partial_x a_2 \partial_x + a_0\,.$$

- The operator L is self-adjoint if it is formally self-adjoint and $D(L) = D(L^*)$.

- For a formally self-adjoint operator L the concomitant has the form

$$J(f, g) = a_2[(\partial_x f)g - f\partial_x g]\,.$$

- Any operator of the form

$$L = a_2\partial_x^2 + a_1\partial_x + a_0\,.$$

  can be made formally self-adjoint by multiplying it by the **integrating factor**

$$\mu = \frac{1}{a_2}\exp\left[\int dx \frac{a_1}{a_2}\right]\,.$$

- Let ω be a **weight function** that is positive except possibly at isolated points where $\omega(x) = 0$.

- We define the Hilbert space $L^2(I, \omega)$ by the inner product

$$(f, g) = \int_a^b dx\, \omega(x) f(x)\bar{g}(x)\,.$$

- The adjoint of the operator

$$L = a_2\partial_x^2 + a_1\partial_x + a_0$$

  with respect to the weighted inner product is

$$\begin{aligned} L^* &= \omega^{-1}\partial_x^2\omega a_2 - \omega^{-1}\partial_x\omega a_1 + a_0 \\ &= a_2\partial_x^2 + \left(2a_2' - a_1 + 2a_2\frac{\omega'}{\omega}\right)\partial_x \\ &\quad + \left[a_2'' - a_1' + a_0 + (2a_2' - a_1)\frac{\omega'}{\omega} + a_2\frac{\omega''}{\omega}\right]\omega\,. \end{aligned}$$

- So, the operator L is formally self-adjoint if

$$a_1 = a_2' + a_2\frac{\omega'}{\omega}\,.$$

- A formally self-adjoint operator can be written in the form

$$L = \frac{1}{\omega}\partial_x\omega p\partial_x + q\,.$$

- The spectrum of the operator L is determined by the equations

$$L f_n = \lambda_n f_n, \qquad B_1(f_n) = B_2(f_n) = 0, \qquad \| f_n \| = 1 .$$

- **Examples.**

1. The operator
$$L = i\partial_x$$
acting on smooth functions of compact support on $\mathbb{R}$, $C_0^\infty(\mathbb{R})$, is symmetric and essentially self-adjoint on $L^2(\mathbb{R})$. That is L^* is self-adjoint and $L \subset L^*$.

2. The operator
$$L = i\partial_x$$
acting on smooth functions of compact support on $\mathbb{R}_+$, $C_0^\infty(\mathbb{R}_+)$, where $\mathbb{R}_+ = (0, \infty)$, is symmetric but does not have any self-adjoint extensions to $L^2(\mathbb{R}_+)$.

3. The operator
$$L = i\partial_x$$
acting on smooth functions of compact support on $[0, 1]$ is symmetric. It has a self-adjoint extension to $L^2([0, 1])$ defined by the boundary conditions
$$f(0) = f(1) = 0 .$$
The general self-adjoint extension is defined by the boundary conditions
$$f(1) = e^{i\varphi} f(0) ,$$
where φ is an angle.

4. Let $I = [0, 1]$ and $a_2(x) \neq 0$ on I. The operator
$$L = a_2\partial_x^2 + a_1\partial_x + a_0$$
acting on smooth functions on $[0, 1]$ with the boundary conditions
$$f(0) = 0, \qquad f'(1) = 0 ,$$
is self-adjoint.

5. The operator
$$L = -\partial_x^2$$
acting on smooth functions on $[0, 1]$ with the boundary conditions
$$f(0) = 0\,, \qquad f'(0) = 0\,,$$
is not self-adjoint.
6. The operator
$$L = -\partial_x^2$$
acting on smooth functions on $\mathbb{R}_+$ with compact support is symmetric. It has a symmetric extension defined by the boundary conditions
$$f(0) = 0\,,$$
which is essentially self-adjoint on $L^2(\mathbb{R}_+)$.
The most general self-adjoint extension of the operator L is defined by the boundary conditions
$$\cos\varphi\, f(0) + \sin\varphi\, f'(0) = 0\,,$$
where φ is an arbitrary angle.
7. Legendre, Associated Legendre, Chebyshev, Jacobi, Laguerre, Associated Laguerre, Bessel, Hermite, operators.

2.1.1 Homework

- Exercises: 5.12[17,19,21,]
- Find the spectrum (eigenvalues and the eigenfunctions) of the operators $L = -\partial_x^2$ on the interval $[a, b]$ with the following boundary conditions
  1. Dirichlet
  $$f(a) = f(b) = 0\,,$$
  2. Neumann
  $$f'(a) = f'(b) = 0\,,$$
  3. Zaremba
  $$f(a) = f'(b) = 0\,,$$
  4. Periodic
  $$f(a) = f(b)\,.$$

2.2 Sturm-Liouville Systems

- Let $I = [a, b]$ be an interval, ω be a smooth real-valued positive function on I, i.e. $\omega(x) > 0$ for any $x \in I$, be a smooth real-valued positive function on I, i.e. $p(x) > 0$ for any $x \in I$, and q be a smooth real-valued function on I.

- Let L be a differential operator acting on smooth functions, $f \in C^\infty(I)$, on I defined by
$$L = -\omega^{-1}\partial_x \omega p \partial_x + q\,.$$

- Let B be the boundary operator defined by
$$Bf = (uf + vf')|_{\partial I},$$
where u and v are some real-valued functions of the boundary of I, consisting of two points, $\partial I = \{a, b\}$, which are not simultaneously zero, that is $u^2 + v^2 > 0$.

- Let $L^2(I, \omega)$ be the Hilbert space with the weight function ω and $\| \cdot \|$ be the corresponding norm. A **regular Sturm-Lioville system** is the boundary value problem
$$Lf = \lambda f\,, \qquad \| f \| = 1\,,$$
with the homogeneous boundary conditions
$$Bf = 0\,.$$

- If the function p is positive only on an open interval (a, b) but vanishes at one or both endpoints of the interval $[a, b]$, then the boundary condition Bf at that point is replaced by the condition that f must be bounded at that point. Such a boundary value problem is called a **singular Sturm-Lioville system**.

- If the functions p, ω, and q are periodic, that is $p(a) = p(b)$, $\omega(a) = \omega(b)$ and $q(a) = q(b)$, and the boundary conditions are periodic, that is,
$$f(a) = f(b), \qquad f'(a) = f'(b)\,,$$
then the boundary value problem is called a **periodic Sturm-Lioville system**.

- **Examples**.

- The domain of L

$$D(L) = \{f \in L^2(I,\omega) | f'' \in L^2(I,\omega), Bf|_{\partial I} = 0\}$$

is the space of all complex valued functions f on I for which f'' is in $L^2(I,\omega)$ and which satisfy the boundary conditions.

- **Theorem 2.2.1 (Lagrange Indentity)** . *For any $f, g \in D(L)$ there holds*
$$fLg - (Lf)g = -\omega^{-1}\partial_x[\omega p(f\partial_x g - (\partial_x f)g)]\,.$$

Proof: Computation.

■

- **Theorem 2.2.2 (Abel's Formula)** . *Let f and g be two eigenfunctions of the operator L corresponding to the same eigenvalue λ,*
$$Lf = \lambda f, \qquad Lg = \lambda g\,.$$
Then
$$pW(f,g) = \text{const}\,,$$
where (f,g) is the Wronskian defined by
$$W(f,g) = f\partial_x g - (\partial_x f)g\,.$$

Proof: Integrate Lagrange identity from a to x and use the boundary conditions.

■

- **Theorem 2.2.3** *The eigenvalues of a regular Sturm-Lioville system are simple, that is, have multiplicity one.*

Proof: Use Abel formula and the boundary conditions.

■

- **Theorem 2.2.4** *L is a self-adjoint operator on $L^2(I,\omega)$.*

Proof: Integrate Lagrange identity from a to b and use the boundary conditions.

■

- **Theorem 2.2.5** *The eigenvalues of a Sturm-Lioville system are real.*

  Proof: Easy.

  ■

- **Remark**. The eigenvalues of a regular Sturm-Liouville system form an infinite sequence, λ_n, such that $\lambda_n \to \infty$. (Will prove later).

- **Theorem 2.2.6** *The eigenfunctions corresponding to distinct eigenvalues of a Sturm-Lioville system are orthogonal in $L^2(I, \omega)$.*

  Proof: Easy.

