THE GROSS-ZAGIER FORMULA ON SINGULAR MODULI FOR
SHIMURA CURVES

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Abstract. The Gross-Zagier formula on singular moduli can be seen as a calculation of the intersection multiplicity of two CM divisors on the integral model of a modular curve. We prove a generalization of this result to a Shimura curve.

1. Introduction

In this paper we study a moduli problem involving false elliptic curves with complex multiplication (CM), generalizing a theorem about the arithmetic degree of a certain moduli stack of CM elliptic curves. This moduli problem is the main arithmetic content of [10]. The result of that paper can be seen as a refinement of the well-known formula of Gross and Zagier on singular moduli in [7]. We begin by describing how the Gross-Zagier formula and the result of [10] can be interpreted as statements about intersection theory on a modular curve. Our generalization of [10] has a similar interpretation as a result about intersection theory, but now on a Shimura curve.

1.1. Elliptic curves. Let \( K_1 \) and \( K_2 \) be non-isomorphic imaginary quadratic fields and set \( K = K_1 \otimes \mathbb{Q} K_2 \). Let \( F \) be the real quadratic subfield of \( K \) and let \( \mathcal{O} \subset \mathcal{O}_F \) be the different of \( F \). We assume \( K_1 \) and \( K_2 \) have relatively prime discriminants \( d_1 \) and \( d_2 \), so \( K/F \) is unramified at all finite places and \( \mathcal{O}_{K_1} \otimes \mathbb{Z} \mathcal{O}_{K_2} \) is the maximal order in \( K \).

Let \( \mathcal{M} \) be the category fibered in groupoids over \( \text{Spec}(\mathcal{O}_K) \) with \( \mathcal{M}(S) \) the category of elliptic curves over the \( \mathcal{O}_K \)-scheme \( S \). The category \( \mathcal{M} \) is an algebraic stack (in the sense of [20], also known as a Deligne-Mumford stack) which is smooth of relative dimension 1 over \( \text{Spec}(\mathcal{O}_K) \) (so it is 2-dimensional). For \( i \in \{1, 2\} \) let \( \mathcal{Y}_i \) be the algebraic stack over \( \text{Spec}(\mathcal{O}_K) \) with \( \mathcal{Y}_i(S) \) the category of elliptic curves over the \( \mathcal{O}_K \)-scheme \( S \) with complex multiplication by \( \mathcal{O}_{K_i} \). When we speak of an elliptic curve \( E \) over an \( \mathcal{O}_K \)-scheme \( S \) with complex multiplication by \( \mathcal{O}_{K_i} \), we are assuming that the action \( \mathcal{O}_{K_i} \hookrightarrow \text{End}_{\mathcal{O}_S}(\text{Lie}(E)) \) is through the structure map \( \mathcal{O}_{K_i} \hookrightarrow \mathcal{O}_{K} \hookrightarrow \mathcal{O}_S(S) \). The stack \( \mathcal{Y}_i \) is finite and étale over \( \text{Spec}(\mathcal{O}_K) \), so in particular it is 1-dimensional and regular. There is a finite morphism \( \mathcal{Y}_i \rightarrow \mathcal{M} \) defined by forgetting the complex multiplication structure.

Even though the morphism \( \mathcal{Y}_1 \rightarrow \mathcal{M} \) is not a closed immersion, we view \( \mathcal{Y}_i \) as a divisor on \( \mathcal{M} \) through its image ([20, Definition 1.7]). A natural question to now ask is: what is the intersection multiplicity, defined in the appropriate sense below, of the two divisors \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \) on \( \mathcal{M} \)? More generally, if \( T_m : \text{Div}(\mathcal{M}) \rightarrow \text{Div}(\mathcal{M}) \) is the \( m \)-th Hecke correspondence on \( \mathcal{M} \), what is the intersection multiplicity of \( T_m \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \)?

If \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) are two prime divisors on \( \mathcal{M} \) intersecting properly, meaning \( \mathcal{D}_1 \cap \mathcal{D}_2 = \mathcal{D}_1 \times_{\mathcal{M}} \mathcal{D}_2 \) is an algebraic stack of dimension 0, define the intersection multiplicity of \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \).
and \( \mathcal{D}_2 \) on \( \mathfrak{M} \) to be

\[
I(\mathcal{D}_1, \mathcal{D}_2) = \sum_{\mathfrak{P} \subset \mathcal{O}_K} \log(|\mathfrak{P}|) \sum_{x \in [(\mathcal{D}_1 \cap \mathcal{D}_2)(\mathfrak{P})]} \frac{\text{length}(\mathcal{O}^\text{sh}_{\mathcal{D}_1 \cap \mathcal{D}_2, x})}{\text{Aut}(x)},
\]

where \([(\mathcal{D}_1 \cap \mathcal{D}_2)(S)]\) is the set of isomorphism classes of objects in the category \((\mathcal{D}_1 \cap \mathcal{D}_2)(S)\) and \(\mathcal{O}^\text{sh}_{\mathcal{D}_1 \cap \mathcal{D}_2, x}\) is the strictly Henselian local ring of \(\mathcal{D}_1 \cap \mathcal{D}_2\) at the geometric point \(x\) (the local ring for the étale topology). Also, the outer sum is over all prime ideals \(\mathfrak{P} \subset \mathcal{O}_K\), \(\mathbb{F}_\mathfrak{P} = \mathcal{O}_K/\mathfrak{P}\), and \(\text{Spec}(\mathbb{F}_\mathfrak{P})\) is an \(\mathcal{O}_K\)-scheme through the reduction map \(\mathcal{O}_K \twoheadrightarrow \mathbb{F}_\mathfrak{P}\). This number is also called the arithmetic degree of the 0-dimensional stack \(\mathcal{D}_1 \cap \mathcal{D}_2\) and is denoted \(\text{deg}(\mathcal{D}_1 \cap \mathcal{D}_2)\). The definition of \(I(\mathcal{D}_1, \mathcal{D}_2)\) is extended to all divisors \(\mathcal{D}_1\) and \(\mathcal{D}_2\) by bilinearity, assuming \(\mathcal{D}_1\) and \(\mathcal{D}_2\) intersect properly.

The intersection multiplicity \(I(\mathcal{Y}_1, \mathcal{Y}_2)\) relates to the Gross-Zagier formula on singular moduli as follows. Let \(L \supset K\) be a number field and suppose \(E_1\) and \(E_2\) are elliptic curves over \(\text{Spec}(\mathcal{O}_L)\). The \(j\)-invariant determines an isomorphism of schemes \(M_{\mathcal{O}_L} \cong \text{Spec}(\mathcal{O}_L[x])\), where \(M \twoheadrightarrow \text{Spec}(\mathcal{O}_K)\) is the coarse moduli scheme associated with \(\mathfrak{M}\), and the elliptic curves \(E_1\) and \(E_2\) determine morphisms \(\text{Spec}(\mathcal{O}_L) \twoheadrightarrow M_{\mathcal{O}_L}\). These morphisms correspond to ring homomorphisms \(\mathcal{O}_L[x] \twoheadrightarrow \mathcal{O}_L\) defined by \(x \mapsto j(E_1)\) and \(x \mapsto j(E_2)\). Let \(D_1\) and \(D_2\) be the divisors on \(M_{\mathcal{O}_L}\) defined by the morphisms \(\text{Spec}(\mathcal{O}_L) \twoheadrightarrow M_{\mathcal{O}_L}\). Then

\[D_1 \cap D_2 = \text{Spec}(\mathcal{O}_L \otimes_{\mathcal{O}_L[x]} \mathcal{O}_L) \cong \text{Spec}(\mathcal{O}_L/(j(E_1) - j(E_2))).\]

For \(\tau\) an imaginary quadratic integer in the complex upper half plane, let \(|\tau|\) be its equivalence class under the action of \(\text{SL}_2(\mathbb{Z})\). As in [7] define

\[
J(d_1, d_2) = \left( \prod_{\text{disc}(\tau_1) = d_1} (j(\tau_1) - j(\tau_2)) \right)^{4/(w_1 w_2)},
\]

where \(w_i = |\mathcal{O}_K^\times|\). It follows from the above discussion that the main result of [7], which is a formula for the prime factorization of the integer \(J(d_1, d_2)^2\), is essentially the same as giving a formula for \(\text{deg}(\mathcal{Y}_1 \cap \mathcal{Y}_2) = I(\mathcal{Y}_1, \mathcal{Y}_2)\).

For each positive integer \(m\) define \(\mathcal{F}_m\) to be the algebraic stack over \(\text{Spec}(\mathcal{O}_K)\) with \(\mathcal{F}_m(S)\) the category of triples \((E_1, E_2, f)\) where \(E_i\) is an object of \(\mathcal{Y}_i(S)\) and \(f \in \text{Hom}_S(E_1, E_2)\) satisfies \(\text{deg}(f) = m\) on every connected component of \(S\). In [10] it is shown there is a decomposition

\[
\mathcal{F}_m = \bigsqcup_{\alpha \in F^\times} \mathcal{F}_\alpha \quad \text{Tr}_{F/\mathbb{Q}(\alpha)}(\alpha) = m
\]

for some 0-dimensional stacks \(\mathcal{F}_\alpha \to \text{Spec}(\mathcal{O}_K)\) and then a formula is given for each term in

\[
\text{deg}(\mathcal{F}_m) = \sum_{\alpha \in D^{-1}, \alpha \geq 0} \text{deg}(\mathcal{F}_\alpha),
\]

with \(\text{deg}(\mathcal{F}_m)\) and \(\text{deg}(\mathcal{F}_\alpha)\) defined just as in (1.1). We will prove later (in the appendix) that

\[
\text{deg}(\mathcal{F}_m) = I(T_m, \mathcal{Y}_1, \mathcal{Y}_2),
\]

so the main result of [10] really is a refinement of the Gross-Zagier formula.
Let $\mathcal{X}$ be the algebraic stack over $\text{Spec}(\mathcal{O}_K)$ with fiber $\mathcal{X}(S)$ the category of pairs $(E_1, E_2)$ where $E_i = (E_i, \kappa_i)$ with $E_i$ an elliptic curve over the $\mathcal{O}_K$-scheme $S$ with complex multiplication $\kappa_i : \mathcal{O}_K \to \text{End}_S(E_i)$. Let $(E_1, E_2)$ be an object of $\mathcal{X}(S)$. The maximal order $\mathcal{O}_K = \mathcal{O}_{K_1} \otimes_{\mathbb{Z}} \mathcal{O}_{K_2}$ acts on the $\mathbb{Z}$-module $L(E_1, E_2) = \text{Hom}_S(E_1, E_2)$ by

$$ (t_1 \otimes t_2) \cdot f = \kappa_2(t_2) \circ f \circ \kappa_1(t_1), $$

where $x \mapsto \pi$ is the nontrivial element of $\text{Gal}(K/F)$. Writing $[\cdot, \cdot]$ for the bilinear form on $L(E_1, E_2)$ associated with the quadratic form $\text{deg}$, there is a unique $\mathcal{O}_F$-bilinear form $[\cdot, \cdot]_{\text{CM}} : L(E_1, E_2) \times L(E_1, E_2) \to \mathcal{O}^{-1}$ satisfying $[f_1, f_2] = \text{Tr}_{F/\mathbb{Q}}[f_1, f_2]_{\text{CM}}$. Let $\text{deg}_{\text{CM}}$ be the totally positive definite $F$-quadratic form on $L(E_1, E_2) \otimes_{\mathbb{Z}} \mathbb{Q}$ corresponding to $[\cdot, \cdot]_{\text{CM}}$, so $\text{deg}(f) = \text{Tr}_{F/\mathbb{Q}} \text{deg}_{\text{CM}}(f)$.

For any $\alpha \in F^\times$ let $\mathcal{X}_\alpha$ be the algebraic stack over $\text{Spec}(\mathcal{O}_K)$ with $\mathcal{X}_\alpha(S)$ the category of triples $(E_1, E_2, f)$ where $(E_1, E_2)$ is an object of $\mathcal{X}(S)$ and $f \in L(E_1, E_2)$ satisfies $\text{deg}_{\text{CM}}(f) = \alpha$ on every connected component of $S$. The category $\mathcal{X}_\alpha$ is empty unless $\alpha$ is totally positive and lies in $\mathcal{O}^{-1}$.

Let $\chi$ be the quadratic Hecke character associated with the extension $K/F$ and for $\alpha \in F^\times$ define $\text{Diff}(\alpha)$ to be the set of prime ideals $\mathfrak{p} \subset \mathcal{O}_F$ satisfying $\chi_{\mathfrak{p}}(\alpha \mathcal{O}) = -1$. The set $\text{Diff}(\alpha)$ is finite and nonempty. For any fractional $\mathcal{O}_F$-ideal $\mathfrak{b}$ let $\rho(\mathfrak{b})$ be the number of ideals $\mathfrak{B} \subset \mathcal{O}_K$ satisfying $N_{K/F}(\mathfrak{B}) = \mathfrak{b}$. For any prime number $\ell$ let $\rho_{\ell}(\mathfrak{b})$ be the number of ideals $\mathfrak{B} \subset \mathcal{O}_{K,\ell}$ satisfying $N_{K,\ell/F}(\mathfrak{B}) = b\mathcal{O}_{K,\ell}$, so there is a product formula

$$ \rho(\mathfrak{b}) = \prod_{\ell} \rho_{\ell}(\mathfrak{b}). $$

The following theorem, which is essentially [10, Theorem A], is the main result we will generalize.

**Theorem 1** (Howard-Yang). Suppose $\alpha \in F^\times$ is totally positive. If $\alpha \in \mathcal{O}^{-1}$ and $\text{Diff}(\alpha) = \{ \mathfrak{p} \}$ then $\mathcal{X}_\alpha$ is of dimension zero, is supported in characteristic $p$ (the rational prime below $p$), and satisfies

$$ \text{deg}(\mathcal{X}_\alpha) = \frac{1}{2} \log(p) \cdot \text{ord}_{\mathfrak{p}}(\alpha \mathcal{O}) \cdot \rho(\alpha \mathcal{O}^{-1}). $$

If $\alpha \notin \mathcal{O}^{-1}$ or if $\# \text{Diff}(\alpha) > 1$, then $\text{deg}(\mathcal{X}_\alpha) = 0$.

1.2. **False elliptic curves**. Our work in generalizing Theorem 1 goes as follows. Let $B$ be an indefinite quaternion algebra over $\mathbb{Q}$, let $\mathcal{O}_B$ be a maximal order of $B$, and let $d_B$ be the discriminant of $B$. A **false elliptic curve** over a scheme $S$ is a pair $(A, i)$ where $A \to S$ is an abelian scheme of relative dimension 2 and $i : \mathcal{O}_B \to \text{End}_S(A)$ is a ring homomorphism. Any false elliptic curve $A$ comes equipped with a principal polarization $\lambda : A \to A^\vee$ uniquely determined by a condition described below. If $A_1$ and $A_2$ are false elliptic curves over a connected scheme $S$ with corresponding principal polarizations $\lambda_1$ and $\lambda_2$, then the map

$$ f \mapsto \lambda_1^{-1} \circ f^\vee \circ \lambda_2 \circ f : \text{Hom}_{\mathcal{O}_B}(A_1, A_2) \to \text{End}_{\mathcal{O}_B}(A_1) $$

has image in $\mathbb{Z} \subset \text{End}_{\mathcal{O}_B}(A_1)$ and defines a positive definite quadratic form, called the **false degree** and denoted $\text{deg}^\star$.

We retain the same notation of $K_1$, $K_2$, $F$, and $K$ as above. We also assume each prime dividing $d_B$ is inert in $K_1$ and $K_2$, so in particular, $K_1$ and $K_2$ split $B$. Let $S$ be an $\mathcal{O}_K$-scheme. A **false elliptic curve** over $S$ with complex multiplication by $\mathcal{O}_{K_j}$, for $j \in \{1, 2\}$, is a triple $\mathbf{A} = (A, i, \kappa)$ where $(A, i)$ is a false elliptic curve over $S$ and $\kappa : \mathcal{O}_{K_j} \to \text{End}_{\mathcal{O}_B}(A)$
is an action such that the induced map $\mathcal{O}_{K_j} \to \text{End}_{\mathcal{O}_B}(\text{Lie}(A))$ is through the structure map $\mathcal{O}_{K_j} \to \mathcal{O}_K \to \mathcal{O}_S(S)$. Let $m_B \subset \mathcal{O}_B$ be the unique ideal of index $d_B$, so $\mathcal{O}_B/m_B \cong \prod_{p \mid d_B} \mathbb{F}_{p^2}$.

