MATH 332: Vector Analysis
Tensors

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Cartesian Coordinate System. First of all, let us introduce a *Cartesian coordinate system* in three-dimensional Euclidean space. We will denote the coordinates by

$$x^1 = x, \qquad x^2 = y, \qquad x^3 = z \tag{1}$$

and the unit vectors in the direction of positive axes (called the *basis vectors*) by

$$\mathbf{e}_1 = \mathbf{i}, \qquad \mathbf{e_2} = \mathbf{j}, \qquad \mathbf{e_3} = \mathbf{k} \tag{2}$$

Index Notation. This can be denoted simply by x^i and $\mathbf{e}_j$, where $i, j = 1, 2, 3$. For the indices one usually uses the lowercase Latin letters i, j, k, l, m, n etc. (do not confuse with $\mathbf{i}, \mathbf{j}, \mathbf{k}$). If you run out of letters, you can use any other letters. The convention is though that the indices are denoted by small (versus capital) Latin (versus Greek) letters, and take values $1, 2, 3$. Greek indices are used in four-dimensional space-time in special relativity, where they take values $0, 1, 2, 3$, with $x^0 = t$ denoting time.

Kronecker Delta Symbol. The scalar products of the basis vectors are:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1, & \text{if} \quad i = j \\ 0, & \text{if} \quad i \neq j \end{cases} \tag{3}$$

One says that they form an *orthonormal system*. This can be written in a compact form by defining so called *Kronecker symbol* δ_{ij}

$$\delta_{ij} = \begin{cases} 1, & \text{if} \quad i = j \\ 0, & \text{if} \quad i \neq j \end{cases} \tag{4}$$

This can also be represented by the unit 3×3 matrix

$$(\delta_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5}$$

Then

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{6}$$

Scalars. Physical quantities, like mass, energy, volume, temperature, density etc., that can be described by one number are called *scalars*. This number does not depend on the coordinate system; it is an *invariant*.

Vectors. *Vectors* are physical quantities, like velocity, position, displacement, force, acceleration, electric field, magnetic field etc., that are described by three numbers.

Tensors. A *tensor* is a geometric object that requires for its full description more than just one number, as scalar, and even more than three numbers, as a vector. Examples of tensors include: stress tensor, strain tensor, inertia tensor, energy-momentum tensor, tensor of the electromagnetic field, metric tensor, curvature tensor etc.

Tensor Components. These numbers are called the *components* of the tensor. The components of a tensor are labeled by indices, for example,

$$\delta_{ij}, \qquad \varepsilon_{ijk}, \qquad T^{ij}, \qquad B^i{}_j, \qquad \sigma_{ij}, \qquad R^i{}_{ijk} \tag{7}$$

A tensor whose *all* components are zero is called a *zero tensor*.

Types of Tensors. The tenswors with upper indices are called *contravariant*, and the ones with lower indices are called *covariant*. If a tensor has both types of indices then it is of mixed type. The total number of indices is called the *rank* of the tensor. A tensor that has p upper indices and q lower indices

$$T^{i_1\dots i_p}{}_{j_1\dots j_q} \tag{8}$$

is called a tensor of type (p, q). So, a scalar is a tensor of rank 0. A vector is a tensor of rank 1.

Transformation Law. The actual numerical values of the components of a tensor *do depend on the coordinate system*. If one changes the coordinate system, for example, rotates it, then the components of a tensor will change. If one goes from the Cartesian coordinate system to a curvilinear coordinate system, for example, a system of spherical or cylindrical coordinates, then the components of a tensor will also change. It is this *transformation law* of the components of the tensor that makes a collection of numbers a tensor. We will not give the formal definition of a tensor, rather we give here a very short review of tensor analysis in Cartesian coordinates along with some very useful formulas and rules that enable one to deal with tensors.