  ■

2.2.1 Homework

- Exercises: 5.12[18,19,21,22,23,24,25]

2.3 Green Functions

- Let L be a differential operator acting on functions on $I = [a, b]$ satisfying given boundary conditions. Then the solution of the boundary value problem
$$L\varphi = f, \qquad B\varphi = 0$$
has the form
$$\varphi = L^{-1} f\,,$$
or, in integral form,
$$\varphi(x) = \int_a^b dy\, G(x, y) f(y)\,,$$
where $G(x, y)$ is the **Green function** of the operator L. It is the kernel of the operator L^{-1}.

- **Theorem 2.3.1** *Let $I = [a, b]$ and $p, q \in C^\infty(I)$ be smooth real-valued functions on I such that $p(x) > 0$ for any $x \in I$. Let L be an operator acting on smooth functions on I defined by*
$$L = -\partial p \partial + q\,.$$
Let B be the boundary operator defined by
$$B\varphi = (u\varphi + v\varphi')|_{\partial I}\,,$$
where u and v are real-valued functions on $\partial I = \{a, b\}$ such that
$$u^2 + v^2 > 0\,.$$
Let φ_1 and φ_2 be the non-zero solutions of the equation
$$L\varphi = 0$$
with the boundary conditions at one point, that is
$$u(a)\varphi_1(a) + v(a)\varphi_1'(a) = 0\,,$$
$$u(b)\varphi_2(b) + v(b)\varphi_2'(b) = 0\,.$$

Theorem 2.3.2 *Suppose that the operator L with the boundary conditions $B\varphi = 0$ does not have zero eigenvalue. Let $W(\varphi_1, \varphi_2)$ be the Wronskian of the solutions φ_1, φ_2,*

$$W(\varphi_1, \varphi_2) = \varphi_1 \varphi_2' - \varphi_1' \varphi_2$$

and C be the constant defined by

$$C = -pW(\varphi_1, \varphi_2)\,.$$

Let $f \in C(I)$ be a continuous function on I. Then the Sturm-Liouville boundary value problem

$$L\varphi = f\,, \qquad B\varphi = 0\,,$$

has a unique solution

$$\varphi(x) = \int_a^b dy\; G(x,y)\varphi(y)\,,$$

where the Green function $G(x,y)$ is defined by

$$G(x,y) = \begin{cases} \dfrac{1}{C}\varphi_1(x)\varphi_2(y), & \text{for } x < y \\ \dfrac{1}{C}\varphi_2(x)\varphi_1(y), & \text{for } x \geq y\,. \end{cases}$$

- **Remark.** Let θ be a function on I defined by

$$\theta(x,y) = \begin{cases} 0, & \text{for } x < y \\ 1, & \text{for } x \geq y\,. \end{cases}$$

Then the Green function $G(x,y)$ can be written in the form

$$G(x,y) = \frac{1}{C}\theta(x,y)\varphi_1(x)\varphi_2(y) + \frac{1}{C}\theta(y,x)\varphi_2(x)\varphi_1(y)\,.$$

- ***Proof:*** Use the method of variation of parameters.

■

- **Theorem 2.3.3** *Let L be as described above. The operator*

  $$L^{-1} : L^2(I) \to C(I)$$

  is a self-adjoint compact operator.

- ***Proof:***

  1. L^{-1} is compact because it is an integral operator with a continuous kernel.
  2. L^{-1} is self-adjoint because its kernel is self-adjoint

     $$G(x, y) = \overline{G(y, x)}\,.$$

  3. The range of the operator L^{-1} is $C(I)$ since the Green function $G(x, y)$ is continuous in the first argument.

  ■

- **Remark**. If the operator L has smooth coefficients, then the operator L^{-1} is a **smoothing operator**

  $$L^{-1} : L^2(I) \to C^\infty(I)\,.$$

- **Theorem 2.3.4** *Let L be the operator with the boundary conditions defined above. Then the operator L^{-1} does not have a zero eigenvalue.*

- ***Proof:*** Calculation.

  ■

- **Theorem 2.3.5** *Let L be the operator with the boundary conditions defined above. Let $\{(\lambda_n, \varphi_n)\}$ be the eigenvalues and the eigenfunctions of the operator L and $\{(\mu_n, \psi_n)\}$ be the eigenvalues and the eigenfunctions of the operator L^{-1}. Then*

  $$\mu_n = \frac{1}{\lambda_n}\,, \qquad \psi_n = \varphi_n\,.$$

- ***Proof:*** Obvious.

  ■

- **Properties of the spectrum**. Let L be as described above. Then:

1. The operator L is self-adjoint.
2. The spectrum of the operator L is real.
3. The spectrum of the operator L is bounded below.
4. The spectrum is discrete.
5. All eigenvalues are simple, that is, have multiplicity one.
6. The eigenvalues form an unbounded monotonically increasing sequence.
7. There are at most finitely many negative eigenvalues.
8. The eigenfunctions form an orthonormal basis in $L^2([a,b])$.
9. The resolvent of the operator L is a Hilbert-Schmidt operator.

- The spectrum λ_n of the operator L is defined by

$$L\varphi_n = \lambda_n\varphi_n\,, \qquad \|\,\varphi_n\,\| = 1\,.$$

- Then

$$(\varphi_n, \varphi_m) = \delta_{nm}$$

and

$$\sum_{n=1}^{\infty} \varphi_n(x)\bar{\varphi}_n(x') = \delta(x - x')\,,$$

where $\delta(x-x')$ is the kernel of the identity operator (the Dirac delta-function).

- Let $\lambda \in \mathbb{C}$ be in the resolvent set of the operator L, that is the operator $(L - \lambda I)^{-1}$ be a bounded operator.

- The resolvent satisfies the equation

$$(L_x - \lambda I)G(\lambda; x, x') = \delta(x - x')\,,$$

the self-adjointness condition

$$\overline{G(\lambda; x, x')} = G(\bar{\lambda}; x' x)\,,$$

and the boundary conditions in both variables.

- The resolvent is given by

$$G(\lambda; x, x') = \sum_{n=1}^{\infty} \frac{1}{\lambda_n - \lambda} \varphi_n(x)\bar{\varphi}_n(x')\,.$$

- Let C be a contour in the complex plane going around the spectrum in the counterclockwise direction from $\infty + i\varepsilon$ to $\infty - i\varepsilon$. Then a function $f(L)$ of the operator L can be defined in terms of the resolvent by the Cauchy formula

$$f(L) = -\frac{1}{2\pi i}\int_C d\lambda\ f(\lambda)G(\lambda)\,.$$

The kernel of this operator is

$$\begin{aligned} f(L)(x,x') &= -\frac{1}{2\pi i}\int_C d\lambda\ f(\lambda)G(\lambda;x,x') \\ &= \sum_{n=1}^{\infty} f(\lambda_n)\varphi_n(x)\bar{\varphi}_n(x')\,. \end{aligned}$$

- In some cases, depending on the function f one can define the L^2-trace by

$$\begin{aligned} \mathrm{Tr}_{L^2} f(L) &= \int_a^b dx\ f(L)(x,x) \\ &= \sum_{n=1}^{\infty} f(\lambda_n)\,. \end{aligned}$$

- In particular, for $t > 0$ we define the heat kernel

$$\begin{aligned} U(t;x,x') &= -\frac{1}{2\pi i}\int_C d\lambda\ e^{-t\lambda}G(\lambda;x,x') \\ &= \sum_{n=1}^{\infty} e^{-t\lambda_n}\varphi_n(x)\bar{\varphi}_n(x') \end{aligned}$$

and its trace

$$\begin{aligned} \Theta(t) &= \mathrm{Tr}_{\,L^2}\exp(-tL) \\ &= \int_a^b dx\ U(t;x,x) \\ &= \sum_{n=1}^{\infty} e^{-t\lambda_n}\,. \end{aligned}$$

- Let $\mu \in \mathbb{R}$ be a real number such that $\mu < \lambda_1$ and $s \in \mathbb{C}$ be a complex number such that $\text{Re}\, s > \frac{1}{2}$. Then we define the kernel of the zeta-function

$$\begin{aligned} G^s(\mu; x, x') &= -\frac{1}{2\pi i}\int_C d\lambda\, \frac{1}{(\lambda-\mu)^s} G(\lambda; x, x') \\ &= \sum_{n=1}^{\infty} \frac{1}{(\lambda_n - \mu)^s} \varphi_n(x)\bar{\varphi}_n(x')\,, \end{aligned}$$

and the zeta-function

$$\begin{aligned} \zeta(s,\mu) &= \text{Tr}_{L^2}(L-\mu I)^{-s} \\ &= \int_a^b dx\, G^s(\mu; x, x) \\ &= \sum_{n=1}^{\infty} \frac{1}{(\lambda_n-\mu)^s}\,. \end{aligned}$$