Let $\mathcal{M}^B$ be the category fibered in groupoids over $\text{Spec}(\mathcal{O}_K)$ with $\mathcal{M}^B(S)$ the category whose objects are false elliptic curves $(A, i)$ over the $\mathcal{O}_K$-scheme $S$ satisfying the following condition for any $x \in \mathcal{O}_B$: any point of $S$ has an affine open neighborhood $\text{Spec}(R) \to S$ such that $\text{Lie}(A/R)$ is a free $R$-module of rank 2 and there is an equality of polynomials

\[
\text{char}(i(x), \text{Lie}(A/R)) = (T - x)(T - x')
\]

in $\mathbb{R}[T]$, where $x \mapsto x'$ is the main involution on $B$. The category $\mathcal{M}^B$ is an algebraic stack which is regular and flat of relative dimension 1 over $\text{Spec}(\mathcal{O}_K)$, smooth over $\text{Spec}(\mathcal{O}_K[d_B])$ (if $B$ is a division algebra, $\mathcal{M}^B$ is proper over $\text{Spec}(\mathcal{O}_K)$). For $j \in \{1, 2\}$ let $\mathcal{Y}^B_j$ be the algebraic stack over $\text{Spec}(\mathcal{O}_K)$ with $\mathcal{Y}^B_j(S)$ the category of false elliptic curves over the $\mathcal{O}_K$-scheme $S$ with complex multiplication by $\mathcal{O}_{K_j}$. The stack $\mathcal{Y}^B_j$ is finite and étale over $\text{Spec}(\mathcal{O}_K)$, so in particular it is 1-dimensional and regular. Any object of $\mathcal{Y}^B_j(S)$ automatically satisfies condition (1.3) (see Corollary 3.13 below). Therefore there is a finite morphism $\mathcal{Y}^B_j \to \mathcal{M}^B$ defined by forgetting the complex multiplication structure.

Our main goal is to calculate the intersection multiplicity of the two divisors $T_m \mathcal{Y}^B_1$ and $\mathcal{Y}^B_2$ on $\mathcal{M}^B$, defined just as in (1.1), where $T_m$ is the $m$-th Hecke correspondence on $\mathcal{M}^B$. In the course of this calculation we prove the following result, which should be of independent interest. Let $k$ be an imaginary quadratic field and let $L$ be any finite extension of $k$. Assume each prime dividing $d_B$ is inert in $k$. Define $\mathcal{Y}$ to be the algebraic stack over $\text{Spec}(\mathcal{O}_L)$ consisting of all elliptic curves over $\mathcal{O}_L$-schemes with CM by $\mathcal{O}_k$, and make the analogous definition of $\mathcal{Y}^B$ for false elliptic curves. Then there is a decomposition

$$\mathcal{Y}^B = \bigsqcup_{\mathcal{O}_k \to \mathcal{O}_B/m_B} \mathcal{Y},$$

where the union is over all ring homomorphisms $\mathcal{O}_k \to \mathcal{O}_B/m_B$ (Theorem 3.12).

A CM pair over an $\mathcal{O}_K$-scheme $S$ is a pair $(A_1, A_2)$ where $A_1$ and $A_2$ are false elliptic curves over $S$ with complex multiplication by $\mathcal{O}_{K_1}$ and $\mathcal{O}_{K_2}$, respectively. For such a pair, set $L(A_1, A_2) = \text{Hom}_{\mathcal{O}_A}(A_1, A_2)$. As before, there is a unique $\mathcal{O}_F$-quadratic form $\deg_{\text{CM}} : L(A_1, A_2) \to \mathbb{D}^{-1}$ satisfying $\text{Tr}_{F/\mathbb{Q}} \deg_{\text{CM}}(f) = \deg^*(f)$. For any false elliptic curve $A$ let $A[m_B]$ be its $m_B$-torsion, defined as a group scheme below. For any ring homomorphism $\theta : \mathcal{O}_K \to \mathcal{O}_B/m_B$ define $\mathcal{X}^B_\theta$ to be the algebraic stack over $\text{Spec}(\mathcal{O}_K)$ where $\mathcal{X}^B_\theta(S)$ is the category of CM pairs $(A_1, A_2)$ over the $\mathcal{O}_K$-scheme $S$ such that the diagram

$$\begin{gathered}
\mathcal{O}_K \to \mathcal{O}_B/m_B \to \text{End}_{\mathcal{O}_B/m_B}(A_j[m_B]), \\
\mathcal{O}_B/m_B \\
\theta|\mathcal{O}_K
\end{gathered}$$

commutes for $j = 1, 2$, where $\mathcal{O}_B/m_B \to \text{End}_{\mathcal{O}_B/m_B}(A_j[m_B])$ is the map induced by the action of $\mathcal{O}_B$ on $A_j$. Note that this map makes sense as $\mathcal{O}_B/m_B$ is commutative. If $B = M_2(\mathbb{Q})$ then $m_B = \mathcal{O}_B$, so any such $\theta$ is necessarily 0 and $\mathcal{X}^B_0$ is the stack of all CM pairs over $\mathcal{O}_K$-schemes.

For any $\alpha \in F^\times$ define $\mathcal{X}^B_{\theta, \alpha}$ to be the algebraic stack over $\text{Spec}(\mathcal{O}_K)$ with $\mathcal{X}^B_{\theta, \alpha}(S)$ the category of triples $(A_1, A_2, f)$ where $(A_1, A_2)$ is an object of $\mathcal{X}^B_\theta(S)$ and $f \in L(A_1, A_2)$
satisfies \( \deg_{CM}(f) = \alpha \) on every connected component of \( S \). Define the arithmetic degree of \( \mathcal{X}^B_{\theta,\alpha} \) as in (1.1) and define a nonempty finite set of prime ideals

\[
\text{Diff}_\theta(\alpha) = \{ p \subset \mathcal{O}_F : \chi_p(\alpha a_\theta \mathcal{D}) = -1 \},
\]

where \( a_\theta = \ker(\theta) \cap \mathcal{O}_F \). Our main result is the following (Proposition 7.2 and Theorems 6.7 and 7.3 in the text; see the appendix for the proof of (b)).

**Theorem 2.** Let \( \alpha \in F^\times \) be totally positive and suppose \( \alpha \in \mathcal{D}^{-1} \). Let \( \theta : \mathcal{O}_K \to \mathcal{O}_B/m_B \) be a ring homomorphism with \( a_\theta = \ker(\theta) \cap \mathcal{O}_F \), suppose \( \text{Diff}_\theta(\alpha) = \{ p \} \), and let \( p\mathbb{Z} = p \cap \mathcal{O}_F \).

(a) The stack \( \mathcal{X}^B_{\theta,\alpha} \) is of dimension zero and is supported in characteristic p.

(b) There is a decomposition

\[
I(T_m, \mathcal{X}^B_1, \mathcal{X}^B_2) = \sum_{\beta \in \mathcal{D}^{-1}, \beta \geq 0} \sum_{\mathcal{O}_K \to \mathcal{O}_B/m_B} \deg(\mathcal{X}^B_{\eta,\beta}).
\]

(c) If \( p \mid d_B \) then

\[
\deg(\mathcal{X}^B_{\theta,\alpha}) = \frac{1}{2} \log(p) \cdot \text{ord}_p(\alpha p \mathcal{D}) \cdot \rho(\alpha a_\theta^{-1} p^{-1} \mathcal{D}).
\]

(d) Suppose \( p \mid d_B \) and let \( \mathfrak{P} \subset \mathcal{O}_K \) be the unique prime over \( p \). If \( \mathfrak{P} \) divides \( \ker(\theta) \) then

\[
\deg(\mathcal{X}^B_{\theta,\alpha}) = \frac{1}{2} \log(p) \cdot \text{ord}_p(\alpha) \cdot \rho(\alpha a_\theta^{-1} p^{-1} \mathcal{D}).
\]

If \( \mathfrak{P} \) does not divide \( \ker(\theta) \) then

\[
\deg(\mathcal{X}^B_{\theta,\alpha}) = \frac{1}{2} \log(p) \cdot \text{ord}_p(\alpha p) \cdot \rho(\alpha a_\theta^{-1} p^{-1} \mathcal{D}).
\]

If \( \alpha \notin \mathcal{D}^{-1} \) or if \( \# \text{Diff}_\theta(\alpha) > 1 \), then \( \deg(\mathcal{X}^B_{\theta,\alpha}) = 0 \).

The proof of this theorem consists of two general parts: counting the number of geometric points of the stack \( \mathcal{X}^B_{\theta,\alpha} \) (Theorem 5.13 and Proposition 5.14) and calculating the length of the local ring \( \mathcal{O}^{sh}_{\mathcal{X}^B_{\theta,\alpha},x} \) (Theorem 6.7).

### 1.3. Eisenstein series.**

Theorem 1 is really only half of a larger story, one that gives a better explanation of the definition of the arithmetic degree of \( \mathcal{X}^B_{\theta,\alpha} \) and provides a surprising connection between arithmetic geometry and analysis. To explain this, let \( K_1, K_2, F \), and \( K \) be as in Section 1.1, let \( D = \text{disc}(F) \), and let \( \sigma_1 \) and \( \sigma_2 \) be the two real embeddings of \( F \). For \( \tau_1, \tau_2 \) in the complex upper half plane and \( s \in \mathbb{C} \) define an Eisenstein series

\[
E^s(\tau_1, \tau_2, s) = \frac{\Gamma \left( \frac{s+2}{2} \right)}{\Gamma \left( \frac{s}{2} \right)}^{2} \sum_{\mathcal{O}^2_F} \chi(\alpha) N(\alpha)^{1+s} \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z}} (v_1 v_2)^{s/2} \left[ \frac{m,n}{(m,n)(\tau_1,\tau_2)} \right]^{s}.
\]

where \( \text{Cl}(\mathcal{O}_F) \) is the ideal class group of \( F \), \( v_1 = \text{Im}(\tau_1) \), and

\[
[m,n](\tau_1,\tau_2) = (\sigma_1(m)\tau_1 + \sigma_1(n))(\sigma_2(m)\tau_2 + \sigma_2(n)).
\]

This series, which is convergent for \( \Re(s) > 0 \), has meromorphic continuation to all \( s \in \mathbb{C} \) and defines a non-holomorphic Hilbert modular form of weight 1 for \( \text{SL}_2(\mathcal{O}_F) \) which is
holomorphic in $s$ in a neighborhood of $s = 0$. The derivative of $E^*(\tau_1, \tau_2, s)$ at $s = 0$ has a Fourier expansion

$$(E^*)'(\tau_1, \tau_2, 0) = \sum_{\alpha \in \mathcal{D}^{-1}} a_\alpha(v_1, v_2) \cdot q^{\alpha},$$

where $e(x) = e^{2\pi i x}$ and $q^\alpha = e(\sigma_1(\alpha) \tau_1 + \sigma_2(\alpha) \tau_2)$. The connection between this analytic theory and the moduli space $\mathcal{X}_\alpha$ lies in the next theorem ([10, Theorem B, Theorem C]).

**Theorem** (Howard-Yang). Suppose $\alpha \in F^\times$ is totally positive. If $\alpha \in \mathcal{D}^{-1}$ then $a_\alpha = a_\alpha(v_1, v_2)$ is independent of $v_1, v_2$ and $a_\alpha = 4 \cdot \deg(\mathcal{X}_\alpha)$.

It seems likely that there is a theorem in the spirit of the one above for the moduli space $\mathcal{X}_{\alpha}^B$, but we do not pursue that direction here. A reasonable next question to address is: can Theorem 2 be extended to the case where $\mathcal{Y}^B$ is defined to be the stack of false elliptic curves with CM by a fixed non-maximal order in $K_0^*$? A result of this type would seemingly extend the results of Lauter and Viray in [12] to false elliptic curves.

1.4. **Notation and conventions.** If $X$ is an abelian variety or a $p$-divisible group over a field $k$, we write $\text{End}(X)$ for $\text{End}_k(X)$. When we say “stack” we mean algebraic stack in the sense of [20], also called a Deligne-Mumford stack. We write $\mathbb{Q}_{p^2}$ for the unique unramified quadratic extension of $\mathbb{Q}_p$ and $\mathbb{Z}_{p^2} \subset \mathbb{Q}_{p^2}$ its ring of integers. By “scheme” we mean locally Noetherian scheme. If $\mathcal{C}$ is a category, we write $C \in \mathcal{C}$ to mean $C$ is an object of $\mathcal{C}$.

2. **False elliptic curves**

In this section we give a brief review of the basic theory of false elliptic curves. For the remainder of this paper fix an indefinite quaternion algebra $B$ over $\mathbb{Q}$ and a maximal order $\mathcal{O}_B$ of $B$. We do not exclude the case where $B$ is split, that is, where $B = M_2(\mathbb{Q})$. As $B$ is split at $\infty$, all maximal orders of $B$ are conjugate by elements of $B^\times$. Let $d_B$ be the discriminant of $B$.

**Definition 2.1.** Let $S$ be a scheme. A false elliptic curve over $S$ is a pair $(A, i)$ where $A \to S$ is an abelian scheme of relative dimension 2 and $i : \mathcal{O}_B \hookrightarrow \text{End}_S(A)$ is an injective ring homomorphism.

**Definition 2.2.** Let $(A_1, i_1)$ and $(A_2, i_2)$ be two false elliptic curves over a scheme $S$. A homomorphism $f : A_1 \to A_2$ of false elliptic curves is a homomorphism of abelian schemes over $S$ satisfying $i_2(x) \circ f = f \circ i_1(x)$ for all $x \in \mathcal{O}_B$. If in addition $f$ is an isogeny of abelian schemes, then $f$ is called an isogeny of false elliptic curves.

In fact, any nonzero homomorphism of false elliptic curves $A_1 \to A_2$ is necessarily an isogeny. For each place $v$ of $\mathbb{Q}$ let $\text{inv}_v : \text{Br}_2(\mathbb{Q}_v) \to \{\pm 1\}$ be the unique isomorphism.

**Definition 2.3.** For each prime number $p$, define $B(p)$ to be the quaternion division algebra over $\mathbb{Q}$ determined by

$$\text{inv}_v(B(p)) = \begin{cases} \text{inv}_v(B) & \text{if } v \notin \{p, \infty\} \\ -\text{inv}_v(B) & \text{if } v \in \{p, \infty\}. \end{cases}$$

**Proposition 2.4.** Suppose $A$ is a false elliptic curve over a field $k$.

(a) If $k = \mathbb{F}_p$ then $\text{End}_\mathcal{O}_B(A) = \text{End}_{\mathcal{O}_B}(A) \otimes_\mathbb{Z} \mathbb{Q}$ is either

1. an imaginary quadratic field $L$ which admits an embedding $L \hookrightarrow B$, or
2. the definite quaternion algebra $B(p)$.
Furthermore, \( A \) is isogenous to \( E^2 \) for some elliptic curve \( E \) over \( \overline{\mathbb{F}}_p \), with \( E \) ordinary in case (1) and supersingular in case (2).

(b) If \( k = \mathbb{C} \) then \( \text{End}^0_{\mathbb{Q}}(A) \) is either \( \mathbb{Q} \) or an imaginary quadratic field which splits \( B \).

Proof. For (a) see [13, Proposition 5.2] and for (b) see [3, Proposition 52]. \( \square \)

**Proposition 2.5.** Suppose \( A \) is a false elliptic curve over a field \( L \supset \mathbb{F}_p \). Then \( \text{End}(A) \) embeds into \( \text{End}(A') \) for some false elliptic curve \( A' \) defined over a finite extension of \( \mathbb{F}_p \).

