Prosthaphaeresis

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The word prosthaphaeresis comes from the Greek words for "addition" and "subtraction". It is a little known method for performing multiplication and division with table lookups and additions and subtractions. The method is based on the trigonometric identity

$$\cos(a)\cos(b) = \frac{\cos(a+b)+\cos(a-b)}{2}$$

or the related identity

$$\sin(a)\sin(b) = \frac{\cos(a-b)-\cos(a+b)}{2}.$$

If we want to multiply x times y, we start by finding angles a and b so that

$$x = \cos(a)$$

and

$$y = \cos(b).$$

Next, we compute $\cos(a+b)$ and $\cos(a-b)$. Finally, we average these two numbers to obtain the product.

For example, suppose that we want to multiply 1.5732 times 3.5762. We start by writing this as $100 \times 0.15732 \times 0.35762$. Next, we use a table of cosines to find that

$$\cos(80^o 56' 55'') = 0.15732$$

and

$$\cos(69^o 2' 45'') = 0.35762.$$

We add and subtract these angles and find the cosines,

$$\cos(149^o 59' 40'') = -0.86598$$

and

$$\cos(11^o 54' 11'') = 0.97850.$$

The average of these two values is 0.056260. Thus the product of 1.5732 and 3.5762 is approximately 5.6260. This value is off slightly because of round–off errors in the calculation. The correct value is 5.6261.

It is also possible to use prosthaphaeresis to divide two numbers. Suppose that we want to divide x by y. Find angles a and b so that $x = \cos(a)$ and $1/y = \cos(b)$. If $1/y = \cos(b)$, then $y = \sec(b)$. Now, simply use prosthaphaeresis to multiply x times $1/y$.

For example, suppose that we want to divide 2.1378 by 3.7521. First, write this as $10 \times 0.21378/3.7521$. Next, using a table of cosines, we find that

$$\cos(77^o 39' 21'') = 0.21378.$$

Using a table of secants, we find that

$$\sec(74^o 32' 34'') = 3.7521.$$

We add and subtract the two angles and lookup their cosines to get

$$\cos(152^o 11' 55'') = -0.88457$$

and

$$\cos(3^o 06' 47'') = 0.99852.$$

The average of these two values is 0.056975, so 2.1378/3.7521 is approximately 0.56975. The correct value is 0.56976.

Some authors have claimed that the trigonometric identity for the product of two cosines was discovered by the eleventh century mathematician and astronomer Ibn Yunus (or Ibn Junis)[1]. However, more recent research has shown that this is not true[2]. The trigonometric identity for the product of two sines first appeared in a paper by Johannes Werner in 1510[3].

The trigonometric identities were not used for multiplication until late in the sixteenth century. The first publication of the method occurred in a book by Nicolai Reymers Ursus that was published in 1588[3]. However, Tycho Brahe wrote of the technique as early as 1580 in a manual of trigonometry written for his assistants[3]. It appears that Brahe in turn learned of the method from his assistant Paul Wittich[3]. Jost Bürgi also gets credit for providing a proof of the two identities[3].

The method of prosthaphaeresis has number of practical disadvantages. First, it requires accurate tables of the cosine and secant functions. In the sixteenth century the degrees–minutes–seconds system of angle measurements was still in use, making the computation of the sum and difference of the two angles somewhat difficult. The division by two is also inconvenient. Powers of a number could not be computed by prosthaphaeresis. In comparison, Briggs' common logarithms eliminate these disadvantages. Thus it should be no surprise that the method of prosthaphaeresis was quickly replaced by logarithms.

Although the slide rule was invented soon after the development of logarithms, no similar device based on prosthaphaeresis was developed at the time. Recently, a prosthaphaeretic slide rule has appeared[4].

[1] Boyer, C. B. and Merzbach, U. C., *A History of Mathematics, 2nd edition.* New York, John Wiley and Sons, 1989.

[2] Berggren, J. L., *Episodes in the Mathematics of Medieval Islam.* New York, Springer Verlag, 1986.

[3] Thoren, V., "Prosthaphaeresis Revisited", Historia Mathematica, 15:32-39, 1988.

[4] Sher, D. B. and Nataro, D. C., "The Prosthaphaeretic Slide Rule: A Mechanical Multiplication Device Base on Trigonometric Identities", Mathematics and Computer Education, 38(1):37-43, 2004.