- Determinant of the differential operator

$$\text{Det}_{L^2}(L-\mu I) = \exp[-\partial_s \zeta(0,\mu)]\,.$$

- Expression of the resolvent and the zeta function in terms of the heat kernel

$$G(\lambda; x, x') = \int_0^{\infty} dt\, e^{t\lambda} U(t; x, x')$$

and

$$G^s(\mu; x, x') = \frac{1}{\Gamma(s)}\int_0^{\infty} dt\, t^{s-1} e^{t\lambda} U(t; x, x')\,.$$

- **Asymptotics of the resolvent and the heat kertnel.**

As $x \to x'$ and $\lambda \to -\infty$

$$\begin{aligned} G(\lambda; x, x') \sim\ & |x-x'| \sum_{n=0}^{\infty}\sum_{k=0}^{n} b_k^{(n)}(x')(-\lambda)^{n-k}(x-x')^n \\ & +(-\lambda)^{-1/2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} c_k^{(n)}(x')(-\lambda)^{n-k}(x-x')^n\,, \end{aligned}$$

As $t \to 0$ and $x \to x'$

$$U(t; x, x') \sim (4\pi t)^{-1/2} \exp\left[-\frac{(x-x')^2}{4t}\right] \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} t^n a_n^{(k)}(x')(x-x')^k\,.$$

2.3.1 Operators with Constant Coefficients

- Let us consider the space $C_0^\infty(\mathbb{R})$ of all smooth functions of compact support (which is dense in $L^2(\mathbb{R})$). Let ∂ be the operator of differentiation.

- A differential operator of order m with constant coefficients is a polynomial in the operator ∂ of degree m with constant coefficients, that is,

$$L = P_m(\partial) = \sum_{k=0}^{m} a_k(\partial)^k ,$$

where a_k are some constants and $a_m \neq 0$.

- Let

$$K(x) = \left[\mathcal{F}^{-1}\left(\frac{1}{P_m(i\omega)}\right)\right](x) .$$

- The Green function of the operator L is

$$G(x, y) = K(x - y) .$$

- Let $f \in L^2(\mathbb{R})$. Then the solution of the equation

$$L\varphi = f ,$$

is given by

$$\varphi = K * f$$

that is

$$\varphi(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} dy\, K(x - y) f(y) .$$

2.3.2 Examples

- Let L be the operator

$$L = -\partial^2$$

on $I \subseteq \mathbb{R}$ with different boundary conditions. Let $\lambda \in \mathbb{C}$ and $G(\lambda; x, y)$ be the resolvent kernel of the operator L, that is, the kernel of the operator $(L - \lambda I)^{-1}$. Let $\mathrm{Re}\,\lambda < 0$ and $\mu = \sqrt{-\lambda}$ such that $\mathrm{Re}\,\mu > 0$.

1. **Dirichlet operator**. Let $I = [a, b]$ with the boundary conditions

$$\varphi(a) = \varphi(b) = 0\,.$$

Eigenfunctions

$$\varphi_n(x) = \sqrt{\frac{2}{b-a}}\sin\left(\frac{x-a}{b-a}n\pi\right), \qquad n = 1, 2, 3, \ldots\,.$$

Eigenvalues

$$\lambda_n = \left(\frac{\pi}{b-a}\right)^2 n^2$$

The lowest eigenvalue

$$\lambda_{\min} = \left(\frac{\pi}{b-a}\right)^2\,.$$

Resolvent

$$\begin{aligned} G(\lambda; x, y) &= \frac{1}{\mu}\left\{-\theta(x-y)\sinh\left[\mu(x-y)\right] + \frac{\sinh\left[\mu(b-y)\right]\sinh\left[\mu(x-a)\right]}{\sinh\left[\mu(b-a)\right]}\right\} \\ &= \frac{1}{2\mu}\left\{\exp\left[-\mu|x-y|\right] + \frac{\cosh\left[\mu(x-y)\right]\exp\left[-\mu(b-a)\right]}{\sinh\left[\mu(b-a)\right]} - \frac{\cosh\left[\mu(x+y-b-a)\right]}{\sinh\left[\mu(b-a)\right]}\right\} \end{aligned}$$

2. **Neumann operator**. Let $I = [a, b]$ with the boundary conditions

$$\varphi'(a) = \varphi'(b) = 0\,.$$

Eigenfunctions

$$\varphi_n(x) = \sqrt{\frac{2}{b-a}}\cos\left(\frac{x-a}{b-a}n\pi\right), \qquad n = 0, 1, 2, \ldots\,.$$

Eigenvalues

$$\lambda_n = \left(\frac{\pi}{b-a}\right)^2 n^2$$

The lowest eigenvalue

$$\lambda_{\min} = 0\,.$$

Resolvent

$$\begin{aligned} G(\lambda; x, y) &= \frac{1}{\mu}\Bigg\{-\theta(x-y)\sinh\left[\mu(x-y)\right] \\ &\qquad + \frac{\cosh\left[\mu(b-y)\right]\cosh\left[\mu(x-a)\right]}{\sinh\left[\mu(b-a)\right]}\Bigg\} \\ &= \frac{1}{2\mu}\Bigg\{\exp\left[-\mu|x-y|\right] \\ &\qquad + \frac{\cosh\left[\mu(x-y)\right]\exp\left[-\mu(b-a)\right]}{\sinh\left[\mu(b-a)\right]} \\ &\qquad + \frac{\cosh\left[\mu(x+y-b-a)\right]}{\sinh\left[\mu(b-a)\right]}\Bigg\} \end{aligned}$$

3. **Zaremba operator**. Let $I = [a, b]$ with the boundary conditions

$$\varphi'(a) = \varphi(b) = 0\,.$$

Eigenfunctions

$$\varphi_n(x) = \sqrt{\frac{2}{b-a}}\cos\left[\frac{x-a}{b-a}\left(n+\frac{1}{2}\right)\pi\right], \qquad n = 0, 1, 2, \ldots\,.$$

Eigenvalues

$$\lambda_n = \left(\frac{\pi}{b-a}\right)^2\left(n+\frac{1}{2}\right)^2$$

The lowest eigenvalue

$$\lambda_{\min} = \left(\frac{\pi}{b-a}\right)^2\frac{1}{4}\,.$$

Resolvent

$$G(\lambda; x, y) = \frac{1}{\mu}\left\{-\theta(x-y)\sinh\left[\mu(x-y)\right] + \frac{\sinh\left[\mu(b-y)\right]\cosh\left[\mu(x-a)\right]}{\cosh\left[\mu(b-a)\right]}\right\}$$

$$= \frac{1}{2\mu}\left\{\exp\left[-\mu|x-y|\right] - \frac{\cosh\left[\mu(x-y)\right]\exp\left[-\mu(b-a)\right]}{\cosh\left[\mu(b-a)\right]} - \frac{\sinh\left[\mu(x+y-b-a)\right]}{\cosh\left[\mu(b-a)\right]}\right\}$$

4. **Laplacian on the real line**. Let $I = \mathbb{R}$ with the boundedness condition at $\pm\infty$. Spectrum is $[0, \infty)$ with no discrete eigenvalues.
   Resolvent
   $$G(\lambda; x, y) = \frac{1}{2\mu}\exp\left[-\mu|x-y|\right]$$
5. **Dirichlet operator on half-line**. Let $I = \mathbb{R}_+$ with the boundedness condition at ∞ and Dirichlet condition at zero
   $$\varphi(0) = 0\,.$$
   Spectrum is $[0, \infty)$ with no discrete eigenvalues.
   Resolvent
   $$G(\lambda; x, y) = \frac{1}{2\mu}\left\{\exp\left[-\mu|x-y|\right] - \exp\left[-\mu(x+y)\right]\right\}\,.$$
6. **Neumann operator on half-line**. Let $I = \mathbb{R}_+$ with the boundedness condition at ∞ and Neumann condition at zero
   $$\varphi'(0) = 0\,.$$
   Spectrum is $[0, \infty)$ with no discrete eigenvalues.
   Resolvent
   $$G(\lambda; x, y) = \frac{1}{2\mu}\left\{\exp\left[-\mu|x-y|\right] + \exp\left[-\mu(x+y)\right]\right\}\,.$$