Metric Tensor In Cartesian coordinates the square of the distance between two infinitesimally close points in space, one with coordinates x^i and another with coordinates $x^i + dx^i$, is

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = \sum_{i=1}^{3} (dx^i)^2 \tag{9}$$

This can be written in the following form

$$(ds)^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij} dx^i dx^j \tag{10}$$

The distance between infinitesimally close points determines a tensor of rank 2, so called *metric tensor* g_{ij}. In general coordinate system one has

$$(ds)^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij} dx^i dx^j, \tag{11}$$

Thus, the covariant components of the metric tensor in Cartesian coordinates are given by Kronecker delta symbol

$$g_{ij} = \delta_{ij} \tag{12}$$

The contravariant components of the metric tensor are defined by

$$g^{ij} = (g_{ij})^{-1} \tag{13}$$

In Cartesian coordinates

$$g^{ij} = \delta_{ij} \tag{14}$$

Tensor Equations.

- In *any* tensor equation an index can appear only *once* (*single index*) or *twice* (*repeated index*). For example, ${A^i}_{ii}$ is impossible.
- A single index can be either covariant in the whole equation or contravariant in the whole equation. It cannot be contravariant in one term and covariant in another term. For example, ${A^j}_i + B_{ij}$ is wrong.
- The repeated indices always appear in pairs, one covariant and another contravariant. For example, ${A^i}_{ij}$.
- A pair of repeated indices cannot appear more than once. For example, ${{{A^i}_i}^i}_i$ is wrong.

Einstein Summation Convention. In tensor analysis one always encounters the sums over the indices that appear *twice* in an equation. For examle, in the formula for the distance above the indices i and j appear twice, and there is summation over i and j running from 1 to 3. According to the standard convention, called Einstein summation convention, one has agreed to *sum over repeated indices and omit the summation signs.* For example,

$$\delta_{ij} dx^i dx^j = \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij} dx^i dx^j \tag{15}$$

$$A_i B^i = \sum_{i=1}^{3} A_i B^i \tag{16}$$

$${T^i}_i = \sum_{i=1}^{3} {T^i}_i \tag{17}$$

$${R^i}_{jik} = \sum_{i=1}^{3} {R^i}_{jik} \tag{18}$$

$$\delta_{ij} R^{ij} = \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij} R^{ij} \tag{19}$$

$$\varepsilon_{ijk} A^{jk} B^i = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} A^{jk} B^i \tag{20}$$

Raising and Lowering Indices The metric tensor can be used to raise and lower indices of tensors. For example, if A^i are contravariant components of a vector then its covariant components are

$$A_i = \delta_{ij} A^j \tag{21}$$

Conversely,

$$A^i = \delta^{ij} A_j \tag{22}$$

This operations, called raising and lowering indices can be applied to any tensor. If one applies it to the metric tensor, one gets

$$\delta^i_j = \delta_{jk} \delta^{ik} \tag{23}$$

By raising and lowering indices any tensor can be put in a covariant or contravariant form. One has to be careful though with the order of indices. For example,

$$A^i{}_j = \delta_{jk} A^{ik} \neq A_j{}^i = \delta_{jk} A^{ki} \tag{24}$$

Remark. In Cartesian coordinates the covariant and contravariant components are equal, since the metric tensor is given by the Kronecker symbol. Therefore, in this case it does not make any difference and all indices can be placed down, for example.

Properties and Identities of Kronecker Delta Symbol

$$\delta_{ij} = \delta_{ji} \tag{25}$$

$$\delta^i_j = \delta^j_i = \delta^{ij} = \delta_{ij} \tag{26}$$

$$\delta^i_i = 3 \tag{27}$$

$$\delta^j_i A_j = A_i \tag{28}$$

$$\delta^{ij} A_i B_j = A^i B_i = \mathbf{A} \cdot \mathbf{B} \tag{29}$$

Addition. One can add tensors of the *same type*. The result is a tensor of the same type.

Multiplication By Scalars. One can multiply tensors by scalars. The result is a tensor of the same type.

