

Math 132 Complex Numbers and Functions

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Many engineering problems can be treated and solved by using complex numbers and complex functions. We will look at complex numbers, complex functions and complex differentiation and integration.

Part 1. Complex Numbers

In algebra we discovered that many equations are not satisfied by any real numbers. Examples are:

$$x^2 = -2 \quad \text{or} \quad x^2 - 10x + 40 = 0$$

We must introduce the concept of complex numbers.

Definition: A complex number is an ordered pair (x,y) of real numbers x and y . We write

$$z = (x, y).$$

We call x the **real part** of z and y the **imaginary part**. and we write

$$\text{Re } z = x, \quad \text{Im } z = y.$$

Example 1 $\text{Re } (4, -3) = 4$ and $\text{Im } (4, -3) = -3$.

We define for two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$,

$$z_1 = z_2 \text{ if and only if } x_1 = x_2 \text{ and } y_1 = y_2$$

Addition and subtraction of complex numbers For two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, Addition is defined by

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$$

and subtraction by $z_1 - z_2 = (x_1 - x_2, y_1 - y_2)$

Multiplication of complex numbers: For two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, multiplication is defined by

$$z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

Example 2 Let $z_1 = (2, 3)$ and $z_2 = (5, -4)$ then

$$z_1 + z_2 = (2 + 5, 3 + (-4)) = (7, -1)$$

and

$$z_1 z_2 = (2 \cdot 5 - 3 \cdot (-4), 2 \cdot (-4) + 3 \cdot 5) = (22, 7)$$

We need to represent complex numbers in a manner that will make addition and multiplication easier to do.

Complex numbers represented as $z = x + i y$

A complex number whose imaginary part is 0 is of the form $(x,0)$ and we have

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0) \quad \text{and} \quad (x_1, 0)(x_2, 0) = (x_1 x_2, 0)$$

which is just addition and multiplication of real numbers. So we identify $(x,0)$ with the real numbers x . We can now consider the real numbers as a subset of the complex numbers.

We let the letter $i = (0, 1)$ and we call $i = (0, 1)$ the purely imaginary number.

Now consider $i^2 = i \cdot i = (0, 1) \cdot (0, 1) = (-1, 0)$ and so we can consider the complex number i^2 the real number -1 . We also get $yi = y(0, 1) = (0, y)$. And so we have

$$(x, y) = (x, 0) + (0, y) = x + i y.$$

Now we can write addition and multiplication as follows:

$$z_1 + z_2 = (x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2) i$$

and

$$\begin{aligned} z_1 z_2 &= (x_1 + y_1 i)(x_2 + y_2 i) \\ &= x_1 x_2 + y_1 y_2 i^2 + (x_1 y_2 + x_2 y_1) i \\ &= x_1 x_2 - y_1 y_2 + (x_1 y_2 + x_2 y_1) i \end{aligned}$$

Example 3. Let $z_1 = (2, 3) = 2 + 3 i$ and $z_2 = (5, -4) = 5 - 4 i$ then

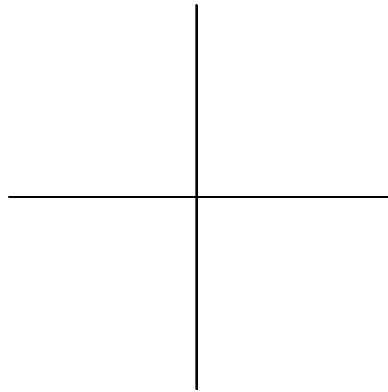
$$z_1 + z_2 = (2 + 3 i) + (5 - 4 i) = 7 - i$$

and

$$z_1 \cdot z_2 = (2 + 3 i) \cdot (5 - 4 i) = 10 - 12i^2 + (-8 i + 15 i) = 22 + 7 i$$

The Complex plane

The geometric representation of complex numbers is to represent the complex number (x,y) as the point (x,y) .



So the real numbers $(x,0)$ are the points on the horizontal x-axis, the purely imaginary numbers $y i = (0,y)$ are on the vertical y-axis. The real numbers x of the complex number $z = x + y i$ is called the *real* part of z and y is called the *imaginary* part.

Example 4 Locate $2-3i$ on the graph above.

How do we divide complex numbers? Lets introduce the conjugate of a complex number then go to division.