Proof. Use induction on the transcendence degree of \( L \) over \( \mathbb{F}_p \). \( \square \)

Let \( x \mapsto x' \) be the main involution of \( B \) and fix \( a \in \mathcal{O}_B \) satisfying \( a^2 = -d_B \). Define another involution on \( B \) by \( x \mapsto x'' = a^{-1}x'a \). The order \( \mathcal{O}_B \) is stable under \( x \mapsto x'' \). If \( (A, i) \) is a false elliptic curve over \( S \), then so is the dual abelian scheme \( A' \), with corresponding homomorphism \( i^\vee : \mathcal{O}_B \to \text{End}_S(A') \) defined by \( i^\vee(x) = i(x)^\vee \). If \( f : A_1 \to A_2 \) is a homomorphism of false elliptic curves, then so is \( f^\vee : A_2' \to A_1' \).

**Proposition 2.6.** Let \( A \) be a false elliptic curve over a scheme \( S \). There is a unique principal polarization \( \lambda : A \to A' \) such that the corresponding Rosati involution \( \varphi \mapsto \varphi^\dagger = \lambda^{-1} \circ \varphi^\vee \circ \lambda \) on \( \text{End}(A) \) induces \( x \mapsto x^\ast \) on \( \mathcal{O}_B \subset \text{End}(A) \).

Proof. See [2, p. 3] and [1, Proposition III.3.3]. \( \square \)

Let \( A_1 \) and \( A_2 \) be false elliptic curves over \( S \) with corresponding principal polarizations \( \lambda_1 : A_1 \to A_1' \) and \( \lambda_2 : A_2 \to A_2' \). Suppose \( f : A_1 \to A_2 \) is an isogeny of false elliptic curves. Using the principal polarizations \( \lambda_1 \) and \( \lambda_2 \), we obtain a map \( f^\dagger : A_2 \to A_1 \) defined as the composition

\[
 f^\dagger = \lambda_1^{-1} \circ f^\vee \circ \lambda_2 : A_2 \to A_1.
\]

This is an isogeny of false elliptic curves, called the dual isogeny to \( f \).

**Proposition 2.7.** Let \( f : A_1 \to A_2 \) be an isogeny of false elliptic curves over a scheme \( S \). The isogeny \( f^\dagger \circ f : A_1 \to A_1 \) is locally on \( S \) multiplication by an integer.

Proof. This can be checked on geometric fibers, so we may assume \( A_1 \) is a false elliptic curve over an algebraically closed field. Viewing \( f^\dagger \circ f \in \text{End}^0_{\mathbb{Q}}(A_1) \), a calculation shows \( f^\dagger \circ f \) is fixed by the Rosati involution corresponding to \( \lambda_1 \). The set of fixed points is \( \mathbb{Q} \), so \( f^\dagger \circ f : A_1 \to A_1 \) is multiplication by an integer. \( \square \)

**Definition 2.8.** If the integer in the previous proposition is constant on \( S \), then it is called the false degree of \( f \), and is denoted \( \text{deg}^* (f) \).

**Proposition 2.9.** Let \( A_1 \) and \( A_2 \) be false elliptic curves over a connected scheme \( S \). The map \( \text{deg}^* : \text{Hom}_{\mathcal{O}_S}(A_1, A_2) \to \mathbb{Z} \) is a positive definite quadratic form.

Proof. Easy exercise. \( \square \)

3. CM false elliptic curves

For this section let \( k \) be an imaginary quadratic field and let \( L \) be a finite extension of \( k \). Assume any prime dividing \( d_B \) is inert in \( k \).
3.1. Definitions.

**Definition 3.1.** Let \( S \) be an \( \mathcal{O}_L \)-scheme. A false elliptic curve over \( S \) with complex multiplication by \( \mathcal{O}_k \) is a triple \( A = (A, i, \kappa) \), where \( (A, i) \) is a false elliptic curve over \( S \) and \( \kappa : \mathcal{O}_k \to \text{End}_{\mathcal{O}_B}(A) \) is a ring homomorphism such that the diagram

\[
\begin{array}{ccc}
\mathcal{O}_k & \xrightarrow{\kappa_{\text{Lie}}} & \text{End}_{\mathcal{O}_B}(\text{Lie}(A)) \\
\downarrow & & \downarrow \\
\mathcal{O}_S(S) & \xrightarrow{} & \mathcal{O}_S(S)
\end{array}
\]

commutes, where \( \mathcal{O}_k \hookrightarrow \mathcal{O}_L \to \mathcal{O}_S(S) \) is the structure map. We call the commutativity of this diagram the CM normalization condition.

When we speak of a CM false elliptic curve over \( \mathbb{F}_p \) for some prime ideal \( \mathfrak{p} \subset \mathcal{O}_L \), it is understood that \( \text{Spec}(\mathbb{F}_p) \) is an \( \mathcal{O}_L \)-scheme through the reduction map \( \mathcal{O}_L \to \mathbb{F}_p \hookrightarrow \mathbb{F}_p \). Less precisely, when we speak of a CM false elliptic curve \( A \) over \( \mathbb{F}_p \) for some prime number \( p \), we really mean \( A \) is a CM false elliptic curve over \( \mathbb{F}_p \) for some prime ideal \( \mathfrak{p} \subset \mathcal{O}_L \) lying over \( p \).

**Definition 3.2.** Define \( \mathcal{Y}^B \) to be the category whose objects are triples \( (A, i, \kappa) \), where \( (A, i) \) is a false elliptic curve over some \( \mathcal{O}_L \)-scheme with complex multiplication \( \kappa : \mathcal{O}_k \to \text{End}_{\mathcal{O}_B}(A) \). A morphism \( (A', i', \kappa') \to (A, i, \kappa) \) between two such triples defined over \( \mathcal{O}_L \)-schemes \( T \) and \( S \), respectively, is a morphism of \( \mathcal{O}_L \)-schemes \( T \to S \) together with an \( \mathcal{O}_k \)-linear isomorphism \( A' \to A \times_S T \) of false elliptic curves.

The category \( \mathcal{Y}^B \) is a stack of finite type over \( \text{Spec}(\mathcal{O}_L) \). In fact, \( \mathcal{Y}^B \to \text{Spec}(\mathcal{O}_L) \) is étale by Proposition 3.6 below, proper by a proof identical to that of [9, Proposition 3.3.5], and quasi-finite by Propositions 3.4 and 3.7 below, so the morphism is finite étale. Let \( [\mathcal{Y}^B(S)] \) denote the set of isomorphism classes of objects in \( \mathcal{Y}^B(S) \).

For each prime \( p \) dividing \( d_B \) there is a unique maximal ideal \( m_p \subset \mathcal{O}_B \) of residue characteristic \( p \), and \( \mathcal{O}_B/m_p \) is a finite field with \( p^2 \) elements. Set \( m_B = \bigcap_{p|d_B} m_p \). We have \( m_B = \prod_{p|d_B} m_p \) because for any two primes \( p \) and \( q \) dividing \( d_B \), \( m_p m_q = m_q m_p \) since these lattices have equal completions at each prime number. Note that

\[
\mathcal{O}_B/m_B \cong \prod_{p|d_B} \mathbb{F}_{p^2}
\]

as rings. Let \( (A, i) \) be a false elliptic curve over a scheme \( S \). The \( d_B \)-torsion \( A[d_B] \) is a finite flat commutative \( S \)-group scheme with a natural action of \( m_B/d_B \mathcal{O}_B \). Let \( x_B \) be any element of \( m_B \) whose image generates the principal ideal \( m_B/d_B \mathcal{O}_B \subset \mathcal{O}_B/d_B \mathcal{O}_B \). Define the \( m_B \)-torsion of \( A \) as

\[
A[m_B] = \ker(i(x_B) : A[d_B] \to A[d_B]),
\]

which again is a finite flat commutative \( S \)-group scheme \( i(x_B) : A \to A \) is an isogeny). This definition does not depend on the choice of \( x_B \). The group scheme \( A[m_B] \) has an action of \( \mathcal{O}_B/m_B \) given on points by \( x \cdot a = i(x)(a) \) for \( x \in \mathcal{O}_B/m_B \) and \( a \in A[m_B](T) \) for any \( S \)-scheme \( T \). All the statements of this paragraph are vacuous if \( B \) is split.
Definition 3.3. Let \( \theta : \mathcal{O}_k \to \mathcal{O}_B/\mathfrak{m}_B \) be a ring homomorphism. Define \( \mathcal{Y}^B(\theta) \) to be the category whose objects are objects \((A, i, \kappa)\) of \( \mathcal{Y}^B \) such that the diagram

\[
\begin{array}{ccc}
\mathcal{O}_k & \xrightarrow{\kappa_{m_B}} & \text{End}_{\mathcal{O}_B/\mathfrak{m}_B}(A[\mathfrak{m}_B]) \\
\theta & & \\
\mathcal{O}_B/\mathfrak{m}_B & \xrightarrow{\kappa_{m_B}} & \text{End}_{\mathcal{O}_B/\mathfrak{m}_B}(A[\mathfrak{m}_B])
\end{array}
\]

commutes, where \( \kappa_{m_B} \) is the map on \( \mathfrak{m}_B \)-torsion induced by \( \kappa \) and

\[
\mathcal{O}_B/\mathfrak{m}_B \to \text{End}_{\mathcal{O}_B/\mathfrak{m}_B}(A[\mathfrak{m}_B])
\]

is the map induced by \( i \). Morphisms are defined in the same way as in the category \( \mathcal{Y}^B \).

Note that \( \mathcal{Y}^B(\theta) = \mathcal{Y}^B \) if \( B \) is split. Recall from the introduction that \( \mathcal{Y} \) is the stack over \( \text{Spec}(\mathcal{O}_L) \) with \( \mathcal{Y}(S) \) the category of elliptic curves over the \( \mathcal{O}_L \)-scheme \( S \) with CM by \( \mathcal{O}_k \). We will prove below there is an isomorphism of stacks over \( \text{Spec}(\mathcal{O}_L) \)

\[
\begin{array}{c}
\square \\
\theta : \mathcal{O}_k \to \mathcal{O}_B/\mathfrak{m}_B
\end{array}
\]

inducing an equivalence of categories \( \mathcal{Y} \to \mathcal{Y}^B(\theta) \) for any \( \theta \) (Theorem 3.12). It follows that \( \mathcal{Y}^B(\theta) \) has the structure of a stack, finite étale over \( \text{Spec}(\mathcal{O}_L) \), and \( \mathcal{Y} \cong \mathcal{Y}^B \) in the case of \( B \) split.

3.2. Group actions. Suppose \((A, i, \kappa)\) is a false elliptic curve over an \( \mathcal{O}_L \)-scheme \( S \) with complex multiplication by \( \mathcal{O}_k \), and let \( \mathfrak{a} \) be a fractional ideal of \( \mathcal{O}_k \). Since there is a ring homomorphism \( \kappa : \mathcal{O}_k \to \text{End}_S(A) \), we may view \( A \) as an \( \mathcal{O}_k \)-module scheme over \( S \), so from \( \mathfrak{a} \) being a finitely generated projective \( \mathcal{O}_k \)-module, locally free of rank 1, there is an abelian scheme \( \mathfrak{a} \otimes_{\mathcal{O}_k} A \to S \) of relative dimension 2 satisfying \((\mathfrak{a} \otimes_{\mathcal{O}_k} A)(X) = \mathfrak{a} \otimes_{\mathcal{O}_k} A(X) \) for any \( S \)-scheme \( X \) (see [4, Section 7]). There are commuting actions

\[
i_{\mathfrak{a}} : \mathcal{O}_B \to \text{End}_S(\mathfrak{a} \otimes_{\mathcal{O}_k} A), \quad \kappa_{\mathfrak{a}} : \mathcal{O}_k \to \text{End}_S(\mathfrak{a} \otimes_{\mathcal{O}_k} A)
\]

defined in the obvious way. Using the isomorphism \( \text{Lie}(\mathfrak{a} \otimes_{\mathcal{O}_k} A) \cong \mathfrak{a} \otimes_{\mathcal{O}_k} \text{Lie}(A) \) of \( \mathcal{O}_S \)-modules, it follows that \( \kappa_{\mathfrak{a}} \) inherits the CM normalization condition from \( \kappa \). This shows \( \mathfrak{a} \otimes_{\mathcal{O}_k} A \) is a false elliptic curve over \( S \) with complex multiplication by \( \mathcal{O}_k \). Therefore the ideal class group \( \text{Cl}(\mathcal{O}_k) \) acts on the set \([\mathcal{Y}^B(S)]\).

The other important group action on \([\mathcal{Y}^B(S)]\) comes from the Atkin-Lehner group \( W_0 \) of \( \mathcal{O}_B \). By definition, \( W_0 = N_{\mathfrak{B}^*}(\mathcal{O}_B)/\mathfrak{B}^* \mathcal{O}_B^* = \langle w_p : p \mid d_B \rangle \), where \( w_p \in \mathcal{O}_B \) has reduced norm \( p \). As an abstract group, \( W_0 \cong \prod_{p \mid d_B} \mathbb{Z}/2\mathbb{Z} \). The group \( W_0 \) acts on the set \([\mathcal{Y}^B(S)]\) for any \( \mathcal{O}_L \)-scheme \( S \) as follows: for \( w \in W_0 \) and \( x = (A, i, \kappa) \in \mathcal{Y}^B(S) \), define \( w \cdot x = (A, i_{w'}(a) = i(waw^{-1}) \). The actions of \( W_0 \) and \( \text{Cl}(\mathcal{O}_k) \) commute, so there is an induced action of \( W_0 \times \text{Cl}(\mathcal{O}_k) \) on \([\mathcal{Y}^B(S)]\).

Proposition 3.4. The group \( W_0 \times \text{Cl}(\mathcal{O}_k) \) acts simply transitively on \([\mathcal{Y}^B(\mathbb{C})]\).

Proof. It is shown in [11] that \( W_0' \times \text{Cl}(\mathcal{O}_k) \) acts simply transitively on \([\mathcal{Y}^B(\mathbb{C})]\), where \( W_0' \subset W_0 \) is the subgroup generated by \( \{w_p : p \mid d_B, p \text{ inert in } k\} \). However, we are assuming each prime \( p \mid d_B \) is inert in \( k \). \(\square\)
3.3. Structure of CM false elliptic curves. The main result of this section states that any CM false elliptic curve arises from a CM elliptic curve through the Serre tensor construction. We will use this in the next section to give a description, in terms of certain coordinates, of the ring \( \text{Hom}_{\mathcal{O}_L}(A) \otimes \mathbb{Z}_p \) for \( A \) a CM false elliptic curve over \( \overline{\mathbb{F}}_p \) for \( p \mid d_B \). Fix a prime ideal \( \mathfrak{p} \subset \mathcal{O}_L \) of residue characteristic \( p \). Let \( \mathcal{W}_{L_{\mathfrak{p}}} \) be the ring of integers in the completion of the maximal unramified extension of \( L_{\mathfrak{p}} \), so in particular \( \mathcal{W}_{L_{\mathfrak{p}}} \) is an \( \mathcal{O}_L \)-algebra. Let \( \text{CLN}_{L_{\mathfrak{p}}} \) be the category whose objects are complete local Noetherian \( \mathcal{W}_{L_{\mathfrak{p}}} \)-algebras with residue field \( \mathbb{F}_{\mathfrak{p}} \), where \( \mathbb{F}_{\mathfrak{p}} = \mathcal{O}_L/\mathfrak{p} \), and morphisms \( R \to R' \) are local ring homomorphisms inducing the identity \( \overline{\mathbb{F}}_{\mathfrak{p}} \to \overline{\mathbb{F}}_{\mathfrak{p}} \) on residue fields.

**Definition 3.5.** Suppose \( \tilde{R} \to R \) is a surjection of \( \mathcal{O}_L \)-algebras and \( x = (A, i, \kappa) \in \mathcal{Y}^B(R) \). A deformation of \( x \) (or just a deformation of \( A \)) to \( \tilde{R} \) is an object \( (\tilde{A}, \tilde{i}, \tilde{\kappa}) \in \mathcal{Y}^B(\tilde{R}) \) together with an \( \mathcal{O}_k \)-linear isomorphism \( \tilde{A} \otimes_R R \to A \) of false elliptic curves.