2.3.3 Homework

- Exercises:

Chapter 3

Distributions and Partial Differential Equations

3.1 Distributions

- Let $x = (x^\mu) = (x^1, \ldots, x^n)$ be Cartesian coordinates in $\mathbb{R}^n$.
- We consider the space $C^\infty(\mathbb{R}^n)$ of smooth functions on $\mathbb{R}^n$.
- The operators
$$D_\mu = \partial_\mu = \frac{\partial}{\partial x^\mu}$$
are partial differential operators acting on smooth functions.
- Let
$$\alpha = (\alpha_1, \ldots, \alpha_n), \qquad \alpha_i \geq 0\,,$$
be a multiindex with α_i be nonnegative integers and
$$|\alpha| = \alpha_1 + \cdots \alpha_n$$
be the norm (length) of the multiindex α.
- We denote by
$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial(x^1)^{\alpha_1} \cdots \partial(x^n)^{\alpha_n}}$$
a partial differential operator of order $|\alpha|$.

- Let $a_\alpha = a_{\alpha_1 \dots \alpha_n}$ be smooth functions on $\mathbb{R}^n$. Then the operator

$$L = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$$

  is a **linear partial differential operator of order** m.

- Alternatively, any differential operator of order m can be written in the form

$$L = \sum_{k=0}^{m} a^{\mu_1 \dots \mu_k} \partial_{\mu_1} \cdots \partial_{\mu_k} ,$$

  where $a^{\mu_1 \dots \mu_k}$ are smooth functions over $\mathbb{R}^n$ and a summation over repeated indices is understood.

- The **formal adjoint** of the operator L with respect to the L^2 inner product with unit weight is

$$L^* = \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha a_\alpha .$$

- Let $\xi = (\xi_\mu) = (\xi_1, \dots, \xi_n)$ be a vector in $\mathbb{R}^n$ and

$$\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} .$$

  The **symbol** of the operator L is a function on $\mathbb{R}^n \times \mathbb{R}^n$ defined by

$$\sigma(L; x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha .$$

- The **principal part** of the operator L is

$$L_p = \sum_{|\alpha| = m} a_\alpha(x) D^\alpha .$$

- The **principal symbol** of the operator L is a function on $\mathbb{R}^n \times \mathbb{R}^n$ defined by

$$\sigma_p(L; x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha .$$

- **Idea of distributions**. Let g be a given function. Consider the differential equation
$$Lf = g\,.$$
Let us multiply it by a "nice" function φ. Then
$$(Lf,\varphi) = (g,\varphi)$$
or
$$(f, L^*\varphi) = (g,\varphi)\,.$$
A function f may satisfy this equation *without being m times differentiable!*

- Notice that if $f \in C(\mathbb{R}^n)$ is continuous and $\varphi \in C_0^\infty(\mathbb{R})$ is smooth with compact support, then the equation $(f,\varphi) = 0$ implies $f = 0$.

- **Definition 3.1.1 (Test Functions.)** *A* **test function** *is a smooth function with compact support in $\mathbb{R}^n$. The space of the test functions is denoted by $\mathcal{D}(\mathbb{R}^n)$. That is $\mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$.*

- **Example**. Let $|x| = \sqrt{(x^1)^2 + \cdots + (x^n)^2}$. Then
$$\varphi(x) = \begin{cases} \exp\left(\dfrac{1}{|x|^2 - 1}\right), & \text{if } |x| < 1 \\ 0, & \text{if } |x| \ge 1 \end{cases}$$
is a test function.

- **Theorem 3.1.1** *The space $\mathcal{D}(\mathbb{R}^n)$ is a vector space. Let $\varphi,\psi \in \mathcal{D}(\mathbb{R}^n)$, $f \in C^\infty(\mathbb{R}^n)$ and $A : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transformation (that is a nondegenerate linear transformation). Then*

  1. *$f\varphi \in \mathcal{D}(\mathbb{R}^n)$, that is $\mathcal{D}(\mathbb{R}^n)$ is a module over $C^\infty(\mathbb{R}^n)$,*
  2. *$\varphi \circ A \in \mathcal{D}(\mathbb{R}^n)$, that is $\mathcal{D}(\mathbb{R}^n)$ is invariant under the action of the group of affine transformations,*
  3. *$\varphi * \psi \in \mathcal{D}(\mathbb{R}^n)$, that is $\mathcal{D}(\mathbb{R}^n)$ is closed under the convolution.*

  Proof: Exercise.

  ■

- **Definition 3.1.2 (Convergence of Test Functions.)** *Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ be a test function and (φ_n) be a sequence of test functions. Then the sequence (φ_n) converges to φ in $\mathcal{D}(\mathbb{R}^n)$ if:*

  1. *there is a compact set $B \subset \mathbb{R}^n$ (independent of n) such that all φ_n vanish outside B, that is*
  $$\bigcup_{n=1}^{\infty} \operatorname{supp} \varphi_n \subset B\,,$$
  2. *for any α*
  $$D^\alpha \varphi_n \to D^\alpha \varphi$$
  uniformly on $\mathbb{R}^n$.

- **Notation**. Convergence in $\mathcal{D}(\mathbb{R}^n)$ is denoted by
$$\varphi_n \stackrel{\mathcal{D}}{\to} \varphi$$

- **Examples.**

- **Theorem 3.1.2** *Let (φ_n) and (ψ_n) be sequences in $\mathcal{D}(\mathbb{R}^n)$ and $\varphi, psi \in \mathcal{D}(\mathbb{R}^n)$ be such that*
$$\varphi_n \stackrel{\mathcal{D}}{\to} \varphi \qquad \text{and} \qquad \psi_n \stackrel{\mathcal{D}}{\to} \psi\,.$$
let $a, b \in \mathbb{C}$, $f \in C^\infty(\mathbb{R}^n)$, A be an affine transformation of $\mathbb{R}^n$, and α be a multiindex. Then:

  1. $a\varphi_n + b\psi_n \stackrel{\mathcal{D}}{\to} a\varphi + b\psi$,
  2. $f\varphi_n \stackrel{\mathcal{D}}{\to} f\varphi$,
  3. $\varphi_n \circ A \stackrel{\mathcal{D}}{\to} \varphi \circ A$,
  4. $D^\alpha\varphi_n \stackrel{\mathcal{D}}{\to} D^\alpha\varphi$.

Proof: Exercise.

■

- **Definition 3.1.3** *A* **distribution** *(or a* **generalized function***) F on $\mathbb{R}^n$ is a continuous linear functional on $\mathcal{D}(\mathbb{R}^n)$.*

  That is, a distribution F on $\mathbb{R}^n$ is a mapping $F : \mathbb{R}^n \to \mathbb{C}$ such that

  1. *$F(a\varphi + b\psi) = aF(\varphi) + bF(\psi)$ for any $a, b \in \mathbb{C}$ and $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$,*
  2. *$F(\varphi_n) \to F(\varphi)$ in $\mathbb{C}$ for any sequence (φ_n) in $\mathcal{D}(\mathbb{R}^n)$ converging to $\varphi \in \mathcal{D}(\mathbb{R}^n)$ in $\mathcal{D}(\mathbb{R}^n)$, i.e. $\varphi_n \xrightarrow{\mathcal{D}} \varphi$.*

- **Notation**. The space of all distributions is denoted by $\mathcal{D}'(\mathbb{R}^n)$.

  The action of the distribution F on a test function φ is denoted by

  $$F(\varphi) = \langle F, \varphi\rangle .$$

- Every locally integrable function f can be identified with a distribution F by

  $$\langle F, \varphi, \rangle = (\varphi, \bar{f}) .$$

- **Definition 3.1.4 (Regular and Singular Distributions.)** *A distribution $F \in \mathcal{D}'(\mathbb{R}^n)$ is called* **regular** *if there is a locally integrable function f such that for any test function $\varphi \in \mathcal{D}(\mathbb{R}^n)$,*

  $$\langle F, \varphi\rangle = (\varphi, \bar{f}) .$$

  A distribution which is not regular is called **singular**.