Tensor Multiplication. If one multiplies a tensor of rank r with a tensor of rank k, one gets a new tensor of rank $r+k$. More precisely, if one multiplies a tensor of type (p,q) with a tensor of type (r,s), then one gets a new tensor of type $(p+r,q+s)$. For example,

$$A_i B_j = C_{ij}, \qquad T^{mn}\sigma_{ij} = R^{mn}{}_{ij} \tag{30}$$

Note: one just multiplies the components of the tensors without any summation.

Contraction. Given a tensor of type (p,q) (that is of rank $r = p+q$) one may select a pair of indices, of which one should be an upper index and another an lower index, and replace them by two identical (repeated indices), summation over the latter being implied by the summation convention. This process is called *contraction*. As a result one gets a new tensor of type $(p-1,q-1)$ of rank $r-2 = p+q-2$. For example,

$$A^i{}_i, \qquad R^{ij}{}_{ki}, \qquad C^{ik}{}_k \tag{31}$$

Clearly,

$$\delta^i_i = \delta^1_1 + \delta^2_2 + \delta^3_3 = 3 \tag{32}$$

Symmetrization and Anti-symmetrization. A tensor A of rank 2 is said to be *symmetric* if

$$A_{ij} = A_{ji} \tag{33}$$

and *anti-symmetric* (or *skew-symmetric*) if

$$A_{ij} = -A_{ji} \tag{34}$$

Any tensor A_{ij} of second rank can be decomposed

$$A_{ij} = A_{(ij)} + A_{[ij]} \tag{35}$$

into its symmetric

$$A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji}) \tag{36}$$

and anti-symmetric parts

$$A_{[ij]} = \frac{1}{2}(A_{ij} - A_{ji}) \tag{37}$$

One can also symmetrize a tensor over three indices:

$$B_{(ijk)} = \frac{1}{6}(B_{ijk} + B_{jki} + B_{kij} + B_{ikj} + B_{jik} + B_{kji}) \tag{38}$$

Correspondingly, the anti-symmetrization over three indices is defined by

$$B_{[ijk]} = \frac{1}{6}(B_{ijk} + B_{jki} + B_{kij} - B_{ikj} - B_{jik} - B_{kji}) \tag{39}$$

What one does here is one sums over all possible permutations of indices and changes sign if the permutation is odd.

Contraction of Symmetric and Anti-symmetric Tensors. Let A_{ij} be a symmetric tensor and B_{ij} an antisymmetric. Then

$$A_{ij}B^{ij} = A_{ji}B^{ij} = -A_{ji}B^{ji} = -A_{ij}B^{ij}\,, \tag{40}$$

and, therefore,

$$A_{ij}B^{ij} = 0\,. \tag{41}$$

Levi Civita Symbol Levi-Civita symbol ε_{ijk} is defined by

$$\varepsilon_{ijk} = \begin{cases} +1, & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1, & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0, & \text{otherwise} \end{cases} \tag{42}$$

If one raises the indices then one sees that in Cartesian coordinates one obtains the same symbol, so that

$$\varepsilon^{ijk} = \varepsilon_{ijk} \tag{43}$$

The Levi-Civita symbol defines a tensor of rank 3, called a Levi-Civita tensor. This is another very important tensor that is purely geometric in nature. It describes not the distances but the volume in three-dimensional space. The volume of a parallelepiped based on three displacement vectors A^i, B^j, C^k is

$$V = \varepsilon_{ijk}A^i\,B^j\,C^k \tag{44}$$

Properties and Identities of Levi-Civita Symbol. The Levi-Civita symbol defines a *completely antisymmetric tensor*. The following properties immediately follow from its definition.

$$\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj} = -\varepsilon_{kji} \tag{45}$$