If \( \tilde{R} \to R \) is a surjection of \( \mathcal{O}_L \)-algebras, \( (A, i, \kappa) \in \mathcal{Y}^B(R) \), and \( (\tilde{A}, \tilde{i}, \tilde{\kappa}) \in \mathcal{Y}^B(\tilde{R}) \) is a deformation of \( (A, i, \kappa) \), then it is easy to check that the principal polarizations \( \tilde{\lambda} : \tilde{A} \to (\tilde{A})^\vee \) and \( \lambda : A \to A^\vee \) defined in Proposition 2.6 are compatible in the sense that \( \tilde{\lambda} \) is the reduction of \( \lambda \). Let \( x = (A, i, \kappa) \in \mathcal{Y}^B(\mathbb{F}_{\mathfrak{p}}) \) and define a functor \( \text{Def}_{\mathcal{O}_B}(A, \mathcal{O}_k) : \text{CLN}_{L_{\mathfrak{p}}} \to \text{Sets} \) that assigns to each object \( R \) of \( \text{CLN}_{L_{\mathfrak{p}}} \) the set of isomorphism classes of deformations of \( x \) to \( R \).

**Proposition 3.6.** The functor \( \text{Def}_{\mathcal{O}_B}(A, \mathcal{O}_k) \) is represented by \( \mathcal{W}_{L_{\mathfrak{p}}} \), so there is a bijection

\[
\text{Def}_{\mathcal{O}_B}(A, \mathcal{O}_k)(R) \cong \text{Hom}_{\text{CLN}_{L_{\mathfrak{p}}}}(\mathcal{W}_{L_{\mathfrak{p}}}, R),
\]

which is a one point set, for any object \( R \) of \( \text{CLN}_{L_{\mathfrak{p}}} \). In particular, the reduction map

\[
[\mathcal{Y}^B(R)] \to [\mathcal{Y}^B(\mathbb{F}_{\mathfrak{p}})]
\]

is a bijection for any \( R \in \text{CLN}_{L_{\mathfrak{p}}} \).

**Proof.** Let \( R \) be an Artinian object of \( \text{CLN}_{L_{\mathfrak{p}}} \), so the reduction map \( R \to \mathbb{F}_{\mathfrak{p}} \) is surjective with nilpotent kernel. By [8, Proposition 2.1.2], \( A \) has a unique deformation \( \hat{A} \), as a abelian scheme with an action of \( \mathcal{O}_k \), to \( R \), and the reduction map \( \text{End}_{\mathcal{O}_k}(\hat{A}) \to \text{End}_{\mathcal{O}_k}(A) \) is an isomorphism. Therefore we can lift the \( \mathcal{O}_k \)-linear action of \( \mathcal{O}_B \) on \( A \) to a unique such action on \( \hat{A} \). This shows that each object of \( \mathcal{Y}^B(\mathbb{F}_{\mathfrak{p}}) \) has a unique deformation to an object of \( \mathcal{Y}^B(R) \) for any Artinian \( R \) in \( \text{CLN}_{L_{\mathfrak{p}}} \). Now let \( R \) be an arbitrary object of \( \text{CLN}_{L_{\mathfrak{p}}} \), so \( R = \lim \frac{R}{m^n} \), where \( m \subset R \) is the maximal ideal. The result now follows from the Artinian case, the bijection

\[
\text{Hom}_{\text{CLN}_{L_{\mathfrak{p}}}}(\mathcal{W}_{L_{\mathfrak{p}}}, R) \cong \lim_{\text{def}} \text{Hom}_{\text{CLN}_{L_{\mathfrak{p}}}}(\mathcal{W}_{L_{\mathfrak{p}}}, \frac{R}{m^n}),
\]

and the fact that the natural map

\[
\text{Def}_{\mathcal{O}_B}(A, \mathcal{O}_k)(R) \to \lim_{\text{def}} \text{Def}_{\mathcal{O}_B}(A, \mathcal{O}_k)(\frac{R}{m^n})
\]

is a bijection by Grothendieck’s existence theorem ([4, Theorem 3.4]). □

**Proposition 3.7.** The group \( W_0 \times \text{Cl}(\mathcal{O}_k) \) acts simply transitively on \( [\mathcal{Y}^B(\mathbb{F}_{\mathfrak{p}})] \).

**Proof.** Let \( \mathbb{C}_p \) be the metric completion of an algebraic closure of \( \mathbb{Q}_p \) and fix a ring embedding \( \mathcal{W}_{L_{\mathfrak{p}}} \to \mathbb{C}_p \). There is a \( W_0 \times \text{Cl}(\mathcal{O}_k) \)-equivariant bijection \( [\mathcal{Y}^B(\mathbb{C}_p)] \to [\mathcal{Y}^B(\mathbb{F}_{\mathfrak{p}})] \) defined by descending to a number field, reducing modulo a prime over \( p \), and then base extending to \( \mathbb{F}_{\mathfrak{p}} \). The inverse to this map is the composition

\[
[\mathcal{Y}^B(\mathbb{F}_{\mathfrak{p}})] \to [\mathcal{Y}^B(\mathcal{W}_{L_{\mathfrak{p}}})] \to [\mathcal{Y}^B(\mathbb{C}_p)],
\]
where the first map is the inverse of the reduction map in Proposition 3.6 and the second is base extension to \( C_p \). The result now follows from Proposition 3.4.

Our next goal is to prove there is an isomorphism as in (3.2). It will be a consequence of this isomorphism that any \( A \in \mathcal{Y}_{B}(S) \) is of the form \( M \otimes_{\mathcal{O}_k} E \) for some \( E \in \mathcal{Y}(S) \) and some \( \mathcal{O}_B \otimes_{\mathbb{Z}} \mathcal{O}_k \)-module \( M \), free of rank 4 over \( \mathbb{Z} \). To prove this result, we will describe a bijection between the set of isomorphism classes of such modules \( M \) and the set \( \{ \mathcal{Y}_{B}(\mathbb{C}) \} \).

For the remainder of this section set \( \mathcal{O} = \mathcal{O}_B \otimes_{\mathbb{Z}} \mathcal{O}_k \), and define \( \mathcal{L} \) to be the set of isomorphism classes of \( \mathcal{O} \)-modules that are free of rank 4 over \( \mathbb{Z} \). Define \( \mathcal{K} \) to be the set of \( \mathcal{O}_B \)-conjugacy classes of ring embeddings \( \mathcal{O}_k \hookrightarrow \mathcal{O}_B \). We begin by examining the local structure of modules in \( \mathcal{L} \).

**Lemma 3.8.** Fix a prime \( p \) and let \( \Delta \) be the maximal order in the unique quaternion division algebra over \( \mathbb{Q}_p \). Fix an embedding \( \mathbb{Z}_p^2 \hookrightarrow \Delta \) so that there is a decomposition \( \Delta = \mathbb{Z}_p^2 \oplus \mathbb{Z}_p \Pi \), where \( \Pi \) is a uniformizer satisfying \( \Pi^2 = p \) and \( \Pi a = \bar{a} \Pi \) for all \( a \in \mathbb{Z}_p^2 \). Then any ring homomorphism \( f : \Delta \rightarrow M_2(\mathbb{Z}_p^2) \) is \( \text{GL}_2(\mathbb{Z}_p^2) \)-conjugate to exactly one of the following two maps:

\[
\begin{align*}
  f_1 : a + b\Pi &\mapsto \begin{bmatrix} a & b \\ b & \bar{a} \end{bmatrix}, \\
  f_2 : a + b\Pi &\mapsto \begin{bmatrix} a & pb \\ 0 & \bar{a} \end{bmatrix}.
\end{align*}
\]

The proof uses the general ideas of the proof of [17, Theorem 1.4].

**Proof.** The group \( M = \mathbb{Z}_p^2 \oplus \mathbb{Z}_p \) is a left \( \mathbb{Z}_p^2 \)-module via componentwise multiplication, and a right \( \Delta \)-module via matrix multiplication \( \begin{bmatrix} a & b \\ b & \bar{a} \end{bmatrix} f(x) \), viewing elements of \( M \) as row vectors. These actions commute, so \( M \) is a \( \Delta \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^2 \)-module. There is an isomorphism of rings \( \Delta \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^2 \cong R_1 \), where \( R_1 \) is the standard Eichler order of level 1 in \( M_2(\mathbb{Z}_p^2) \). Any \( R_1 \)-module which is free of finite rank over \( \mathbb{Z}_p \) is a direct sum of copies of \( \Delta \) and \( m_\Delta \), where \( m_\Delta \in \Delta \) is the unique maximal ideal ([16, Chapter 9]). By comparing \( \mathbb{Z}_p \)-ranks, we see that there is an isomorphism of \( \Delta \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^2 \)-modules \( M \rightarrow \Delta \) or \( M \rightarrow m_\Delta \). The rest of the proof is an easy exercise.

**Lemma 3.9.** Let \( p \) be a prime number. For \( p \nmid d_B \) there is a unique isomorphism class of \( \mathcal{O}_p \)-modules free of rank 4 over \( \mathbb{Z}_p \) and for \( p \mid d_B \) there are two isomorphism classes.

**Proof.** First suppose \( p \nmid d_B \). In this case,

\[
\mathcal{O}_p \cong \mathcal{O}_{B,p} \otimes_{\mathbb{Z}_p} \mathcal{O}_k,p \cong M_2(\mathcal{O}_k,p),
\]

and any \( \mathcal{O}_p \)-module that is free of rank 4 over \( \mathbb{Z}_p \) is isomorphic to \( \mathcal{O}_k,p \oplus \mathcal{O}_k,p \), with the natural left action of \( M_2(\mathcal{O}_k,p) \). Now suppose \( p \mid d_B \), so \( \mathcal{O}_p \cong \Delta \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^2 \). By the proof of Lemma 3.8 there are two isomorphism classes of modules over this ring that are free of rank 4 over \( \mathbb{Z}_p \).

Now we will show that the three sets \( \mathcal{K} \), \( \mathcal{L} \), and \( \{ \mathcal{Y}_{B}(\mathbb{C}) \} \) are all in bijection.

**Proposition 3.10.** There is a bijection \( \mathcal{K} \rightarrow \mathcal{L} \).

**Proof.** Let \( \Theta : \mathcal{O}_k \rightarrow \mathcal{O}_B \) be a representative of an \( \mathcal{O}_B \)-conjugacy class of embeddings and define \( f : \mathcal{K} \rightarrow \mathcal{L} \) by sending \( \Theta \) to the \( \mathbb{Z}_p \)-module \( L_\Theta = \mathcal{O}_B \), viewed as a right \( \mathcal{O}_k \)-module through \( \Theta \) (and multiplication on the right) and a left \( \mathcal{O}_B \)-module through multiplication on the left. The isomorphism class of this \( \mathcal{O} \)-module only depends on \( \Theta \) through its \( \mathcal{O}_B \)-conjugacy class. The map \( f \) is easily seen to be a bijection, using that the group \( \text{Cl}(\mathcal{O}_k) \) acts on the sets \( \mathcal{K} \) and \( \mathcal{L} \).
Proposition 3.11. There is a bijection $\mathcal{L} \to [\mathcal{Y}^B(\mathbb{C})]$.

Proof. Let $M \in \mathcal{L}$. Then $V = M \otimes_{\mathbb{Z}} \mathbb{R}$ is a 4-dimensional $\mathbb{R}$-vector space with $M$ a $\mathbb{Z}$-lattice in $V$. The action of $O_k$ on $M$ induces a map $k \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C} \to \text{End}(V)$, turning $V$ into a $\mathbb{C}$-vector space. Define a function $\mathcal{L} \to [\mathcal{Y}^B(\mathbb{C})]$ by sending $M$ to the CM false elliptic curve with complex points $V/M$. The inverse $[\mathcal{Y}^B(\mathbb{C})] \to \mathcal{L}$ is given by $A \mapsto H_1(A(\mathbb{C}), \mathbb{Z})$. □

Define an equivalence relation on the set $\mathcal{K}$ according to $\Theta \sim \Theta'$ if and only if the induced maps $\tilde{\Theta}, \tilde{\Theta}' : O_k \to O_B/m_B$ are equal. Let $\mathcal{K}'$ be the set of equivalence classes under this relation. Under the bijection $\mathcal{K} \to \mathcal{L}$, this equivalence relation corresponds to the following equivalence relation on $\mathcal{L}'$: $M \sim M'$ if and only if $M_t \cong M'_t$ as $O_t$-modules for all primes $\ell$ (note by Lemma 3.9 that this really is only a condition at each prime dividing $d_B$). Let $\mathcal{L}'$ be the set of equivalence classes under this relation. We know that the group $W_0 \times \text{Cl}(O_k)$ acts simply transitively on the set $[\mathcal{Y}^B(\mathbb{C})]$, so its natural actions on $\mathcal{K}$ and $\mathcal{L}$ are also simply transitive.

The elements of $\mathcal{L}'$ can be thought of as collections of $O_t$-modules $\{M_t\}_t$ indexed by the prime numbers. The action of $W_0$ on $\mathcal{L}'$ induces an action on $\mathcal{L}'$. Explicitly, for $\ell \mid d_B$, the Atkin-Lehner operator $w_\ell \in W_0$ interchanges the two isomorphism classes of modules $M_t$ over $O_t$. It follows that under the action of $W_0 \times \text{Cl}(O_k)$ on $\mathcal{L}'$, the group $\text{Cl}(O_k)$ acts simply transitively on each equivalence class under $\sim$ and the group $W_0$ acts simply transitively on the set of equivalence classes $\mathcal{L}'$. The corresponding results hold for the set $\mathcal{K}$, so in particular $\#\mathcal{K}' = |W_0| = 2^r$, where $r$ is the number of primes dividing $d_B$. Since there are $2^r$ ring homomorphisms $O_k \to O_B/\mathfrak{m}_B$, each such homomorphism arises as the reduction of a homomorphism $O_k \to O_B$.

The equivalence relation $\sim$ on $\mathcal{K}$ induces an equivalence relation on the set $[\mathcal{Y}^B(\mathbb{C})]$ determined by the following: if $[\Theta]$ is the equivalence class of $\Theta \in \mathcal{K}$, then $[\Theta]$ is in bijection with $[\mathcal{Y}^B(\tilde{\Theta})(\mathbb{C})]$. It follows that the natural action of $\text{Cl}(O_k)$ on $[\mathcal{Y}^B(\tilde{\Theta})(\mathbb{C})]$ is simply transitive. The same statements hold with $[\mathcal{Y}^B(\tilde{\Theta})(\mathbb{F}_p)]$ in place of $[\mathcal{Y}^B(\tilde{\Theta})(\mathbb{C})]$.

Suppose $(E, \kappa)$ is an elliptic curve over an $O_L$-scheme $S$ with CM by $O_k$ and let $M \in \mathcal{L}$. From $M$ being a finitely generated projective $O_k$-module, locally free of rank 2, there is an abelian scheme $M \otimes_{O_k} E \to S$ of relative dimension 2 with $(M \otimes_{O_k} E)(X) = M \otimes_{O_k} E(X)$ for any $S$-scheme $X$. There are commuting actions

$$i_M : O_B \to \text{End}_S(M \otimes_{O_k} E), \quad \kappa_M : O_k \to \text{End}_S(M \otimes_{O_k} E)$$

given on points by

$$i_M(x)(m \otimes z) = x \cdot m \otimes z, \quad \kappa_M(a)(m \otimes z) = m \otimes \kappa(a)(z),$$

so $M \otimes_{O_k} E$ is a false elliptic curve over $S$ with complex multiplication by $O_k$.

If $\Theta : O_k \to O_B$ is a false elliptic curve, we will sometimes write $\mathcal{Y}^B(\Theta)$ for $\mathcal{Y}^B(\tilde{\Theta})$. Recall that $\mathcal{Y}$ is the stack of all elliptic curves over $O_L$-schemes with CM by $O_k$.

Theorem 3.12. Fix representatives $\Theta_1, \ldots, \Theta_m \in \mathcal{K}$ of the $m = 2^r$ classes in $\mathcal{K}'$. There is an isomorphism of stacks over $\text{Spec}(O_L)$

$$f : \bigsqcup_{d=1}^m \mathcal{Y} \to \mathcal{Y}^B,$$

defined by $(E, \mathfrak{d}) \mapsto L_{\Theta_d} \otimes_{O_k} E$. This isomorphism induces an equivalence of categories $\mathcal{Y} \to \mathcal{Y}^B(\Theta)$ for any $[\Theta] \in \mathcal{K}'$. 