- **Heaviside distribution**. Let $\Omega \subset \mathbb{R}^n$ be an open set in $\mathbb{R}^n$ and χ_Ω be the characteristic function of Ω. The **Heaviside distribution** H is a regular distribution defined by

  $$\langle F, \varphi\rangle = (\varphi, \chi_\Omega) = \int_\Omega \varphi .$$

  If $\Omega = (0, \infty) \times \cdots \times (0, \infty)$, then the distribution

  $$\langle H, \varphi\rangle = \int_0^\infty \cdots \int_0^\infty dx^1 \cdots dx^n \varphi(x) .$$

- **Dirac distribution** δ is a singular distribution defined by

  $$\langle \delta, \varphi\rangle = \varphi(0) .$$

- Let A be an operator defined on test functions, i.e. on $\mathcal{D}(\mathbb{R}^n)$. We can extend it to distributions, i.e. to $\mathcal{D}'(\mathbb{R}^n)$,by

$$\langle AF, \varphi \rangle = \langle F, A^* \varphi \rangle .$$

- **Definition 3.1.5 (Derivative of Distributions)** *The derivative $\partial_\mu F$ of a distribution $F \in \mathcal{D}'(\mathbb{R}^n)$ is defined by*

$$\langle \partial_\mu F, \varphi \rangle = \langle F, (-\partial_\mu) \varphi \rangle .$$

  More generally,

$$\langle D^\alpha F, \varphi \rangle = \langle F, (-1)^{|\alpha|} D^\alpha \varphi \rangle ,$$

  and for any differential operator L

$$\langle LF, \varphi \rangle = \langle F, L^* \varphi \rangle .$$

- **Theorem 3.1.3** *Let F be a distribution and L be a differential operator with smooth coefficients. Then LF is a distribution.*

Proof: Prove the linearity and continuity of $D^\alpha F$.

■

- **Example**.

$$\langle D^\alpha \delta, \varphi \rangle = (-1)^{|\alpha|} D^\alpha \varphi(0) .$$

- **Definition 3.1.6 (Weak Distributional Convergence.)** *A sequence of distributions (F_n) in $\mathcal{D}'(\mathbb{R}^n)$ converges to a distribution $F \in \mathcal{D}'(\mathbb{R}^n)$ if for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$*

$$\langle F_n, \varphi \rangle \to \langle F, \varphi \rangle .$$

  This convergence is called weak distributional convergence.

- **Examples**.

  1. Let $f, f_n \in C(\mathbb{R}^n)$ and $f_n \to f$ uniformly on any compact subset of $\mathbb{R}^n$. Let F, F_n be the corresponding regular distributions. Then $F_n \to F$.
  2. Let $f, f_n \in L^1(\mathbb{R}^n)$ and $f_n \to f$ in $L^1(\mathbb{R}^n)$. Let F, F_n be the corresponding regular distributions. Then $F_n \to F$.

- **Terminology**. If a sequence of regular distributions F_n corresponding to a sequence of functions f_n converges to a distribution F, then we say that the sequence of functions f_n **converges in the distributional sense (or converges distributionally)** to the distribution F.

- **Example.** Sequences of functions converging distributionally to the Dirac distribution. Let $n \in \mathbb{N}$.

  1. $$f_n(x) = \frac{1}{\pi}\frac{n}{1+n^2x^2}\,.$$

  2. $$f_n(x) = \frac{n}{\sqrt{\pi}}e^{-n^2x^2}\,.$$

  3. $$f_n(x) = \frac{\sin(nx)}{\pi x}\,.$$

- **Theorem 3.1.4** *Let (F_n) be a sequence in $\mathcal{D}'(\mathbb{R}^n)$ converging to $F \in \mathcal{D}'(\mathbb{R}^n)$, that is $F_n \to F$. Then for any multiindex α*
$$D^\alpha F_n \to D^\alpha F\,.$$

  Proof: Note that
$$\langle D^\alpha F_n, \varphi\rangle \to (-1)^{|\alpha|}\langle F, D^\alpha\varphi\rangle = \langle D^\alpha F, \varphi\rangle\,.$$

  ■

- **Definition 3.1.7 (Antiderivative of Distributions.)** *Let $F, G \in \mathcal{D}'(\mathbb{R})$ be distributions on $\mathbb{R}$. Then G is an* **anti-derivative** *of F if $G' = F$.*

- **Theorem 3.1.5** *Every distribution on $\mathbb{R}$ has an anti-derivative.*

  Proof:

  1. Let $\varphi_0 \in \mathcal{D}(\mathbb{R})$ be a test function such that
$$\int \varphi_0 = 1\,.$$

2. Let $\varphi \in \mathcal{D}(\mathbb{R})$ and
$$K = \int \varphi\,.$$
Then there is a decomposition
$$\varphi = K\varphi_0 + \varphi_1\,,$$
where
$$\int \varphi_1 = 0\,.$$

3. Let
$$\psi(x) = \int_{-\infty}^{x} dt\ \varphi_1(t)\,.$$
Let $F, G \in \mathcal{D}'(\mathbb{R})$ such that
$$\langle G, \varphi\rangle = KC_0 - \langle F, \psi\rangle\,,$$
where C_0 is a constant. Then
$$G' = F\,.$$

■

• **Theorem 3.1.6** *Let $F \in {}'(\mathbb{R})$ be a distribution on $\mathbb{R}$. If $F' = 0$, then F is a constant function.*

Proof:

1. Let
$$\varphi = K\varphi_0 + \varphi_1 \in \mathcal{D}(\mathbb{R})\,,$$
where
$$\int \varphi_0 = 1\,, \qquad \int \varphi_1 = 0\,,$$
and
$$K = \int \varphi\,.$$
Let
$$\psi(x) = \int_{-\infty}^{x} dt\ \varphi_1(t)\,.$$

2. Then
$$\langle F, \varphi_1 \rangle = \langle F, \psi' \rangle = -\langle F', \psi \rangle = 0\,.$$
3. Therefore,
$$\langle F, \varphi \rangle = K \langle F, \varphi_0 \rangle = \int C\varphi\,,$$
where $C = \langle F, \varphi_0 \rangle$.
4. Thus, F is a regular distribution generated by the constant function C.

■

- **Definition 3.1.8 (The Space $\mathcal{D}(\Omega)$) . Test Functions on an Open Set Ω of $\mathbb{R}^n$.** *Let $\Omega \subset \mathbb{R}^n$ be an open set in $\mathbb{R}^n$. The space $\mathcal{D}(\Omega)$ of test functions is the space $C_0^\infty(\Omega)$ of smooth functions with compact support contained in Ω.*

- **Definition 3.1.9 (Convergence in $\mathcal{D}(\Omega)$) .** *Let $\varphi \in \mathcal{D}(\Omega)$ be a test function and (φ_n) be a sequence of test functions on Ω. Then the sequence (φ_n) converges to φ in $\mathcal{D}(\Omega)$, denoted $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$, if:*

  1. *there is a compact set $B \subset \Omega$ (independent of n) such that all φ_n vanish outside B, that is*
  $$\bigcup_{n=1}^{\infty} \operatorname{supp} \varphi_n \subset B\,,$$
  2. *for any α*
  $$D^\alpha \varphi_n \to D^\alpha \varphi$$
  uniformly on Ω.

- **Definition 3.1.10 (The Space $\mathcal{D}'(\Omega)$) . Distributions on an Open Set Ω of $\mathbb{R}^n$.** *The space $\mathcal{D}'(\Omega)$ of distributions is the space of all continuous linear functionals on $\mathcal{D}(\Omega)$.*

3.1.1 Homework

- Exercises:

3.2 Fundamental Solutions of Partial Differential Equations

3.2.1 Boundary Value Problems

- **Definition 3.2.1** *Let $m \in \mathbb{N}$ and let $L : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ be a differentiable operator of order m with smooth coefficients acting on smooth functions on $\mathbb{R}^n$.*

  1. *Let $f \in C^m(\mathcal{R}^n)$ be a m times differentiable function on $\mathbb{R}^n$. Then a function $u \in C^m(\mathbb{R}^n)$ such that*
  $$(Lu)(x) = f(x) , \qquad \text{for all } x \in \mathbb{R}^n$$
  is the **classical solution** *of the equation $Lu = f$.*

  2. *Let f be a function or a distribution on $\mathbb{R}^n$. Then a function u on $\mathbb{R}^n$ such that*
  $$(u, L^*\varphi) = (f, \varphi)$$
  for every test function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is the **weak solution** *of the equation $Lu = f$.*

  3. *Let $f \in \mathcal{D}'(\mathcal{R}^n)$ be a distribution on $\mathbb{R}^n$. Then a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ such that*
  $$Lu = f$$
  is the **distributional solution** *of the equation $Lu = f$.*

- **Examples**.