Let $z = x + y i$. We define the conjugate $\bar{z} = \overline{x + y i} = x - y i$

We can divide by using the following:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{x_1 + y_1 i}{x_2 + y_2 i} = \frac{(x_1 + y_1 i)(x_2 - y_2 i)}{(x_2 + y_2 i)(x_2 - y_2 i)} \\ &= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} i \end{aligned}$$

Example 5
$$\frac{2 + 3 i}{3 - 4 i} = \frac{(2 + 3 i)(3 + 4 i)}{(3 - 4 i)(3 + 4 i)} = \frac{6 + 12 i^2 + 9 i - 8 i}{9 - 16 i^2} = -\frac{6}{25} + \frac{1}{25} i$$

Problem Set I

Find

- | | | |
|-------------|-------------|--------------|
| 1. i^2 | 2. i^3 | 3. i^4 |
| 4. i^{21} | 5. i^{-1} | 6. i^{-2} |
| 7. i^{-3} | 8. i^{-4} | 9. i^{-25} |

Let $z_1 = 5 + 6i$, $z_2 = 3 - 2i$, and $z_3 = 1 + 3i$. Find

- | | | |
|-----------------|-----------------------|---------------------|
| 10. $z_1 + z_2$ | 11. $z_1 - z_3$ | 12. \bar{z}_1 |
| 13. $z_1 z_2$ | 14. $\frac{z_1}{z_3}$ | 15. $\frac{i}{z_1}$ |

Graph the following:

- | | | |
|--------------|---------|--|
| 16. $3 + 4i$ | 17. i | 18. $2 - 3i$ and the conjugate of $2 - 3i$ |
|--------------|---------|--|

Evaluate

- | | |
|------------------|----------------------|
| 19. $(3 + 4i)^3$ | 20. $\frac{1}{2+5i}$ |
|------------------|----------------------|

Complex Numbers in Polar Form

It is possible to express complex number in polar form. These equations are defined by the following:

$$x = r \cos \theta, \quad \text{and} \quad y = r \sin \theta$$

We get

$$z = x + yi = r \cos \theta + r \sin \theta i = r(\cos \theta + i \sin \theta)$$

called the polar form of z . r is called the absolute value or modulus of z and written $|z|$.

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

$|z|$ is the distance from the origin to the point (x, y) . θ is called the argument of z and is given by $\theta = \arctan \frac{y}{x}$. Geometrically, θ is the directed angle from the positive x-axis to line segment from the origin through the point (x, y) . Note that when $z = 0$ the angle θ is undefined and θ has the property $-\pi < \theta \leq \pi$. This is the principal value of the argument of θ .

Example 6. For $z = 1 + i$, we get $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$. The principal value of θ is $\frac{\pi}{4}$ but $\theta = \frac{9\pi}{4}$ would work also.

Multiplication and Division in Polar Form

Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ then we have

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$$

Example 7: Let $z_1 = 1 + i = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}) = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$ and $z_2 = \sqrt{3} - i = 2(\cos \frac{\pi}{6} - i \sin \frac{\pi}{6})$

Then

$$(1+i)(\sqrt{3}-i) = 2\sqrt{2}(\cos(\frac{\pi}{4} + \frac{\pi}{6}) + i \sin(\frac{\pi}{4} + \frac{\pi}{6})) = 2\sqrt{2}(\cos(\frac{5\pi}{12}) + i \sin(\frac{5\pi}{12}))$$

And

$$\frac{\sqrt{3}-i}{1+i} = \frac{\sqrt{2}}{2}(\cos(\frac{\pi}{6} - \frac{\pi}{4}) + i \sin(\frac{\pi}{6} - \frac{\pi}{4})) = \frac{\sqrt{2}}{2}(\cos(-\frac{\pi}{12}) + i \sin(-\frac{\pi}{12}))$$

We can use

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$$

when $z_1 = z_2$ to get

$$z^2 = r^2 (\cos(\theta + \theta) + i \sin(\theta + \theta)) = r^2 (\cos 2\theta + i \sin 2\theta)$$

and

$$z^n = r^n (\cos n\theta + i \sin n\theta)$$

Example 8: Compute $(1 + i)^6$.

$$(1 + i)^6 = (\sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}))^6$$

$$= (\sqrt{2})^6 (\cos \frac{6\pi}{4} + i \sin \frac{6\pi}{4})$$

$$= 8(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2})$$

$$= -8i$$

We can use $z^n = r^n (\cos n\theta + i \sin n\theta)$ when $r = 1$ to get (**De Moivre's Theorem**)

$$(\cos \theta + i \sin \theta)^n = (\cos n\theta + i \sin n\theta)$$

Consider $z = r(\cos \theta + i \sin \theta) = w^n = R^n(\cos n\phi + i \sin n\phi)$

where $w = R(\cos \phi + i \sin \phi)$

Then $R = \sqrt[n]{r}$ and $n\theta = \phi$ so $\theta = \frac{\phi}{n}$.

However $n\theta = \phi + 2\pi$ implies $\theta = \frac{\phi+2\pi}{n}$.