The notation \((E, d)\) means \(E\) is an object of the \(d\)-th copy of \(\mathcal{Y}\) in the disjoint union. Therefore we obtain an isomorphism

\[
\bigcup_{\theta : \mathcal{O}_k \to \mathcal{O}_B/m_B} \mathcal{Y}^B(\theta) \to \mathcal{Y}^B.
\]

In particular, any \(A \in \mathcal{Y}^B(S)\) is isomorphic to \(L_{\Theta} \otimes_{\mathcal{O}_k} E\) for some \(\Theta : \mathcal{O}_k \to \mathcal{O}_B\) and some \(E \in \mathcal{Y}(S)\).

**Proof.** The idea of the proof is to introduce level structure to the stacks \(\mathcal{Y}\) and \(\mathcal{Y}^B\), show that these new spaces are schemes, and then show \(f\) induces an isomorphism between these schemes. We begin by showing \(f\) induces a bijection on geometric points. Let \(k = \mathbb{C}\) or \(k = \mathbb{F}_p\) and let \(X \subset [\mathcal{Y}^B(k)]\) be the image of the map

\[
f_k : \bigcup_{d=1}^m [\mathcal{Y}(k)] \to [\mathcal{Y}^B(k)]
\]

on \(k\)-points determined by \(f\). The group \(W_0 \times \text{Cl}(\mathcal{O}_k)\) acts simply transitively on \([\mathcal{Y}^B(k)]\) and this action preserves the subset \(X\), so \(f_k\) is surjective. Now, it is well-known that \(\text{Cl}(\mathcal{O}_k)\) acts simply transitively on \([\mathcal{Y}(k)]\), and thus \(f_k\) is a bijection since

\[
\# \bigcup_{d=1}^m [\mathcal{Y}(k)] = m \cdot \# [\mathcal{Y}(k)] = |W_0| \cdot |\text{Cl}(\mathcal{O}_k)| = \# [\mathcal{Y}^B(k)].
\]

Fix an integer \(n \geq 1\) and set \(S = \text{Spec}(\mathcal{O}_L)\) and \(S_n = \text{Spec}(\mathcal{O}_L[n^{-1}])\). For \(n\) prime to \(d_B\) define \(\mathcal{Y}^B(n)\) to be the category fibered in groupoids over \(S_n\) with \(\mathcal{Y}^B(n)(T)\) the category of quadruples \((A, i, \kappa, \nu)\) where \((A, i, \kappa) \in \mathcal{Y}^B(T)\) and \(\nu : (\mathcal{O}_B/(n))_T \to A[n]\)

is an \(\mathcal{O}\)-linear isomorphism of schemes, where \((\mathcal{O}_B/(n))_T\) is the constant group scheme over the \(S_n\)-scheme \(T\) associated with \(\mathcal{O}_B/(n)\). Here we are viewing \(\mathcal{O}_B/(n)\) as a left \(\mathcal{O}_B\)-module through multiplication on the left and a right \(\mathcal{O}_k\)-module through a fixed inclusion \(\mathcal{O}_k \hookrightarrow \mathcal{O}_B\) and multiplication on the right. Forgetting \(\nu\) defines a finite étale representable morphism \(\mathcal{Y}^B(n) \to \mathcal{Y}^B \times_S S_n\), so \(\mathcal{Y}^B(n)\) is a stack, finite étale over \(S_n\). A similar argument to that used in the proof of [2, Lemma 2.2] shows that for \(n \geq 3\) prime to \(d_B\), any object of \(\mathcal{Y}^B(n)\) has no nontrivial automorphisms. It follows from this fact, as in the proof of [2, Corollary 2.3], that \(\mathcal{Y}^B(n)\) is a scheme.

For any \(n \geq 1\) define \(\mathcal{Y}(n)\) to be the category fibered in groupoids over \(S_n\) with \(\mathcal{Y}(n)(T)\) the category of triples \((E, \kappa, \nu)\) where \((E, \kappa) \in \mathcal{Y}(T)\) and \(\nu : (\mathcal{O}_k/(n))_T \to E[n]\)

is an \(\mathcal{O}_k\)-linear isomorphism of schemes. As above, \(\mathcal{Y}(n)\) is a scheme, finite étale over \(S_n\). Let \(G_n = \text{Aut}_{\mathcal{O}_k}(\mathcal{O}_k/(n)) \cong (\mathcal{O}_k/(n))^\times\). There is an action of the finite group scheme \((G_n)_{S_n}\) on the scheme \(\mathcal{Y}(n)\), defined on \(T\)-points, for any connected \(S_n\)-scheme \(T\), by

\[
g \cdot (E, \kappa, \nu) = (E, \kappa, \nu \circ g^{-1}).
\]

There is an associated quotient stack \(\mathcal{Y}(n)/(G_n)_{S_n} \to S_n\), defined in [20, Example 7.17], and there is an isomorphism of stacks \(\mathcal{Y}(n)/(G_n)_{S_n} \to \mathcal{Y} \times_S S_n\) such that the composition

\[
\mathcal{Y}(n) \to \mathcal{Y}(n)/(G_n)_{S_n} \cong \mathcal{Y} \times_S S_n
\]

is the morphism defined by forgetting the level structure.
Note that there is an isomorphism of groups \( \operatorname{Aut}_O(\mathcal{O}_B/(n)) \cong (\mathcal{O}_k/(n))^\times \), so \((G_n)_{S_n}\) also acts on \( \mathcal{Y}_B(n) \), the action defined in the same way as above. As before there is an isomorphism of stacks \( \mathcal{Y}_B(n)/(G_n)_{S_n} \rightarrow \mathcal{Y}_B \times_S S_n \) such that the composition

\[
\mathcal{Y}_B(n) \rightarrow \mathcal{Y}_B(n)/(G_n)_{S_n} \rightarrow \mathcal{Y}_B \times_S S_n
\]

is the forgetful morphism. The base change

\[
f_n = f \times \text{id} : \bigsqcup_{d=1}^m \mathcal{Y} \times_S S_n \rightarrow \mathcal{Y}_B \times_S S_n
\]

induces a morphism of schemes over \( S_n \)

\[
f'_n : \bigsqcup_{d=1}^m \mathcal{Y}(n) \rightarrow \mathcal{Y}(n)
\]

given on \( T \)-points by \((E, \nu, d) \mapsto (L_{\Theta_d} \otimes_{\mathcal{O}_k} E, \nu')\), where \( \nu' \) is the composition

\[
(\mathcal{O}_B/(n))_T \cong L_{\Theta_d} \otimes_{\mathcal{O}_k} \mathcal{O}_k/(n) \xrightarrow{\text{id} \otimes \nu} L_{\Theta_d} \otimes_{\mathcal{O}_k} E[n] \cong (L_{\Theta_d} \otimes_{\mathcal{O}_k} E)[n].
\]

For \( k = \mathbb{C} \) or \( k = \mathbb{F}_p \), it follows easily from \( f_k \) being a bijection that \( f'_n \) defines a bijection

\[
(f'_n)_k : \bigsqcup_{d=1}^m [\mathcal{Y}(n)(k)] \rightarrow [\mathcal{Y}_B(n)(k)].
\]

The morphism \( f'_n \) is \((G_n)_{S_n}\)-equivariant, so there is a morphism of stacks

\[
\bigsqcup_{d=1}^m \mathcal{Y}(n)/(G_n)_{S_n} \rightarrow \mathcal{Y}_B(n)/(G_n)_{S_n}
\]

inducing \( f_n \) under the isomorphisms described above. It follows that to show \( f_n \) is an isomorphism, it suffices to show \( f'_n \) is an isomorphism. As \( f'_n \) is a finite étale morphism of \( S_n \)-schemes inducing a bijection on geometric points, it is an isomorphism. Choosing relatively prime integers \( n, n' \geq 3 \) prime to \( d_B \), the morphisms \( f_n \) and \( f_{n'} \) being isomorphisms implies \( f \) is an isomorphism.

For the final statement of the theorem, let \( S \) be any \( \mathcal{O}_L \)-scheme and fix an integer \( 1 \leq d \leq m \). It follows directly from the definitions that any CM false elliptic curve of the form \( L_{\Theta_d} \otimes_{\mathcal{O}_k} E \) for some \( E \in \mathcal{Y}(S) \) lies in \( \mathcal{Y}_B([\Theta_d])(S) \). Conversely, suppose \((A, i, \kappa) \in \mathcal{Y}_B([\Theta_d])(S) \). Then \( A \cong L_{\Theta_d'} \otimes_{\mathcal{O}_k} E \) for some \( E \in \mathcal{Y}(S) \) and a unique \( 1 \leq d' \leq m \), so the diagram

\[
\begin{array}{ccc}
\mathcal{O}_k & \xrightarrow{\kappa_{\mathcal{O}_B/m_B}} & \operatorname{End}_{\mathcal{O}_B/m_B}(A[m_B]) \\
\downarrow{\eta} & & \downarrow{\nu} \\
\mathcal{O}_B/m_B & \xrightarrow{f} & \mathcal{O}_B/m_B
\end{array}
\]

commutes for \( \eta = \tilde{\Theta}_d \) and \( \eta = \tilde{\Theta}_{d'} \). Picking any geometric point \( \pi \) of \( S \), the above diagram still commutes with \( A \) replaced with \( A_{\pi} \). But the map \( \mathcal{O}_B/m_B \rightarrow \operatorname{End}_{\mathcal{O}_B/m_B}(A_{\pi}[m_B]) \) is an isomorphism by Corollary 5.9, proved below only using the first paragraph of this proof. Therefore \( \tilde{\Theta}_d = \tilde{\Theta}_{d'} \), so \( d = d' \), which shows \( f \) defines an equivalence of categories \( \mathcal{Y} \rightarrow \mathcal{Y}_B([\Theta_d]) \). \( \square \)
Corollary 3.13. Suppose $S$ is an $\mathcal{O}_L$-scheme and let $(A, i, \kappa) \in \mathcal{W}(S)$. Then the trace of $i(x)$ acting on $\text{Lie}(A)$ is equal to $\text{Trd}(x)$ for any $x \in \mathcal{O}_B$.

Proof. We have $A \cong M \otimes_{\mathcal{O}_k} E$ for some $\mathcal{O}$-module $M$ and $E \in \mathcal{W}(S)$. Then $\text{Lie}(A) \cong M \otimes_{\mathcal{O}_k} \text{Lie}(E)$ as $\mathcal{O}$-modules, with $\mathcal{O}_B$ acting on $M \otimes_{\mathcal{O}_k} \text{Lie}(E)$ through its action on $M$. As $M \cong \mathcal{O}_B$ as a left $\mathcal{O}_B$-module, the result easily follows. □

Corollary 3.14. Suppose $\tilde{R} \to R$ is a surjection of $\mathcal{O}_L$-algebras, $x = (A, i, \kappa) \in \mathcal{W}(R)$, and $\tilde{x} = (\tilde{A}, \tilde{i}, \tilde{\kappa}) \in \mathcal{W}(\tilde{R})$ is a deformation of $x$. Let $\theta : \mathcal{O}_k \to \mathcal{O}_B/\mathfrak{m}_B$ be a ring homomorphism. Then $x \in \mathcal{W}^{\tilde{B}}(\theta)(\check{R})$ if and only if $\tilde{x} \in \mathcal{W}^{\tilde{B}}(\theta)(\tilde{R})$.

Proof. This is a direct consequence of Theorem 3.12. □

3.4. The Dieudonné module. Fix a prime number $p$ and let $W = W(\mathbb{F}_p)$ be the ring of Witt vectors over $\mathbb{F}_p$, so $W$ is the ring of integers in the completion of the maximal unramified extension of $\mathbb{Q}_p$. If $A$ is a false elliptic curve over $\mathbb{F}_p$, we write $D(A)$ for the covariant Dieudonné module of $A$ (that is, the Dieudonné module of $A[p^\infty]$), which is a module over the Dieudonné ring $\mathcal{D}$, free of rank 4 over $W$. Recall that there is a unique continuous ring automorphism $\sigma$ of $W$ inducing the absolute Frobenius $x \mapsto x^p$ on $W/pW \cong \mathbb{F}_p$, and $\mathcal{D} = W\{\mathcal{F}, \mathcal{Y}\}/(\mathcal{F}\mathcal{Y} - p)$ where $W\{\mathcal{F}, \mathcal{Y}\}$ is the non-commutative polynomial ring in two commuting variables $\mathcal{F}$ and $\mathcal{Y}$ satisfying $\mathcal{F}x = \sigma(x)\mathcal{F}$ and $\mathcal{Y}x = \sigma^{-1}(x)\mathcal{Y}$ for all $x \in W$.

Let $A \in \mathcal{W}(\mathbb{F}_p)$, so $A \cong M \otimes_{\mathcal{O}_k} E$ for some $E \in \mathcal{W}(\mathbb{F}_p)$ and some module $M$ over $\mathcal{O} = \mathcal{O}_B \otimes \mathbb{Z}_p \mathcal{O}_k$, free of rank 4 over $\mathbb{Z}_p$. Let $p$ be the rational prime below $\mathfrak{p}$. There is an isomorphism of $W \otimes_{\mathbb{Z}_p} \mathcal{O}_p$-modules

$$D(A) \cong M_p \otimes_{\mathcal{O}_k,p} D(E).$$

However, $M_p \cong \mathcal{O}_{k,p} \oplus \mathcal{O}_{k,p}$ as $\mathcal{O}_{k,p}$-modules and thus $D(A) \cong D(E) \oplus D(E)$ as modules over $W \otimes_{\mathbb{Z}_p} \mathcal{O}_{k,p}$, where $\mathcal{O}_{k,p}$ acts on $D(E) \oplus D(E)$ diagonally through its action on $D(E)$. We still have to determine the possibilities for the actions of $\mathcal{O}_{B,p}$ and $\mathcal{D}$ on $D(A)$.