- **Definition 3.2.2** *Let L be a differential operator. A* **fundamental solution** *G of L is a distributional solution of the equation*
  $$LG = \delta .$$

- A fundamental solution is not unique.

- Let f be a function, G be a fundamental solution of the operator L and $u = f * G$. Then u satisfies the equation
  $$Lu = f .$$

- The Heaviside distribution is the fundamental solution of the operator $L = \partial_x$.

- Let $\Omega \subset \mathbb{R}^n$ be a bounded open set in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Let N be the outward pointing unit normal to Ω and ∇_N be the normal derivative. Let f be a function on $\mathbb{R}^n$. The **homogeneous boundary value problem** is the equation
$$Lu = f \qquad \text{in } \Omega\,,$$
with the boundary conditions
$$Bu\Big|_{\partial\Omega} = 0\,,$$
where B is the boundary operator. One considers the following boundary operators:
$$\begin{aligned} \textbf{Dirichlet}: \qquad B &= 1\,, \\ \textbf{Neumann}: \qquad B &= \nabla_N\,, \\ \textbf{Robin}: \qquad B &= \nabla_N + g\,, \end{aligned}$$
where g is a smooth function on $\partial\Omega$.

- **Definition 3.2.3** *A* **Green function** *G of the boundary value problem (L, B) is a fundamental solution of the operator L that satisfies the homogeneous boundary conditions.*

3.2.2 Equations of Mathematical Physics

- Let $x \in \mathbb{R}^n$ be the space coordinates and $t \in \mathbb{R}$ be the time.

- Let $g_{\mu\nu}$ be the components of the metric tensor in some local coordinates. The **Laplace Operator** (or **Laplacian**) is a second-order partial differential operator acting on smooth functions of the form
$$\Delta = g^{-1/2}\partial_\mu g^{1/2} g^{\mu\nu}\partial_\nu\,.$$
In Cartesian coordinates in $\mathbb{R}^n$ the Laplacian has the form
$$\Delta = \delta^{\mu\nu}\partial_\mu\partial_\nu = \sum_{\mu=1}^{n} \partial_\mu^2\,.$$

- The **D'Alambert Operator** (or **D'Alambertian**) is a second-order partial differential operator
$$\Box = -\partial_t^2 + \Delta\,.$$

- **Laplace Equation**
$$\Delta u = 0\,.$$

- **Poisson Equation**
$$\Delta u = f\,.$$

- **Wave Equation**
$$-\Box u = (\partial_t^2 - \Delta)u = f\,.$$
Homogeneous wave equation
$$\Box u = (\partial_t^2 - \Delta)u = 0\,.$$

- **Heat Equation**
$$(\partial_t - \Delta)u = f\,.$$
Homogeneous heat equation
$$(\partial_t - \Delta)u = 0\,.$$

- **Telegrapher Equation**
$$(\partial_t^2 + 2\partial_t - \Delta + q)u = 0\,,$$
where q is a constant.

- **Helmholtz Equation**
$$(k^2 + \Delta)u = f\,,$$
Homogeneous Helmholtz equation
$$(k^2 + \Delta)u = 0\,,$$
where k is a constant.

- **Biharmonic Wave Equation**.
$$(\partial_t^2 + \Delta^2)u = 0\,.$$

- **Biharmonic Equation**.
$$\Delta^2 u = 0\,.$$

- **Schrödinger Equation**.
$$(i\partial_t - \Delta + V)u = 0\,,$$
where V is a function on $\mathbb{R}^n$.

- **Stationary Schrödinger Equation**.
$$(-\Delta + V - E)u = 0\,,$$
where E is a constant.

- **Klein-Gordon Equation**.
$$(-\Box + m^2)u = (\partial_t^2 - \Delta + m^2)u = 0\,,$$
where m is a constant.

3.2.3 Green's Formulas

- Let Ω be a bounded open set in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Let $\partial\Omega$ be parametrized by
$$x^\mu = x^\mu(\hat{x})\,,$$
where $\hat{x}^i$, $i = 1, \dots, n-1$, are the parameters (local coordinates on $\partial\Omega$).

- Let $g_{\mu\nu}$ be the metric in Ω (equal to $\delta_{\mu\nu}$ in Cartesian coordinates). The induced metric $\hat{g}_{ij}$ on $\partial\Omega$ is defined by
$$\hat{g}_{ij}(\hat{x}) = g_{\mu\nu}(x(\hat{x}))\frac{\partial x^\mu}{\partial \hat{x}^i}\frac{\partial x^\nu}{\partial \hat{x}^j}\,.$$

- The volume elements in Ω and $\partial\Omega$ are defined by
$$dvol(x) = dx\,\sqrt{g(x)}\,, \qquad dvol(\hat{x}) = d\hat{x}\,\sqrt{\hat{g}(\hat{x})}$$
where
$$g = \det g_{\mu\nu}\,, \qquad \hat{g} = \det g_{ij}\,.$$

- The vectors
$$e^\mu_i = \frac{\partial x^\mu}{\partial \hat{x}^i}$$
are tangent to $\partial\Omega$ and the vector
$$N^\nu = g^{\nu\mu}\varepsilon_{\mu_1\cdots\mu_{n-1}\mu}e^{\mu_1}_1\cdots e^{\mu_{n-1}}_{n-1}$$
defines the normal to $\partial\Omega$. The unit normal is defined by
$$n^\mu = \frac{N^\mu}{\sqrt{g_{\alpha\beta}N^\alpha N^\beta}}\,.$$

- The normal derivative is defined by
$$\nabla_N = n^\mu \nabla_\mu\,.$$

- Let
$$(u,v) = \int_\Omega dx u(x)\bar{v}(x)$$
denote the L^2 inner product in Ω and
$$\langle u,v\rangle = \int_{\partial\Omega} dvol(\hat{x})u(\hat{x})\bar{v}(\hat{x})$$
be the L^2 inner product on $\partial\Omega$.

- **Green's First Identity**.
$$(u,\Delta v) = -(\nabla u,\nabla v) + \langle u,\nabla_N v\rangle\,.$$
In particular,
$$(u,\Delta u) = -(\nabla u,\nabla u) + \langle u,\nabla_N u\rangle\,.$$

- If the boundary conditions are such that the boundary contribution vanishes, then
$$-(u,\Delta u) = (\nabla u,\nabla u) > 0\,,$$
so that, $(-\Delta)$ is a non-negative operator.

- **Green's Second Identity**.
$$(u,\Delta v) - (\Delta u, v) = \langle u,\nabla_N v\rangle - \langle \nabla_N u, v\rangle\,.$$

- More generally, let L be the operator

$$L = \Delta + a^\mu \partial_\mu + b\,.$$

Then

$$(Lu, v) - (u, L^* v) = \langle \nabla_N u, v \rangle - \langle u, \nabla_N v \rangle + \langle a_N u, v \rangle\ ,$$

where $a_N = a^\mu n_\mu$.

- In particular case $a = 0$, we obtain a self-adjoint operator (Schrödinger operator)

$$L = \Delta + b\,.$$

- For a self-adjoint operator $L = \Delta + b$ we have

$$(Lu, v) + (\nabla u, \nabla v) - (bu, v) = \langle \nabla_N u, v \rangle\ .$$

- The bilinear form

$$E(u, v) = (\nabla u, \nabla v) - (bu, v)$$

is called the **Dirichlet integral** of the operator L (the energy functional). If $b > 0$, then for any $u \neq 0$

$$E(u, u) > 0\,.$$

- Let

$$L = \sum_{|\alpha| \le m} a_\alpha D^\alpha\,.$$

Then

$$(Lu, v) - (u, L^* v) = \langle Ju, v \rangle - \langle u, J^* v \rangle\ ,$$

where J is a differential operator of order $m - 1$.