And $n\theta = \phi + 4\pi$ implies $\theta = \frac{\phi+4\pi}{n}$.

However $n\theta = \phi + 6\pi$ implies $\theta = \frac{\phi+6\pi}{n}$.

And continuing $n\theta = \phi + 2k\pi$ implies $\theta = \frac{\phi+2k\pi}{n}$ for k any integer up to n .

We get

$$\sqrt[n]{z} = \sqrt[n]{r} \left(\cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n} \right), \quad k = 0, 1, \dots, n-1$$

Example 9: Compute $\sqrt[4]{-4i}$.

Solution: $\sqrt[4]{-4i} = \sqrt[4]{4(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2})} = \sqrt{2}(\cos \theta + i \sin \theta)$,

where $\theta = \frac{3\pi+2k\pi}{4}$.

So $\theta = \frac{3\pi}{4}$ or $\theta = \frac{5\pi}{4}$ or $\theta = \frac{7\pi}{4}$ or $\theta = \frac{9\pi}{4}$

$$\sqrt[4]{-4i} = \pm \frac{\sqrt{2}}{2} \pm i \frac{\sqrt{2}}{2}.$$

Example 10: Compute $\sqrt[3]{1}$.

Solution: $\sqrt[3]{1} = \sqrt[3]{\cos 0 + i \sin 0}$

$$= \cos\left(\frac{0+2k\pi}{3}\right) + i \sin\left(\frac{0+2k\pi}{3}\right)$$

$$= \cos 0 + i \sin 0 \text{ or } \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \text{ or } \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}.$$

$$\text{So } \sqrt[3]{1} = 1 \text{ or } -\frac{\sqrt{3}}{2} + \frac{i}{2} \text{ or } -\frac{\sqrt{3}}{2} - \frac{i}{2}$$

Problem Set II:

Write in polar form

1. $3i$

2. $2 + 2i$

3. $\sqrt{2} + 3i$

Write in rectangular form

4. $\sqrt{32}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$

5. $3(\cos \pi + i \sin \pi)$

6. $\sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$

7. Find $(2 + 2i)^4$

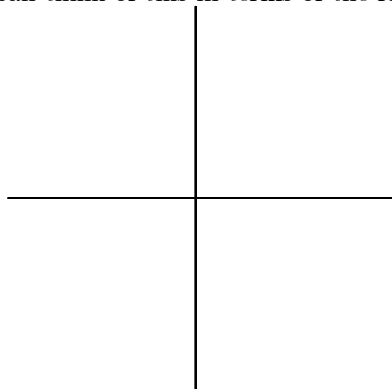
8. Find $\sqrt[4]{16 + 16i}$

Part II Functions

We consider the concept of a function of a complex number. For $y = f(x) = x^2$, where x and y are real numbers, we know about limits, continuity and derivatives. Let the complex number $w = z^2$ where $z = x + iy$ and $w = (x + iy)^2 = x^2 - y^2 + i2xy$. Here x and y are independent variables and $w = u + iv$ where $u = x^2 - y^2$ and $v = 2xy$. This is an example of a complex valued function, w , of a complex variable z . In general, let

$$w = u + i v \quad \text{where } u = u(x, y) \quad \text{and } v = v(x, y)$$

We can think of this in terms of the following figure



In the complex plane let $z_1 = x_1 + iy_1$ and $z_0 = x_0 + iy_0$ then

$$|z_1 - z_0| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

is the distance between the complex numbers z_1 and z_0 .

Definition A **neighborhood** $N(z_0, r)$ of the point z_0 in the complex plane is the set of points z where $r > 0$ and

$$\{z \mid |z - z_0| < r\}$$

Definition A function $w = f(z)$ is said to have a **limit** L as z approaches z_0 (and written $\lim_{z \rightarrow z_0} f(z) = L$) iff is defined in a neighborhood $N(z_0, r)$, $r > 0$ except at z_0 itself and the values of f are close to L for all z close to z_0 . Mathematically: for every $\epsilon > 0$, we can find $\delta > 0$ such that for all $z \neq z_0$ in the neighborhood $N(z_0, r)$, i.e. $0 < |z - z_0| < \delta$, we have $|f(z) - L| < \epsilon$

Notice that this definition of a limit is similar to the definition in calculus. The difference here is that z can approach z_0 from any direction in the complex plane.

A function $w = f(z)$ is **continuous** at $z = z_0$ if $f(z_0)$ is defined and

$$\lim_{z \rightarrow z_0} f(z) = f(z_0)$$

By definition a complex function that is continuous at z_0 is defined in a neighborhood of z_0 . $f(z)$ is continuous in a domain D if it is continuous at each point in the domain D .

The **derivative** of a function $f(z)$ at a point z is defined as

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

provided that limit exists.