Proposition 3.15. Suppose $A \in \mathcal{W}(\mathbb{F}_p)$ for $p \mid d_B$, with $A \cong M \otimes_{\mathcal{O}_k} E$ for some supersingular $E$. Fix an isomorphism $\mathcal{O}_{B,p} \cong \Delta$ and a uniformizer $\Pi \in \Delta$ satisfying $\Pi^2 = p$ and $\Pi a = a \Pi$ for all $a \in \mathbb{Z}_p$, where we are viewing $\mathbb{Z}_p \to \Delta$ through the CM action $\mathcal{O}_{k,p} \to \text{End}(E) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p$. Then there is an isomorphism of rings $\text{End}_{\mathcal{O}_{B,p}}(A) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \cong R_{11}$, where

$$R_{11} = \left\{ \begin{bmatrix} x & y\Pi \\ py\Pi & x \end{bmatrix} : x, y \in \mathbb{Z}_p \right\} \subset M_2(\Delta).$$

Proof. There is the $\Delta$-action on $D(A)$

$$D(i) : \Delta \to \text{End}_{\mathcal{O}_{B,p}}(D(A)) \cong M_2(\text{End}_{\mathcal{O}_{B,p}}(D(E))) \cong M_2(\mathbb{Z}_{p^{2}}).$$

By Lemma 3.8 there are two possibilities for $D(i)$ up to $\text{GL}_2(\mathbb{Z}_p)$-conjugacy, $f_1$ and $f_2$, and we may assume $D(i)$ is equal to $f_1$ or $f_2$ in computing

$$\text{End}_{\mathcal{O}_{B,p}}(A) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \cong \text{End}_{\mathcal{O}_{B,p}}(D(A)) \cong C_{M_2(\Delta)}(\Delta).$$

If $D(i) = f_1$ then a computation shows $C_{M_2(\Delta)}(\Delta) = R_{11}$. In the case of $D(i) = f_2$ we have $C_{M_2(\Delta)}(\Delta) = R_{22}$, where

$$R_{22} = \left\{ \begin{bmatrix} x & py\Pi \\ y\Pi & x \end{bmatrix} : x, y \in \mathbb{Z}_p \right\} \cong R_{11}. \quad \Box$$
We know that for \( p \mid d_B \) there are two isomorphism classes of modules over \( W \otimes_{\mathbb{Z}_p} \mathcal{O}_p \) that are free of rank 4 over \( W \), and the proof of the previous proposition gives us explicit coordinates for each of these modules (which we will use for the \( W \otimes_{\mathbb{Z}_p} \mathcal{O}_p \)-module \( D(A) \)). To describe this, identify \( \Delta \) with a subring of \( M_2(\mathbb{Z}_p^2) \subset M_2(W) \) by

\[
a + b\Pi \mapsto \begin{bmatrix} a & pb \\ b & \pi \end{bmatrix},
\]

and use this to view \( \mathbb{Z}_p^2 \subset \Delta \) inside \( M_2(\mathbb{Z}_p^2) \). Then there is a basis \( \{e_n\} \) for the free of rank 4 \( W \)-module \( D(A) \cong D(E) \oplus D(E) \) relative to which the \( \Delta \)-action on \( D(A) \) is given by one of the two maps \( f_1, f_2 : \Delta \to \text{End}_W(D(A)) \cong M_4(W) \) of Lemma 3.8:

\[
f_1(a + b\Pi) = \begin{bmatrix} a & 0 & b & 0 \\ 0 & \pi & 0 & \pi b \\ \pi b & 0 & \pi & 0 \\ 0 & pb & 0 & a \end{bmatrix}, \quad f_2(a + b\Pi) = \begin{bmatrix} a & 0 & pb & 0 \\ 0 & \pi & 0 & \pi b \\ \pi b & 0 & \pi & 0 \\ 0 & b & 0 & a \end{bmatrix}.
\]

The action of \( \mathcal{O}_{k,p} \cong \mathbb{Z}_p^2 \) on \( D(A) \) is necessarily given in this basis by

\[
a \mapsto \text{diag}(a, \pi, a, \pi).
\]

Furthermore, using the basis \( \{e_n\} \) to view \( R_{11} \cong \text{End}_{\mathcal{O}_B \otimes \mathbb{Z}_p}(D(A)) \subset M_4(W) \), we can express any

\[
f = \begin{bmatrix} x & py\Pi \\ py\Pi & x \end{bmatrix} \in R_{11}
\]

as an element of \( M_4(W) \) by

\[
f = \begin{bmatrix} x & 0 & 0 & py \\ 0 & \pi & 0 & \pi y \\ 0 & p^2y & x & 0 \\ py & 0 & 0 & \pi \end{bmatrix}.
\]

Note that (3.3) comes from choosing a basis \( \{v_1, v_2\} \) of \( D(E) \) with \( \mathcal{F} = \mathcal{V} \) satisfying \( \mathcal{F}(v_1) = v_2 \) and \( \mathcal{F}(v_2) = pv_1 \), so we have proved the following.

**Proposition 3.16.** With notation as above, there is a \( W \)-basis \( \{e_1, e_2, e_3, e_4\} \) for \( D(A) \) relative to which the action of \( \Delta \) on \( D(A) \) is given by one of the matrices (3.4), the action of \( \mathcal{O}_{k,p} \) is given by (3.5), the action of \( \mathcal{F} = \mathcal{V} \) is determined by

\[
\mathcal{F}(e_1) = e_2, \quad \mathcal{F}(e_2) = pe_1, \quad \mathcal{F}(e_3) = e_4, \quad \mathcal{F}(e_4) = pe_3,
\]

and any \( f \in \text{End}_{\mathcal{O}_B \otimes \mathbb{Z}_p}(D(A)) \) is given by a matrix of the form (3.6).

Proposition 3.15 gives a description of \( \text{End}_{\mathcal{O}_p}(A) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \) in terms of coordinates, which is best suited for computations. The next result gives the abstract structure of this ring.

**Proposition 3.17.** There is an isomorphism of rings \( R_{11} \cong R_2 \), where

\[
R_2 = \begin{bmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^2\mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix}
\]

is the standard Eichler order of level 2 in \( M_2(\mathbb{Q}_p) \).

**Proof.** The proof is identical to a calculation carried out in [5, pp. 26-27]. \( \square \)
4. Moduli spaces

We continue with the same notation of $K_1$, $K_2$, $F$, and $K$ as in Section 1.1. Recall that we assume any prime dividing $d_B$ is inert in $K_1$ and $K_2$. In particular, each $p | d_B$ is nonsplit in $K_1$ and $K_2$, which implies $K_1$ and $K_2$ embed into $B$, or equivalently, they split $B$. If a prime number $p$ is inert in both $K_1$ and $K_2$, then $p$ is split in $F$ and each prime of $F$ lying over $p$ is inert in $K$. If $p$ is ramified in one of $K_1$ or $K_2$, then $p$ is ramified in $F$ and the unique prime of $F$ lying over $p$ is inert in $K$.

**Definition 4.1.** A CM pair over an $\mathcal{O}_K$-scheme $S$ is a pair $(\mathbf{A}_1, \mathbf{A}_2)$ where $\mathbf{A}_1$ and $\mathbf{A}_2$ are false elliptic curves over $S$ with complex multiplication by $\mathcal{O}_{K_1}$ and $\mathcal{O}_{K_2}$, respectively. An isomorphism between CM pairs $(\mathbf{A}_1', \mathbf{A}_2') \to (\mathbf{A}_1, \mathbf{A}_2)$ is a pair $(f_1, f_2)$ where each $f_j : A'_j \to A_j$ is an $\mathcal{O}_{K_j}$-linear isomorphism of false elliptic curves.

Given a CM pair $(\mathbf{A}_1, \mathbf{A}_2)$ over an $\mathcal{O}_K$-scheme $S$ and a morphism of $\mathcal{O}_K$-schemes $T \to S$, there is a CM pair $(\mathbf{A}_1, \mathbf{A}_2)|_T$ over $T$ defined as the base change to $T$. For every CM pair $(\mathbf{A}_1, \mathbf{A}_2)$ over an $\mathcal{O}_K$-scheme $S$, set

\[ L(\mathbf{A}_1, \mathbf{A}_2) = \text{Hom}_\mathcal{O}_B(A_1, A_2), \quad V(\mathbf{A}_1, \mathbf{A}_2) = L(\mathbf{A}_1, \mathbf{A}_2) \otimes \mathbb{Q}. \]

If $S$ is connected we have the quadratic form $\deg^*$ on $L(\mathbf{A}_1, \mathbf{A}_2)$. Let $[f, g] = f^t \circ g + g^t \circ f$ be the associated bilinear form. Then $\mathcal{O}_K = \mathcal{O}_{K_1} \otimes \mathcal{O}_{K_2}$ acts on the $\mathbb{Z}$-module $L(\mathbf{A}_1, \mathbf{A}_2)$ by

\[(x_1 \otimes x_2) \cdot f = \kappa_2(x_2) \circ f \circ \kappa_1(\mathfrak{m}).\]

**Proposition 4.2.** Let $(\mathbf{A}_1, \mathbf{A}_2)$ be a CM pair. There is a unique $F$-bilinear form $[\cdot, \cdot]_{\text{CM}}$ on $V(\mathbf{A}_1, \mathbf{A}_2)$ satisfying $[f, g] = \text{Tr}_{F/\mathbb{Q}}[f, g]_{\text{CM}}$. Under this pairing,

\[ [L(\mathbf{A}_1, \mathbf{A}_2), L(\mathbf{A}_1, \mathbf{A}_2)]_{\text{CM}} \subset \mathcal{D}^{-1}. \]

The quadratic form $\deg_{\text{CM}}(f) = \frac{1}{2}[f, f]_{\text{CM}}$ is the unique $F$-quadratic form on $V(\mathbf{A}_1, \mathbf{A}_2)$ satisfying $\deg^*(f) = \text{Tr}_{F/\mathbb{Q}} \deg_{\text{CM}}(f)$.

**Proof.** This is the same as the proof of [10, Proposition 2.2]. \[\square\]

**Definition 4.3.** For $j \in \{1, 2\}$ define $\mathcal{Y}_j^B$ to be the stack $\mathcal{Y}^B$ with $k = K_j$ and $L = K$. For any ring homomorphism $\theta_j : \mathcal{O}_{K_j} \to \mathcal{O}_B/\mathfrak{m}_B$, define $\mathcal{Y}_j^B(\theta_j)$ to be the stack $\mathcal{Y}^B(\theta_j)$ with $k = K_j$ and $L = K$.

From now on, we write $\mathcal{Y}^B$ to mean the category defined in Definition 3.2 for some fixed imaginary quadratic field $k$ and finite extension $L$.

**Definition 4.4.** Let $\theta : \mathcal{O}_K \to \mathcal{O}_B/\mathfrak{m}_B$ be a ring homomorphism. Define $\mathcal{Y}_\theta^B$ to be the category whose objects are CM pairs $(\mathbf{A}_1, \mathbf{A}_2)$ over $\mathcal{O}_K$-schemes such that $\mathbf{A}_j$ is an object of $\mathcal{Y}_j^B(\theta_j)$ for $j = 1, 2$, where $\theta_j = \theta|\mathcal{O}_{K_j}$. A morphism $(\mathbf{A}_1', \mathbf{A}_2') \to (\mathbf{A}_1, \mathbf{A}_2)$ between two such pairs defined over $\mathcal{O}_K$-schemes $T$ and $S$, respectively, is a morphism of $\mathcal{O}_K$-schemes $T \to S$ together with an isomorphism of CM pairs $(\mathbf{A}_1', \mathbf{A}_2') \cong (\mathbf{A}_1, \mathbf{A}_2)|_T$ over $T$.

**Definition 4.5.** Let $\theta : \mathcal{O}_K \to \mathcal{O}_B/\mathfrak{m}_B$ be a ring homomorphism. For any $\alpha \in F^\times$ define $\mathcal{Y}_{\theta, \alpha}^B$ to be the category whose objects are triples $(\mathbf{A}_1, \mathbf{A}_2, f)$ where $(\mathbf{A}_1, \mathbf{A}_2) \in \mathcal{Y}_\theta^B(S)$ for some $\mathcal{O}_K$-scheme $S$ and $f \in L(\mathbf{A}_1, \mathbf{A}_2)$ satisfies $\deg_{\text{CM}}(f) = \alpha$ on every connected component of $S$. A morphism

\[(\mathbf{A}_1', \mathbf{A}_2', f') \to (\mathbf{A}_1, \mathbf{A}_2, f)\]
between two such triples, with \((A_1', A_2')\) and \((A_1, A_2)\) CM pairs over \(O_K\)-schemes \(T\) and \(S\), respectively, is a morphism of \(O_K\)-schemes \(T \rightarrow S\) together with an isomorphism
\[
(A_1', A_2') \rightarrow (A_1, A_2)/T
\]
of CM pairs over \(T\) compatible with \(f\) and \(f'\).

The categories \(\mathcal{X}_a^B\) and \(\mathcal{X}_{a,\alpha}^B\) are stacks of finite type over \(\text{Spec}(O_K)\). For each positive integer \(m\) define \(\mathcal{X}_m^B\) to be the stack over \(\text{Spec}(O_K)\) with \(\mathcal{X}_m^B(S)\) the category of triples \((A_1, A_2, f)\) where \(A_j \in \mathcal{Y}_j^B(S)\) and \(f \in L(A_1, A_2)\) satisfies \(\text{deg}^\ast(f) = m\) on every connected component of \(S\). It follows from Theorem 3.12 that there is a decomposition
\[
\mathcal{X}_m^B = \bigsqcup_{\alpha \in F^\times} \bigsqcup_{\theta: O_K \rightarrow O_B/mB} \mathcal{X}_{\theta,\alpha}^B,
\]
(4.1)

A false elliptic curve \((A, i)\) over \(\overline{F}_p\) is supersingular if the underlying abelian variety \(A\) is supersingular. A CM pair \((A_1, A_2)\) over \(\overline{F}_p\) is supersingular if the underlying abelian varieties \(A_1\) and \(A_2\) are supersingular. If \(p\) is a prime dividing \(d_B\), or more generally, a prime nonsplit in \(K_j\), then any \(A \in \mathcal{Y}_j^B(\overline{F}_p)\) is necessarily supersingular.

**Proposition 4.6.** Let \(k\) be an algebraically closed field of characteristic \(p \geq 0\) and let \(\theta: O_K \rightarrow O_B/mB\) be a ring homomorphism. Let \(\alpha \in F^\times\) and suppose \((A_1, A_2, f) \in \mathcal{X}_{\theta,\alpha}(k)\).

(a) We have \(p > 0\) and \((A_1, A_2)\) is a supersingular CM pair.

(b) There is an isomorphism of \(F\)-quadratic spaces
\[
(V(A_1, A_2), \text{deg}_{CM}) \cong (K, \beta \cdot N_K/F)
\]
for some totally positive \(\beta \in F^\times\), determined up to multiplication by a norm from \(K^\times\).

(c) There is an isomorphism of \(Q\)-quadratic spaces
\[
(V(A_1, A_2), \text{deg}^\ast) \cong (B^{(p)}, Nrd),
\]
where \(Nrd\) is the reduced norm on \(B^{(p)}\).

(d) If \(p\) does not divide \(d_B\) then it is nonsplit in \(K_1\) and \(K_2\).

**Proof.** The proof is very similar to that of [10, Proposition 2.6]. \(\square\)

For any \(O_K\)-scheme \(S\) and any ring homomorphism \(\theta: O_K \rightarrow O_B/m_B\), the group \(\Gamma = \text{Cl}(O_{K_1}) \times \text{Cl}(O_{K_2})\) acts on the set \(\mathcal{X}_\theta^B(S)\) by
\[
(a_1, a_2) \cdot (A_1, A_2) = (a_1 \otimes_{O_{K_1}} A_1, a_2 \otimes_{O_{K_2}} A_2).
\]

The only thing to note is that the diagram (3.1) commutes for the CM false elliptic curve \(a_j \otimes_{O_{K_j}} A_j\) since it commutes for \(A_j\) and there is an isomorphism of \(O_{K_j}\)-module schemes over \(S\)
\[
(a_j \otimes_{O_{K_j}} A_j)[m_B] \cong a_j \otimes_{O_{K_j}} A_j[m_B].
\]

**Lemma 4.7.** Let \(S\) be an \(O_K\)-scheme and for \(j \in \{1, 2\}\) set \(w_j = |O_{K_j}^\times|\). Every \(x \in \mathcal{X}_\theta^B(S)\), viewed as an element of the set \([\mathcal{X}_\theta^B(S)]\), has trivial stabilizer in \(\Gamma\) and satisfies \(|\text{Aut}_{\mathcal{X}_\theta^B}(x)| = w_1 w_2\).

**Proof.** Set \(O_j = O_B \otimes_{\mathbb{Z}} O_{K_j}\). By [14, Corollary 6.2] and our classification of endomorphism rings of false elliptic curves over algebraically closed fields, \(\text{End}_{O_j}(A_j) \cong O_{K_j}\) as an \(O_{K_j}\)-algebra. The first claim then follows as in the proof of [10, Lemma 2.16]. Next, by definition, an automorphism of \(x\) in \(\mathcal{X}_\theta^B(S)\) is a pair \((a_1, a_2)\) with \(a_j \in \text{Aut}_{O_j}(A_j) \cong O_{K_j}^\times\). \(\square\)
5. Local quadratic spaces

This section and the next form the technical core of this paper. In this section we (essentially) count the number of geometric points of \( \mathcal{X}_{\theta, o} \). This comes from a careful examination of the quadratic spaces \((V_t(A_1, A_2), \degCM)\) for each prime \( \ell \), where

\[
L_t(A_1, A_2) = L(A_1, A_2) \otimes \mathbb{Z}_\ell, \quad V_t(A_1, A_2) = V(A_1, A_2) \otimes \mathbb{Q}_\ell.
\]

The methods of the proofs follow [10] quite closely. Suppose \( \ell \) is a prime dividing \( d_B \), let \( k \) be an algebraically closed field, and let \( A \in \mathcal{Y}^B(k) \). Define the \( m_\ell \)-torsion of \( A \) as

\[
A[m_\ell] = \ker(i(x)) : A[\ell] \to A[\ell],
\]

where \( x \) is any element of \( m_\ell \) whose image generates the principal ideal \( m_\ell / O_B \subset O_B / \ell O_B \). This is a flat commutative group scheme over \( \text{Spec}(k) \) of order \( \ell^2 \).