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} \tag{46}$$

$$\varepsilon_{ij}{}^{j} = \varepsilon^{j}{}_{ij} = \varepsilon^{j}{}_{ji} = 0 \tag{47}$$

$$\varepsilon_{ijk}\delta^{ij} = \varepsilon_{ijk}\delta^{ik} = \varepsilon_{ijk}\delta^{jk} = 0 \tag{48}$$

$$\varepsilon^{ijk}A_jA_k = \varepsilon^{ijk}A_iA_k = \varepsilon^{ijk}A_iA_j = 0 \tag{49}$$

$$\varepsilon^{ijk} = \varepsilon_{ijk} \tag{50}$$

$$\varepsilon_{ijk}\varepsilon^{mnl} = 6\delta^m_{[i}\delta^n_j\delta^l_{k]} \tag{51}$$

$$= \delta^m_i\delta^n_j\delta^l_k + \delta^m_j\delta^n_k\delta^l_i + \delta^m_k\delta^n_i\delta^l_j - \delta^m_i\delta^n_k\delta^l_j - \delta^m_j\delta^n_i\delta^l_k - \delta^m_k\delta^n_j\delta^l_i \tag{52}$$

$$\begin{aligned} \varepsilon_{ijk}\varepsilon^{mnk} &= 2\delta^m_{[i}\delta^n_{j]} \\ &= \delta^m_i\delta^n_j - \delta^m_j\delta^n_i \end{aligned} \tag{53}$$

$$\varepsilon_{ijk}\varepsilon^{mjk} = 2\delta^m_i \tag{54}$$

$$\varepsilon_{ijk}\varepsilon^{ijk} = 6 \tag{55}$$

Vector Operations in Tensor Notation. Tensor notation is very useful in vector analysis, in particular when manipulating the multiple vector products and vector identities.

First of all, the scalar and vector products of two vectors $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} \cdot \mathbf{B} = A_iB^i \tag{56}$$

and

$$(\mathbf{A} \times \mathbf{B})_i = \varepsilon_{ijk}A^jB^k \tag{57}$$

The triple product of three vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ is then given by

$$[\mathbf{A}, \mathbf{B}, \mathbf{C}] = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \varepsilon^{ijk}A_iB_jC_k \tag{58}$$

Note that the position of indices (up versus down) in Cartesian coordinates is not important. However, it is still more clear, when you see one index up and the same index down then you should immediately notice that this is a contraction and there is a summation over this index from 1 to 3. We repeat once again that the name of such repeated indices is not important, they are *dummy indices*; one can rename them to any other letter if needed (make sure that there are no other indices with that name in the given tensor equation!).

By using the properties of Levi-Civita symbol and Kronecker symbol one can derive now all vector identities. For example,

$$
\begin{aligned}
[(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})]_i &= \varepsilon_{ijk}(\mathbf{A}\times\mathbf{B})^j(\mathbf{C}\times\mathbf{D})^k \\
&= \varepsilon_{ijk}\varepsilon^{jmn}A_mB_n\varepsilon^{kpq}C_pD_q \\
&= (\delta_i^p\delta_j^q-\delta_j^p\delta_i^q)\varepsilon^{jmn}A_mB_nC_pD_q \\
&= (\delta_i^p\varepsilon^{qmn}-\delta_i^q\varepsilon^{pmn})A_mB_nC_pD_q \\
&= \varepsilon^{qmn}A_mB_nC_iD_q-\varepsilon^{pmn}A_mB_nC_pD_i \qquad (59)\\
&= [\mathbf{D},\mathbf{A},\mathbf{B}]C_i-[\mathbf{C},\mathbf{A},\mathbf{B}]D_i \qquad (60)
\end{aligned}
$$

In other words, we have just proven the following vector identity

$$
(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})=[\mathbf{D},\mathbf{A},\mathbf{B}]\mathbf{C}-[\mathbf{C},\mathbf{A},\mathbf{B}]\mathbf{D} \qquad (61)
$$

(do not forget that the triple product is a scalar!)