Example 11. Let $f(z) = z^2$. To compute $f'(z)$ we consider

$$f'(z) = \lim_{h \rightarrow 0} \frac{(z+h)^2 - z^2}{h} = \lim_{h \rightarrow 0} (2z+h) = 2z$$

The rules for derivatives are the same for calculus:

$$(cf)' = cf' \quad (f+g)' = f'+g' \quad (fg)' = f'g+fg' \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Now consider the function $f(z) = e^z$, and the derivative $f'(z) = e^z$.

We use the Taylor series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

Let $f(z) = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$ and so we have $f'(z) = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$

We can now see that $e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \dots$

$$= (1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots) + i(y - \frac{y^3}{3!} + \frac{y^5}{5!} + \dots) = \cos y + i \sin y$$

So we use

$$e^{iy} = \cos y + i \sin y$$

We now have

$$e^{x+iy} = e^x (\cos y + i \sin y)$$

which is called **Euler's Formula**.

Note that $|e^{iy}| = |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1$. And so we have

$$|e^z| = |e^x(\cos y + i \sin y)| = |e^x| |\cos y + i \sin y| = |e^x|$$

Example 12. $|e^{3-5i}| = e^3$ and $|e^{-1+5i}| = e^{-1}$.

Change y to $-y$ in $e^{iy} = \cos y + i \sin y$ to get $e^{-iy} = \cos -y + i \sin -y = \cos y - i \sin y$.

and solving for $\cos y$ and $\sin y$ to get

$$\cos y = \frac{e^{iy} + e^{-iy}}{2} \qquad \sin y = \frac{e^{iy} - e^{-iy}}{2}$$

So now we can define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2}$$

Facts:

1. For $z = x + i0$ we get $e^z = e^x(\cos 0 + i \sin 0) = e^x$.
2. The derivative of e^z is e^z .
3. $e^{z_1+z_2} = e^{z_1} e^{z_2}$
4. For $z = iy$ we get $e^{iy} = \cos y + i \sin y$
5. In polar form $z = r(\cos \theta + i \sin \theta)$ we get $z = re^{i\theta}$.
6. It now follows that $e^{z+2\pi i} = e^z$
7. It now follows that $e^{2\pi i} = e^0 = 1$. Also $e^{\pi i} = -1$

Problem Set III

Sketch the point sets defined by the following

1. $|z - 1| \leq 2$
2. $|z + 2| \leq 3$
3. $|z - 1 + i| \leq 2$

Show that

4. $\sin(-z) = -\sin z$
5. $\cosh(-z) = \cosh z$
6. $(e^z)^2 = e^{2z}$

Evaluate each of the following

7. $e^{2+\pi i}$
8. e^{i-1}
9. $\sin(3 + \pi i)$
10. $e^{\frac{\pi i}{4}}$
11. $\cos(\pi i)$

Logarithmic Functions If $z = e^w$ then we write $w = \ln z$, called the natural logarithm of z . Thus the natural logarithm function is the inverse of the exponential function and can be defined by

$$w = \ln z = \ln r + i(\theta + 2k\pi) \quad k = 0, \pm 1, \pm 2, \dots$$

where $z = re^{i\theta} = re^{i\theta+2k\pi}$

So we have that $\ln z$ is multiple-valued, and in fact is infinitely many-valued. If we let $k = 0$ then $w = \ln z = \ln r + i(\theta)$ where $0 \leq \theta \leq 2\pi$ is called the principal value or principal branch of $\ln z$.

To find the values of $\ln(1-i)$, let $z = e^w$ where $z = r(\cos \theta + i \sin \theta)$ and $w = u + iv$, and so $z = e^w = e^{u+iv} = e^u(\cos v + i \sin v)$. We equate the real and imaginary parts:

$$(1) \quad e^u \cos v = r \cos \theta \quad (2) \quad e^u \sin v = r \sin \theta$$

Squaring (1) and (2) and adding, we find $e^{2u} = r^2$ or $e^u = r$. Thus, from (1) and (2),

$$r \cos v = r \cos \theta, \quad r \sin v = r \sin \theta$$

from which we see $v = \theta + 2k\pi$. Hence $w = u + iv = \ln r + i(\theta + 2k\pi)$

Since $1-i = \sqrt{2}e^{\frac{7\pi i}{4} + 2k\pi i}$, we have $\ln(1-i) = \ln \sqrt{2} + (\frac{7\pi i}{4} + 2k\pi i) = \frac{1}{2} \ln 2 + \frac{7\pi i}{4} + 2k\pi i$. The principle value is $\frac{1}{2} \ln 2 + \frac{7\pi i}{4}$ (let $k = 0$).