**Lemma 5.1.** Suppose \( A \in \mathcal{Y}^B(k) \) for \( k = \mathbb{C} \) or \( k = \overline{\mathbb{F}}_p \) and \( \ell \neq p \) is a prime dividing \( d_B \). There is an isomorphism of \( O_B / m_\ell \)-algebras \( \text{End}_{O_B / m_\ell}(A[m_\ell]) \cong O_B / m_\ell \).

**Proof.** Since \( \ell \neq p \), the group scheme \( A[\ell] \) is étale over \( k \), so \( A[m_\ell] \) is étale over \( k \) and thus constant. It follows that the natural map

\[
\text{End}_{O_B / m_\ell}(A[m_\ell]) \to \text{End}_{O_B / m_\ell}(A[m_\ell](k))
\]

is an isomorphism. The group \( A[m_\ell](k) \) is a vector space of dimension 1 over \( O_B / m_\ell \), which proves the result. \( \square \)

5.1. The case of \( \ell \neq p \). Fix a prime ideal \( \mathfrak{P} \subset O_K \) of residue characteristic \( p \), where \( p \) is nonsplit in \( K_1 \) and \( K_2 \), a ring homomorphism \( \theta : O_K \to O_B / m_\ell \), and a CM pair \((A_1, A_2) \in \mathcal{Y}^B(\overline{\mathbb{F}}_p)\) (necessarily supersingular).

**Proposition 5.2.** Let \( \ell \neq p \) be a prime. There is a \( K_\ell \)-linear isomorphism of \( F_\ell \)-quadratic spaces

\[
(V_t(A_1, A_2), \degCM) \cong (K_\ell, \beta_\ell \cdot N_{K_\ell/F_\ell})
\]

for some \( \beta_\ell \in F_\ell^* \) satisfying \( \beta_\ell O_{F, \ell} = \mathcal{D}_\ell^{-1} = \mathcal{D}_\ell^{-1} O_{F, \ell} \) if \( \ell \mid d_B \) and \( \beta_\ell O_{F, \ell} = \mathcal{D}_\ell^{-1} \) if \( \ell \nmid d_B \), where \( \ell \) is the prime over \( \ell \) dividing \( \text{ker}(\theta) \cap O_F \). This map takes \( L_t(A_1, A_2) \) isomorphically to \( O_{K, \ell} \).

**Proof.** We will write \( L_t \) and \( V_t \) for \( L_t(A_1, A_2) \) and \( V_t(A_1, A_2) \). The existence of an isomorphism of quadratic spaces for some \( \beta_\ell \in F_\ell^* \) follows from Proposition 4.6(b). Under this isomorphism, \( L_t \) is sent to a finitely generated \( O_{K, \ell} \)-submodule of \( K_\ell \), that is, a fractional \( O_{K, \ell} \)-ideal. Then since every ideal of \( O_{K, \ell} \) is principal, there is an isomorphism \( V_t \cong K_\ell \) inducing an isomorphism \( L_t \cong O_{K, \ell} \). The \( O_{F, \ell} \)-bilinear form

\[
\{\cdot, \cdot\}_{CM} : L_\ell \times L_\ell \to \mathcal{D}_\ell^{-1}
\]

induces an \( O_{F, \ell} \)-bilinear form \( O_{K, \ell} \times O_{K, \ell} \to \mathcal{D}_\ell^{-1} \) given by \( (x, y) \mapsto \beta_\ell \text{Tr}_{K_\ell/F_\ell}(x \overline{y}) \).

The dual lattice of \( O_{K, \ell} \cong L_\ell \) with respect to this pairing is \( L_\ell^* \cong O_{K, \ell} \beta_\ell^{-1} \mathcal{D}_\ell^{-1} O_{K, \ell} \).

First suppose \( \ell \nmid d_B \). There are isomorphisms of \( \mathbb{Z}_\ell \)-modules

\[
L_\ell \cong \text{Hom}_{O_B}(T_\ell(A_1), T_\ell(A_2)) \cong M_2(\mathbb{Z}_\ell)
\]

Under this isomorphism the quadratic form \( \deg^* \) on \( L_\ell \) is identified with the quadratic form \( u \cdot \text{det} \) on \( M_2(\mathbb{Z}_\ell) \) for some \( u \in \mathbb{Z}_\ell^* \). The lattice \( M_2(\mathbb{Z}_\ell) \subset M_2(\mathbb{Q}_\ell) \) is self dual relative to \( \text{det} \), so from the isomorphism

\[
L_\ell^* / L_\ell \cong \beta_\ell^{-1} \mathcal{D}^{-1} O_{K, \ell} / O_{K, \ell},
\]
Lemma 5.3. If results. Proof. The case of 5.2. Recall that  \( \phi \) on \( \Lambda \) must be of the form  \( \Lambda = \Lambda' \) : \( \Lambda \) is equal to the composition  \( \Lambda \) with the quadratic form \( Nrd \) on \( \mathcal{O} \). Therefore we may reduce to the case where the CM false elliptic curves \( A_1 \) and \( A_2 \) have the same underlying false elliptic curve \( A \). There are isomorphisms of \( \mathbb{Z}_\ell \)-algebras \( L_\ell \cong \text{End}_{\mathcal{O}_A}(T_\ell(A)) \cong \mathcal{O}_{B,\ell} \), and this isomorphism identifies the quadratic form deg\(^*\) on \( L_\ell \) with the quadratic form \( Nrd \) on \( \mathcal{O}_{B,\ell} \). The rest of the proof is very similar to that of [10, Lemma 2.11], replacing \( \text{Lie}(E) \) and \( \Delta \) there with \( A[\mathfrak{m}_\ell] \) and \( \mathcal{O}_{B,\ell} \), and using the fact that if  
\[
\kappa_{ij}^{m_i} : \mathcal{O}_{K_j} \rightarrow \text{End}_{\mathcal{O}_B/m_B}(A[m_i]) \cong \mathcal{O}_B/m_\ell
\]
is the action on the \( m_\ell \)-torsion, then the map \( \mathcal{O}_K \rightarrow \mathbb{F}_{\ell^2} \) defined by \( t_1 \otimes t_2 \mapsto \kappa_{ij}^{m_i}(t_1)\kappa_{ij}^{m_i}(t_2) \) is equal to the composition  
\[
\mathcal{O}_K \xrightarrow{\phi} \mathcal{O}_B/m_B \rightarrow \mathcal{O}_B/m_\ell,
\]
by definition of \( (A_1, A_2) \) being in \( \mathcal{B}_\ell^B(\mathbb{F}_\ell) \). \( \square \)

5.2. The case of \( \ell = p \). In order to prove a similar result for \( \ell = p \) we need a few preliminary results.

Lemma 5.3. If \( A \in \mathcal{B}^B(\mathbb{F}_p) \) with \( p \mid d_B \), then \( \text{End}_{\mathcal{O}_B}(\text{Lie}(A)) \cong \mathbb{F}_p \) as \( \mathbb{F}_p \)-algebras.

Proof. This is an easy computation in coordinates using Proposition 3.16 and the isomorphisms \( \text{Lie}(A) \cong \text{Lie}(D(A)) \cong D(\mathcal{O}_B) \). \( \square \)

Proposition 5.4. Suppose \( (A, i) \in \mathcal{B}^B(\mathbb{F}_p) \) with \( p \mid d_B \). Under the isomorphism  
\[
\text{End}_{\mathcal{O}_B}(A) \otimes \mathbb{Z}_p \rightarrow R_{11}
\]
in Proposition 3.15, the \( \mathbb{Z}_p \)-quadratic form deg\(^*\) on \( \text{End}_{\mathcal{O}_B}(A) \otimes \mathbb{Z}_p \) is identified with the \( \mathbb{Z}_p \)-quadratic form \( Q \) on \( R_{11} \) given by  
\[
Q \left[ \begin{array}{cc} x & y \\ p_y y & x \end{array} \right] = x^2 - p^2 y^2.
\]

Proof. Recall that \( f^\ell = \lambda^{-1} \circ f^\ell \circ \lambda \), where \( \lambda : A \rightarrow A' \) is the unique principal polarization satisfying \( \lambda^{-1} \circ i(x)^\ell \circ \lambda = i(x) \ell \) for all \( x \in \mathcal{O}_B \). The polarization \( \lambda \) then induces a map \( \Lambda = D(\lambda) : D(A) \rightarrow D(A^\ell) \cong D(A)^\ell \), which determines a nondegenerate, alternating, bilinear pairing \( (\cdot, \cdot) : D(A) \times D(A) \rightarrow W \) satisfying \( (\mathcal{F} x, y) = \sigma((x, y)) \) for all \( x, y \in D(A) \).

Let \( \{e_n\} \) be a \( W \)-basis for \( D(A) \) as in Proposition 3.16. First suppose \( D(i) = f_1 \), in the notation of (3.4). A computation shows \( \Lambda \) must be of the form  
\[
\Lambda = \begin{bmatrix} 0 & 0 & 0 & b \\ 0 & 0 & b & 0 \\ -b & 0 & 0 & 0 \end{bmatrix}
\]
for some \( b \in \mathbb{Z}_p^\times \).

The involution \( \varphi \mapsto \varphi^T \) on \( \text{End}_W(D(A)) \cong M_4(W) \) corresponding to the Rosati involution \( f \mapsto \lambda^{-1} \circ f^\ell \circ \lambda \) on \( \text{End}^B(A) \) (which restricts to \( f \mapsto f^\ell \) on \( \text{End}_{\mathcal{O}_B}(A) \otimes \mathbb{Z}_p \)) is given by \( \varphi^T = \Lambda^{-1} \varphi^T \Lambda \), where \( \varphi^T \) is the transpose of the matrix \( \varphi \). If  
\[
\varphi = \begin{bmatrix} x & y \\ p_y y & x \end{bmatrix} \in R_{11},
\]
then viewing it as an element of $M_4(W)$ as in (3.6), applying the involution $\dagger$, and then viewing it again in $R_{11}$, gives

$$\varphi_{\rho^1} = \begin{bmatrix} x\bar{x} - p^2 y\bar{y} & 0 \\ 0 & x\bar{x} - p^2 y\bar{y} \end{bmatrix},$$

so we obtain $Q(\varphi) = x\bar{x} - p^2 y\bar{y}$. A similar computation gives the same result if $D(i) = f_2$. □

For $j = 1, 2$ let $\theta_j : O_{K_j} \to O_B/\mathfrak{m}_B$ be a ring homomorphism and let $A_j \in \mathcal{B}_j^B(\theta_j)(\overline{F}_p)$ for $p \mid d_B$. There is a unique ring isomorphism $O_{K_1,p} \to O_{K_2,p}$ making the diagram

\[
\begin{array}{ccc}
O_{K_1,p} & \to & O_{K_2,p} \\
\theta_1 & \downarrow & \theta_2 \\
O_B/\mathfrak{m}_B & \to & O_B/\mathfrak{m}_B
\end{array}
\]

commute. We use this to identify the rings $O_{K_1,p}$ and $O_{K_2,p}$, and call this ring $O_K$.

**Definition 5.5.** With notation as above, if $D(A_1)$ and $D(A_2)$ are isomorphic as $\Delta \otimes \mathbb{Z}_p O_K$-modules, we say that $A_1$ and $A_2$ are of the same type.

Note that there are two isomorphism classes of $\Delta \otimes \mathbb{Z}_p O_K$-modules free of rank 4 over $\mathbb{Z}_p$, and $A_1$ and $A_2$ being of the same type just means $D(A_1)$ and $D(A_2)$ lie in the same isomorphism class, and not being of the same type means they lie in the two separate classes. This definition is a bit misleading because we will see below that $A_1$ and $A_2$ are of the same type if and only if $\mathfrak{p}$ divides $\ker(\theta)$, where $\theta : O_K \to O_B/\mathfrak{m}_B$ is the map induced by $\theta_1$ and $\theta_2$, so this “type” is really a property between $\mathfrak{p}$ and $\theta$, independent of $A_1$ and $A_2$.

However, the above definition is the easier one to start with in proving the next few results.

**Proposition 5.6.** Suppose $(A_j, i_j) \in \mathcal{B}_j^B(\theta_j)(\overline{F}_p)$ for $j = 1, 2$, where $p \mid d_B$, and $A_1$ and $A_2$ are not of the same type. There are isomorphisms of $\mathbb{Z}_p$-modules

$$\text{Hom}_{O_B \otimes \mathfrak{p}}(D(A_1), D(A_2)) \cong \text{Hom}_{O_B \otimes \mathfrak{p}}(D(A_2), D(A_1)) \cong R_{12},$$

where

$$R_{12} = \left\{ \begin{bmatrix} px & y\Pi \\ y\Pi & x \end{bmatrix} : x, y \in \mathbb{Z}_p^2 \right\} \subset M_2(\Delta)$$

and we have fixed an embedding $\mathbb{Z}_p^2 \hookrightarrow \Delta$ so that $\Delta = \mathbb{Z}_p^2 \oplus \mathbb{Z}_p^2 \Pi$. Under the isomorphism

$$\text{Hom}_{O_B}(A_1, A_2) \otimes \mathbb{Z} \mathbb{Z}_p \xrightarrow{D} \text{Hom}_{O_B}(A_1, A_2) \cong R_{12},$$

the $\mathbb{Z}_p$-quadratic form $\deg^*$ on $\text{Hom}_{O_B}(A_1, A_2) \otimes \mathbb{Z} \mathbb{Z}_p$ is identified with the $\mathbb{Z}_p$-quadratic form $u \cdot Q'$ on $R_{12}$, where $u \in \mathbb{Z}_p^\times$ and

$$Q' \begin{bmatrix} px \\ y\Pi \\ x \end{bmatrix} = p(x\bar{x} - y\bar{y}).$$

Under the isomorphism

$$\text{Hom}_{O_B}(A_2, A_1) \otimes \mathbb{Z} \mathbb{Z}_p \xrightarrow{D} \text{Hom}_{O_B}(D(A_2), D(A_1)) \cong R_{12},$$

the quadratic form $\deg^*$ is identified with the quadratic form $u^{-1} \cdot Q'$. 

Proof. The first claim follows from a computation in coordinates. Now let \( \lambda_j : A_i \to A_j^\vee \) be the unique principal polarization satisfying \( i_j(x) = \lambda_j^{-1} \circ i(x)^\vee \circ \lambda_j \) for all \( x \in O_B \). In the proof of Proposition 5.4 we showed

\[
\Lambda_j = D(\lambda_j) = \begin{bmatrix} 0 & 0 & 0 & \bar{b}_j \\ 0 & 0 & \bar{b}_j & 0 \\ 0 & -\bar{b}_j & 0 & 0 \\ -\bar{b}_j & 0 & 0 & 0 \end{bmatrix} \in \text{M}_4(W)
\]

for some \( b_j \in \mathbb{Z}_{p}^\times \) satisfying \( b_1^{-1}b_2 \in \mathbb{Z}_{p}^\times \). We have \( D(f^\dagger) = \Lambda^{-1}D(f)^\vee \Lambda \), where \( D(f)^\vee \in \text{Hom}_{O_B \otimes \mathbb{Z}\mathfrak{q}}(D(A_2)^\vee, D(A_2)^\vee) \) is the dual linear map. Therefore, through the map \( D \), the assignment \( f \mapsto f^\dagger \) corresponds to the assignment \( \varphi \mapsto \varphi^\dagger = \Lambda^{-1}\varphi^T \Lambda \). If

\[
\varphi = \begin{bmatrix} px & y\Pi \\ y\Pi & x \end{bmatrix} \in R_{12}
\]

then

\[
\varphi^\dagger \varphi = \begin{bmatrix} p(x\Xi - y\bar{\Pi})u & 0 \\ 0 & p(x\Xi - y\bar{\Pi})u \end{bmatrix},
\]

where \( u = b_1^{-1}b_2 \).

Recall that \( (A_1, A_2) \in \mathcal{A}_B^\theta(D, \mathbb{Z}_p) \) and for \( p \mid d_B \) we are using \( \theta \) to identify \( O_{K_{1,p}} \) and \( O_{K_{2,p}} \) as in (5.1).