3.2.4 Green Functions

- The **Green function of the Laplace operator** $L = -\Delta$ **in** $\mathbb{R}^n$. It can be obtained by the Fourier transform

$$G(x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} dp\ e^{ip(x-y)} \frac{1}{|p|^2}\ ,$$

where $px = p_\mu x^\mu$ and

$$|p| = \sqrt{\delta^{\mu\nu} p_\nu p_\mu}\,.$$

For $n \geq 3$ we compute

$$G(x,y) = (4\pi)^{-n/2}\Gamma\left(\frac{n}{2}-1\right)\frac{2^{n-2}}{|x-y|^{(n-2)}}\,.$$

For $n = 3$ one obtains

$$G(x,y) = \frac{1}{4\pi}\frac{1}{|x-y|}\,.$$

For $n = 2$ one needs to use a regularization to get

$$G(x,y) = -\frac{1}{4\pi}\ln|x-y|^2\,.$$

- The **Green function of the Helmholtz operator** $L = -\Delta + m^2$ **in** $\mathbb{R}^n$. It can be obtained by the Fourier transform

$$G(x,y) = (2\pi)^{-n}\int_{\mathbb{R}^n} dp\; e^{ip(x-y)}\frac{1}{|p|^2+m^2}\,,$$

Further, by using

$$\frac{1}{|p|^2+m^2} = \int_0^\infty ds\; e^{-s(|p|^2+m^2)}$$

and computing the Gaussian integral over p

$$(2\pi)^{-n}\int_{\mathbb{R}^n} dp\; e^{ip(x-y)-s|p|^2} = (4\pi s)^{-n/2}\exp\left[-\frac{|x-y|^2}{4s}\right]$$

we obtain

$$G(x,y) = \int_0^\infty ds\;(4\pi s)^{-n/2}\exp\left[-sm^2 - \frac{|x-y|^2}{4s}\right]$$

- The **Green function of the Helmholtz operator** $L = \Delta + k^2$ **in** $\mathbb{R}^3$ with the radiation asymptotic condition

$$\lim_{r\to\infty} r(\partial_r + ik)u = 0$$

is obtained by the direct solution of the radial Laplace equation

$$\left(\partial_r^2 + \frac{2}{r}\partial_r + k^2\right)G = \frac{1}{4\pi r^2}\delta(r)\,.$$

It has the form

$$G(x,y) = -\frac{1}{4\pi}\frac{e^{-ik|x-y|}}{|x-y|}\,.$$

- The **Green function of the heat operator** $\partial_t - \Delta$ **in** $\mathbb{R}^n$. It can be obtained by the Fourier transform

$$G(t, x; t', x') = (2\pi)^{-n} \int_{\mathbb{R}^n} dp\; e^{ip(x-x')-(t-t')|p|^2} .$$

It has the form

$$G(t, x; t', x') = (4\pi(t - t'))^{-n/2} \exp\left[-\frac{|x - y|^2}{4(t - t')}\right] .$$

The solution of the problem

$$(\partial_t - \Delta)u = q$$

with the initial condition

$$u|_{t=0} = f ,$$

is given by

$$u(t, x) = \int_{\mathbb{R}^n} dx'\; G(t, x; 0, x') f(x') + \int_0^t dt' \int_{\mathbb{R}^n} dx'\; G(t, x; t', x') q(t', x') .$$

- The **Green function of the wave operator** $-\partial_t^2 + \Delta$ **in** $\mathbb{R}^n$. It can be obtained by the Fourier transform

$$G(t, x; t', x') = (2\pi)^{-n} \int_{\mathbb{R}^n} dp\; e^{ip(x-x')} \left\{ A\frac{\sin(t|p|)}{|p|} + B\cos(t|p|) \right\} .$$

where A and B are constants determined from the supplementary conditions. The solution of the problem

$$(-\partial_t^2 + \Delta)u = -p$$

with the initial conditions

$$u|_{t=0} = 0 , \qquad \partial_t u|_{t=0} = 0 ,$$

and the asymptotic condition

$$\lim_{|x|\to 0} u = 0$$

is given by

$$u(t, x) = \int_0^t dt' \int_{\mathbb{R}^n} dx'\; G(t, x; t', x') p(t', x') .$$

- The **Green function of the Klein-Gordon operator** $-\partial_t^2 + \Delta - m^2$ **in** $\mathbb{R}^n$ with the initial conditions

$$u|_{t=0} = \partial_t u|_{t=0} = 0\,,$$

and the boundary conditions

$$\lim_{|x|\to\infty} = 0\,.$$

It can be obtained by the Fourier transform

$$\begin{aligned} G(t,x;t',x') &= (2\pi)^{-n}\int_{\mathbb{R}^n} dp\; e^{ip(x-x')}\Bigg\{A\frac{\sin(t\sqrt{|p|^2+m^2})}{\sqrt{|p|^2+m^2}} \\ &\qquad + B\cos(t\sqrt{|p|^2+m^2})\Bigg\}\,, \end{aligned}$$

where A and B are constants determined from the supplementary conditions. The solution of the problem

$$(-\partial_t^2 + \Delta - m^2)u = -p$$

with the initial conditions

$$u|_{t=0} = f\,, \qquad \partial_t u|_{t=0} = g\,,$$

can be written as

$$\begin{aligned} u(t,x) &= \int_0^t dt' \int_{\mathbb{R}^n} dx'\; G(t,x;t',x')p(t',x') \\ &\quad + \int_{\mathbb{R}^n} dx'\; \left[G(t,x;0,x')g(x') - \partial_{t'}G(t,x;0,x')f(x')\right]\,. \end{aligned}$$

Retarded Green function

Advanced Green function

Symmetric Green function

Feynmann Green function

Schwinger function

Pauli-Jordan function

Hadamard function

3.2.5 Homework

- Exercises: 6.6[27,28,29,32]

3.3 Weak Solutions of Elliptic Boundary Value Problems

- Let $\Omega \subset \mathbb{R}^n$ be a bounded open set in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Let $L = -\Delta$ be the Laplacian acting on smooth functions on Ω and $f \in C^{(}\Omega)$ be a continuous function. We consider the **Dirichlet boundary value problem**:

$$Lu = f \qquad \text{in } \Omega$$

  with the boundary condition

$$u|_{\partial\Omega} = 0\,.$$

- A classical solution of this boundary value problem is a function $u \in C^2(\bar{\Omega})$ that satisfies the equation at every point $x \in \Omega$.

- Let $\varphi \in \mathcal{D}(\Omega)$. Then for any classical solution we have

$$(\nabla u, \nabla\varphi) = (f, \varphi)\,.$$

- If f is not continuous then there is no classical solution.

- Let $H^1_0(\Omega)$ be the subspace of the Sobolev space $H^1(\Omega)$ consisting of functions satisfying Dirichlet boundary conditions. The space $H^1_0(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$.

- For L^2 functions the derivatives are understood in the generalized sense. Then for any $u \in H^1(\Omega_0)$, it follows $\nabla u \in L^2(\Omega)$.

- Let $f \in L^2(\Omega)$, then a $u \in H^1_0(\Omega)$ satisfying the equation

$$(\nabla u, \nabla\varphi) = (f, \varphi),$$

  for every $\varphi \in H^1_0(\Omega)$ is a **weak solution** of the boundary value problem.

  This is the **variational formulation** (or **weak formulation**) of the boundary value problem.

Theorem 3.3.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a boundary $\partial\Omega$. Let $H_0^1(\Omega)$ be the Sobolev space of functions u such that $\nabla u \in L^2(\Omega)$ and $u|_{\partial\Omega} = 0$. Let $f \in L^2(\Omega)$ and $J : H_0^1(\Omega) \to \mathbb{R}$ be a functional on $H_0^1(\Omega)$ defined by*

$$J(v) = \frac{1}{2}(\nabla v, \nabla v) - (f, v)\,.$$

Then:

- 1. *For any $v \in H_0^1(\Omega)$ there is a unique $u \in H_0^1(\Omega)$ satisfying the equation*

$$(\nabla u, \nabla v) = (f, v)\,.$$

  2. *A function $u \in H_0^1(\Omega)$ is a solution of the equation $(\nabla u, \nabla v) = (f, v)$, where $v \in H_0^1(\Omega)$, if and only if u is a function at which the minimum of the functional J is attained.*

Proof:

1. Let a be a bilinear form on $H_0^1(\Omega)$ defined by

$$a(u, v) = (\nabla u, \nabla v)\,.$$

2. Let

$$\| u \|_1^2 = (\nabla u, \nabla u) + (u, u)\,.$$

3. We have the Friedrichs' first inequality: there is constants $\alpha > 0$, such that for any $u \in H_0^1(\Omega)$

$$(u, u) \leq \alpha(\nabla u, \nabla u) \leq \alpha \| u \|_1^2\,.$$

4. Therefore,

$$a(u, u) = (\nabla u, \nabla u) \geq K \| u \|_1^2\,,$$

where $K = \min\{1/2, 1/(2\alpha)\}$.

5. We also have

$$a(u, u) \leq \| u \|_1^2\,.$$

6. Thus, a is a bounded symmetric and coercive (elliptic) bilinear form on $H_0^1(\Omega)$.

7. By the Lax-Milgram theorem there exists a solution in $H_0^1(\Omega)$ of the equation

$$a(u, v) = (f, v)\,.$$

■

- **Neumann Boundary Value Problem**. Let $b > 0$ be a positive constant and $L = -\Delta + b$. The Neumann boundary value problem is the problem

$$Lu = f \qquad \text{in } \Omega$$

with the boundary condition

$$\nabla_N u|_{\partial\Omega} = 0\,.$$

Every classical solution $u \in H^1(\Omega)$ satisfies the equation

$$(\nabla u, \nabla v) + (bu, v) = (f, v)$$

for every $v \in H^1(\Omega)$.

- Let $f \in L^2(\Omega)$. A weak solution of the Neumann boundary value problem is a $u \in H^1(\Omega)$ satisfying the equation

$$(\nabla u, \nabla v) + (bu, v) = (f, v)$$

for every $v \in H^1(\Omega)$.

- Let a be the bilinear form on $H^1(\Omega)$ defined by

$$a(u, v) = (\nabla u, \nabla v) + (bu, v)\,.$$

- We estimate

$$a(u, v) \le M \parallel u \parallel \parallel v \parallel,$$

where $M = \max\{1, b\}$.

- So, a is continuous and coercive bilinear form in $H^1(\Omega)$.

- So, by Lax-Milgram theorem there exists a unique solution $u \in H^1(\Omega)$ satisfying the equation

$$a(u, v) = (f, v)$$

for all $v \in H^1(\Omega)$.

- The weak solution of the Neumann boundary value problem minimizes on $H^1(\Omega)$ the functional

$$J(v) = \frac{1}{2}(\nabla v, \nabla v) + (bv, v) - (f, v)\,.$$

- **Example**.

- **General Elliptic Boundary Value Problem**. Let $a^{\mu\nu}, q \in C^1(\overline{\Omega})$ be bounded differentiable functions such that

$$a^{\mu\nu} = a^{\nu\mu}\,.$$

  We will assume for simplicity that the matrix $a^{\mu\nu}$ is positive definite and $q \geq 0$. Let L be a second-order operator of the form

$$L = -\partial_\mu a^{\mu\nu}\partial_\nu + q\,.$$

  We consider the Dirichlet boundary value problem (L, B) for this operator

$$Lu = f \qquad \text{in } \Omega$$

  with the boundary condition

$$u|_{\partial\Omega} = 0\,.$$

- The operator L is called **uniformly elliptic** if there is a constant C such that for any $x \in \Omega$, $\xi \in \mathbb{R}^n$, $\xi \neq 0$,

$$|a^{\mu\nu}(x)\xi_\mu\xi_\nu| \geq C|\xi|^2\,,$$

  where $|\xi| = \sqrt{\delta^{\mu\nu}\xi_\mu\xi_\nu}$.

- Let $f \in L^2(\Omega)$. A weak solution of the boundary value problem is a function $u \in H^1_0(\Omega)$ satisfying the equation

$$(a^{\mu\nu}\partial_\mu u, \partial_\nu v) + (qu, v) = (f, v)$$

  for every $v \in H^1_0(\Omega)$.

- Every classical solution is a weak solution. Every sufficiently regular weak solution is a classical solution.

- Let a be a bilinear form on $H_0^1(\Omega)$ defined by

$$a(u, v) = (a^{\mu\nu}\partial_\mu u, \partial_\nu v) + (qu, v)$$

- We estimate

$$a(u, u) \geq (a^{\mu\nu}\partial_\mu u, \partial_\nu) \geq K(\nabla u, \nabla u) \, .$$

- Next, we show that a is bounded in $H_0^1(\Omega)$.

- Then by Lax-Milgram theorem there exists a unique solution $u \in H_0^1(\Omega)$ of the equation

$$a(u, v) = (f, v)$$

  for all $v \in H_0^1(\Omega)$.

- This solution minimizes the functional

$$J(v) = \frac{1}{2}(a^{\mu\nu}\partial_\mu u, \partial_\nu) + \frac{1}{2}(qv, v) - (f, v)$$

  on $H_0^1(\Omega)$.

3.3.1 Homework

- Exercises: 6.6[23,24,25]

Chapter 4

Wavelets and Optimization

4.1 Wavelets

4.1.1 Wavelet Transforms

-

4.1.2 Homework

- Exercises:

4.2 Calculus of Variations

4.2.1 Gateaux and Fréchet Differentials

•

4.2.2 Euler-Lagrange Equations

•

4.2.3 Homework

- Exercises:

4.3 Dynamical Systems

4.3.1 Optimal Control

•

4.3.2 Stability

•

4.3.3 Bifurcations

•

4.3.4 Homework

- Exercises:

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Notation

Logic

$A \Longrightarrow B$	A implies B
$A \Longleftarrow B$	A is implied by B
iff	if and only if
$A \Longleftrightarrow B$	A implies B and is implied by B
$\forall x \in X$	for all x in X
$\exists x \in X$	there exists an x in X such that

Sets and Functions (Mappings)

$x \in X$	x is an element of the set X
$x \notin X$	x is not in X
$\{x \in X \mid P(x)\}$	the set of elements x of the set X obeying the property $P(x)$
$A \subset X$	A is a subset of X
$X \setminus A$	complement of A in X
$\overline{A}$	closure of set A
$X \times Y$	Cartesian product of X and Y
$f : X \to Y$	mapping (function) from X to Y
$f(X)$	range of f
χ_A	characteristic function of the set A
$\emptyset$	empty set
$\mathbb{N}$	set of natural numbers (positive integers)
$\mathbb{Z}$	set of integer numbers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of positive real numbers
$\mathbb{C}$	set of complex numbers

Vector Spaces

$H \oplus G$	direct sum of H and G
H^*	dual space
$\mathbb{R}^n$	vector space of n-tuples of real numbers
$\mathbb{C}^n$	vector space of n-tuples of complex numbers
l^2	space of square summable sequences
l^p	space of sequences summable with p-th power

Normed Linear Spaces

$\|\|x\|\|$	norm of x
$x_n \longrightarrow x$	(strong) convergence
$x_n \overset{w}{\longrightarrow} x$	weak convergence

Function Spaces

supp f	support of f
$H \otimes G$	tensor product of H and G
$C_0(\mathbb{R}^n)$	space of continuous functions with bounded support in $\mathbb{R}^n$
$C(\Omega)$	space of continuous functions on Ω
$C^k(\Omega)$	space of k-times differentiable functions on Ω
$C^\infty(\Omega)$	space of smooth (infinitely diffrentiable) functions on Ω
$\mathcal{D}(\mathbb{R}^n)$	space of test functions (Schwartz class)
$L^1(\Omega)$	space of integrable functions on Ω
$L^2(\Omega)$	space of square integrable functions on Ω
$L^p(\Omega)$	space of functions integrable with p-th power on Ω
$H^m(\Omega)$	Sobolev spaces
$C_0(V, \mathbb{R}^n)$	space of continuous vector valued functions with bounded support in $\mathbb{R}^n$
$C^k(V, \Omega)$	space of k-times differentiable vector valued functions on Ω
$C^\infty(V, \Omega)$	space of smooth vector valued functions on Ω

$\mathcal{D}(V, \mathbb{R}^n)$	space of vector valued test functions (Schwartz class)
$L^1(V, \Omega)$	space of integrable vector valued functions functions on Ω
$L^2(V, \Omega)$	space of square integrable vector valued functions on Ω
$L^p(V, \Omega)$	space of vector valued functions integrable with p-th power on Ω
$H^m(V, \Omega)$	Sobolev spaces of vector valued functions

Linear Operators

D^α	differential operator
$L(H, G)$	space of bounded linear transformations from H to G
$H^* = L(H, \mathbb{C})$	space of bounded linear functionals (dual space)