**Proposition 5.7.** There is a \( K_p \)-linear isomorphism of \( F_p \)-quadratic spaces

\[
(V_p(A_1, A_2), \text{deg}_{CM}) \cong (K_p, \beta_p \cdot N_{K_p/F_p})
\]

for some \( \beta_p \in F_p^\times \) satisfying

\[
\beta_p O_{F,p} = \begin{cases} 
p\mathbb{O}_p^{-1} & \text{if } p \nmid d_B \\
p^2\mathbb{O}_p^{-1} & \text{if } p \mid d_B \text{ and } A_1, A_2 \text{ are of the same type} \\
p\mathfrak{T}\mathbb{O}_p^{-1} & \text{if } p \mid d_B \text{ and } A_1, A_2 \text{ are not of the same type},
\end{cases}
\]

where \( \mathbb{O}_p = \mathcal{O}_{F,p}, \mathfrak{p} = \mathcal{P} \cap O_F, \) and \( \mathfrak{p} \) is the other prime ideal of \( O_F \) lying over \( p \). This map takes \( L_p(A_1, A_2) \) isomorphically to \( O_{K,p} \).

**Proof.** First suppose \( p \nmid d_B \). We will write \( L_p \) for \( L_p(A_1, A_2) \). The proof of the existence of the isomorphism taking \( L_p \) to \( O_{K,p} \) is the same as for \( \ell \neq p \). We may reduce to the case where the CM false elliptic curves \( A_1 \) and \( A_2 \) have the same underlying false elliptic curve \( A \) because the idempotents \( \varepsilon, \bar{\varepsilon} \in M_2(W) \cong O_B \otimes \mathbb{Z} W \) provide a splitting \( D(A_j) \cong \varepsilon D(A_j) \oplus \bar{\varepsilon} D(A_j) \), which means \( D(A_1) \cong D(A_2) \) as \( O_B \otimes \mathbb{Z} \mathcal{P} \)-modules and thus

\[
L_p \cong \text{End}_{O_B \otimes \mathbb{Z}\mathcal{P}}(D(A)) \cong \Delta,
\]

where \( \Delta \) is the maximal order in the quaternion division algebra over \( \mathbb{Q}_p \). The rest of the proof is the same as [10, Lemma 2.11].

Next suppose \( p \mid d_B \), and first assume \( A_1 \) and \( A_2 \) are of the same type. As mentioned above we identify \( O_{K_{1,p}} \) and \( O_{K_{2,p}} \), and call this ring \( O_K \). In this case we may assume \( A_1 \) and \( A_2 \) have the same underlying false elliptic curve \( A \cong M \otimes_{O_K} E \) and \( \kappa_1 = \kappa_2 = \kappa \). If we fix the embedding \( O_K \cong \mathbb{Z}_{p^2} \hookrightarrow \Delta \cong \text{End}_{\mathcal{P}}(D(E)) \), then there is an isomorphism \( L_p = \text{End}_{O_B}(A) \otimes \mathbb{Z}_{p^2} \cong R_{11} \) with \( \kappa : O_K \to R_{11} \) given by \( \kappa(x) = \text{diag}(x, x) \), and
the quadratic form \( \text{deg}^* \) on \( L_p \) is identified with the quadratic form \( Q \) on \( R_{11} \) defined in Proposition 5.4. The dual lattice of \( R_{11} \) relative to \( Q \) is

\[
R_{11}' = \left\{ \begin{bmatrix} x & y \\ p^{-1}y & x \end{bmatrix} : x, y \in \mathbb{Z}_{p^2} \right\},
\]

so \([R_{11}': R_{11}] = p^4\). As before, we obtain \([\mathcal{O}_{K,p} : \beta_p \mathcal{O}_{K,p}] = p^4\).

Under the isomorphism \( L_p \cong R_{11} \) there is an action \( R_{11} \to \text{End}_\Delta(\text{Lie}(A)) \cong \mathbb{F}_p \), and any element of

\[
\mathfrak{M} = \left\{ \begin{bmatrix} px & y\Pi \\ py\Pi & px \end{bmatrix} : x, y \in \mathbb{Z}_{p^2} \right\} \subset R_{11},
\]

a maximal ideal of \( R_{11} \), acts trivially on \( D(A)/\mathfrak{Y}D(A) \cong \text{Lie}(A) \), so \( \mathfrak{M} = \ker(R_{11} \to \mathbb{F}_p) \). Hence, \( R_{11} \to \text{End}_\Delta(\text{Lie}(A)) \) determines an isomorphism \( \gamma : R_{11}/\mathfrak{M} \to \mathbb{F}_p \), which allows us to identify \( \kappa_{\text{Lie}} : \mathcal{O}_K \to \text{End}_\Delta(\text{Lie}(A)) \) with the composition

\[
\mathcal{O}_K \xrightarrow{\sim} R_{11} \to R_{11}/\mathfrak{M} \xrightarrow{\sim} \mathbb{F}_p.
\]

However, the map \( \mathcal{O}_K \to \mathbb{F}_p \) defined by \( t_1 \otimes t_2 \to \kappa_{\text{Lie}}(t_1)\kappa_{\text{Lie}}(t_2) \) is the structure map \( \mathcal{O}_K \to \mathbb{F}_p \hookrightarrow \mathbb{F}_p \) by the CM normalization condition, so its kernel is \( \mathfrak{P} \). It follows from the factorization of \( \kappa_{\text{Lie}} \) above that an element of \( \mathcal{O}_{K,p} \) acts trivially on \( R_{11}'/R_{11} \) if and only if it is in \( \mathfrak{P}^2 \). Hence there is an \( \mathcal{O}_{K,p} \)-linear map \( \mathcal{O}_{K,p}/\mathfrak{P}^2 \mathcal{O}_{K,p} \to R_{11}'/R_{11} \) given by \( x \mapsto x \bullet 1 \).

But \( \mathfrak{P}^2 \) has norm \( p^4 \). For \( R_{11}' : R_{11} \), so there are isomorphisms of \( \mathcal{O}_{K,p} \)-modules

\[
\mathcal{O}_{K,p}/\mathfrak{P}^2 \mathcal{O}_{K,p} \cong R_{11}'/R_{11} \cong \beta_p^{-1} \mathcal{D}^{-1} \mathcal{O}_{K,p}/\mathcal{O}_{K,p}.
\]

It follows that \( \beta_p \mathcal{D} \mathcal{O}_{K,p} = \mathfrak{P}^2 \mathcal{O}_{K,p} \) and thus \( \beta_p \mathcal{O}_{F,p} = \mathfrak{P}^2 \mathcal{D}_p^{-1} \).

Next assume \( A_1 \) and \( A_2 \) are not of the same type, with \( A_j \cong M_j \otimes_{\mathcal{O}_K} E_j \). As before we identify \( \mathcal{O}_{K_1,p} \) with \( \mathcal{O}_{K_2,p} \) and call this ring \( \mathcal{O}_K \). Let \( \mathfrak{g} \) be the connected \( p \)-divisible group of height 2 and dimension 1 over \( \mathbb{F}_p \). Isomorphisms \( E_j[p^\infty] \cong \mathfrak{g} \) may be chosen in such a way that the CM actions \( g_1 : \mathcal{O}_K \to \text{End}(E_1[p^\infty]) \cong \Delta \) and \( g_2 : \mathcal{O}_K \to \text{End}(E_2[p^\infty]) \cong \Delta \) have the same image in \( \Delta \). Fix an embedding \( \mathbb{Z}_p \hookrightarrow \Delta \) and a uniformizer \( \Pi \in \Delta \) satisfying \( \Pi g_1(x) = g_1(x) \Pi \) for all \( x \in \mathcal{O}_K \). By Proposition 5.6 there are isomorphisms of \( \mathbb{Z}_p \)-modules

\[
L_\nu \cong \text{Hom}_{\mathcal{O}_p \otimes_{\mathcal{O}_K} D(A_1), D(A_2)} \cong R_{12},
\]

and the quadratic form \( \text{deg}^* \) on \( L_p \) is identified with the quadratic form \( uQ' \) on \( R_{12} \) defined in Proposition 5.6. The dual lattice of \( R_{12} \) relative to \( uQ' \) is

\[
R_{12}' = \kappa^{-1} \left\{ \begin{bmatrix} x & y \Pi \\ p^{-1}y \Pi & x \end{bmatrix} : x, y \in \mathbb{Z}_{p^2} \right\},
\]

so \([R_{12}' : R_{12}] = p^4\). As before this gives \([\mathcal{O}_{K,p} : \beta_p \mathcal{D} \mathcal{O}_{K,p}] = p^4\). Fixing ring isomorphisms

\[
\text{End}_{\mathcal{O}_p}(A_1) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong R_{11} \cong \text{End}_{\mathcal{O}_p}(A_2) \otimes_{\mathbb{Z}} \mathbb{Z}_p,
\]

it makes sense to take the product \( \kappa_2(t_2)\kappa_1(t_1) \) in \( R_{11} \) for \( t_1, t_2 \in \mathcal{O}_K \). As in the case of \( A_1 \) and \( A_2 \) having the same type, we have \( t_1 \otimes t_2 \in \mathfrak{P} \) if and only if \( \kappa_2(t_2)\kappa_1(t_1) \in \mathfrak{M} \).

Let \( \mathfrak{P} \) be the other prime ideal of \( \mathcal{O}_K \) lying over \( p \). For \( t_1 \otimes t_2 \in \mathcal{O}_K \),

\[
(t_1 \otimes t_2) \bullet \varphi \in R_{12} \text{ for all } \varphi \in R_{12}' \iff g_2(t_2)g_1(t_1) \in p\mathbb{Z}_p \text{ and } g_2(t_2)g_1(t_1) \in p\mathbb{Z}_p \iff \kappa_2(t_2)\kappa_1(t_1) \in \mathfrak{M} \text{ and } \kappa_2(t_2)\kappa_1(t_1) \in \mathfrak{M} \iff t_1 \otimes t_2 \in \mathfrak{P} \cap \mathfrak{M} = \mathfrak{P}\mathfrak{M}.
\]
This shows an element of $\mathcal{O}_{K,p}$ acts trivially on $R_{12}'/R_{12}$ if and only if it is in $\mathfrak{P}\mathfrak{F}$. Since $[R_{12}':R_{12}] = p^2$ is the norm of $\mathfrak{P}\mathfrak{F}$, similar to above we obtain $\beta_p\mathcal{O}_{F,p} = \mathfrak{P}\mathfrak{D}^{-1}_p$. □

If $A \in \mathcal{Y}^B(\mathbb{F}_p)$ for $p \mid d_B$, the $m_p$-torsion $A[m_p]$ is defined just as $A[m]$.

**Lemma 5.8.** Suppose $A \in \mathcal{Y}^B(\mathbb{F}_p)$ with $p \mid d_B$. There is an isomorphism of $\mathcal{O}_B/m_p$-algebras $\text{End}_{\mathcal{O}_B/m_p}(A[m_p]) \cong \mathcal{O}_B/m_p$.

**Proof.** This is a computation using Dieudonné modules and Proposition 3.16. □

**Corollary 5.9.** Suppose $A \in \mathcal{Y}^B(k)$ for $k = \mathbb{C}$ or $k = \mathbb{F}_p$. There is an isomorphism of $\mathcal{O}_B/m_B$-algebras $\text{End}_{\mathcal{O}_B/m_B}(A[m_B]) \cong \mathcal{O}_B/m_B$.

**Proof.** Combine Lemmas 5.1 and 5.8. □

**Proposition 5.10.** Let $(A_1, A_2) \in \mathcal{Y}^B(\mathcal{P}\mathfrak{F})$ with $\mathcal{P}$ lying over $p \mid d_B$. Then $\mathcal{P}$ divides $\ker(\theta)$ if and only if $A_1$ and $A_2$ are of the same type.

**Proof.** Suppose $A_1$ and $A_2$ are of the same type. Following the proof of Proposition 5.7 starting around (5.2), replacing $\text{Lie}(A)$ with $A[m_p]$ and using Lemma 5.8, we find that an element of $\mathcal{O}_{K,p}$ acts trivially on $L'/L_p$ if and only if it is in $\mathfrak{Q}^2$, where $\mathfrak{Q} \subset \mathcal{O}_K$ is the prime over $p$ dividing $\ker(\theta)$. However, the same is true for $\mathcal{P}$ in place of $\mathfrak{Q}$, so $\mathfrak{Q} = \mathfrak{Q}$.

Now suppose $A_1$ and $A_2$ are not of the same type. Define a ring homomorphism $\eta : \mathcal{O}_K \rightarrow \mathcal{O}_B/m_B$ according to $\eta_j^{m_p} : \mathcal{O}_{K_j} \rightarrow \mathcal{O}_B/m_p$ being defined by $\eta_j^{m_p} = \theta_j^{m_p}$ for all $\ell \neq p$ and $j = 1, 2$, $\eta_1^{m_p} = \theta_1^{m_p}$, and $\eta_2^{m_p}(x) = \theta_2^{m_p}(x)$. Consider the CM pair $(A_1, A_2')$, where $A_2' = w_p \cdot A_2$ and $w_p$ is the Atkin-Lehner operator at $p$. The map

$$(\kappa_2'')^{m_p} : \mathcal{O}_{K_2} \rightarrow \text{End}_{\mathcal{O}_B/m_p}(A_2'[m_p]) \cong \mathcal{O}_B/m_p$$

is given by $(\kappa_2'')^{m_p}(x) = \kappa_2^{m_p}(x)$. The resulting map $\mathcal{O}_K \rightarrow \mathcal{O}_B/m_p$ for the pair $(A_1, A_2')$ is given by $t_1 \otimes t_2 \mapsto \kappa_1^{m_p}(t_1) \kappa_2^{m_p}(t_2)$, so $(A_1, A_2') \in \mathcal{Y}^B(\mathcal{P}\mathfrak{F})$ and the kernel of this map is $\mathfrak{Q}$, where $\mathfrak{Q}$ is the prime over $p$ dividing $\ker(\theta)$. As $A_1$ and $w_p \cdot A_2$ are of the same type, $\mathfrak{Q} = \mathfrak{Q}$ by the first part of the proof applied to $(A_1, A_2')$, so $\mathfrak{Q}$ does not divide $\ker(\theta)$. □

**5.3. Cases combined.** Let $(A_1, A_2) \in \mathcal{Y}^B(\mathcal{P}\mathfrak{F})$ with $\mathcal{P}$ lying over some prime $p$, and let $p = \mathfrak{P} \cap \mathcal{O}_F$. Set $a_\theta = \ker(\theta) \cap \mathcal{O}_F$.

**Theorem 5.11.** For any finite idele $\beta \in \hat{F}^\times$ satisfying $\beta \hat{\mathcal{O}}_F = a_\theta \mathfrak{D}^{-1} \hat{\mathcal{O}}_F$, there is a $\hat{K}$-linear isomorphism of $\hat{F}$-quadratic spaces

$$(\hat{V}(A_1, A_2), \text{deg}_{\text{CM}}) \cong (\hat{K}, \beta \cdot N_{K/F})$$

taking $\hat{L}(A_1, A_2)$ isomorphically to $\hat{\mathcal{O}}_K$.

**Proof.** Combining Propositions 5.2 and 5.7, and Proposition 5.10 proves the claim for some $\beta \in \hat{F}^\times$ satisfying $\beta \hat{\mathcal{O}}_F = a_\theta \mathfrak{D}^{-1} \hat{\mathcal{O}}_F$, and the surjectivity of the norm map $\hat{\mathcal{O}}_K^\times \rightarrow \hat{\mathcal{O}}_F^\times$ gives the result for all such $\beta$. □

Recall the definitions of the functions $\rho$ and $\rho_e$ from the introduction.

**Definition 5.12.** For each prime number $\ell$ and $\alpha \in F_{\ell}^\times$ define the orbital integral at $\ell$ by

$$O_\ell(\alpha, A_1, A_2) = \begin{cases} 
\rho_\ell(\alpha \mathfrak{D}_\ell) & \text{if } \ell \neq p, \ell \nmid d_B \\
\rho_\ell(\alpha \mathfrak{D}_\ell^{-1} \mathfrak{D}_\ell) & \text{if } \ell \neq p, \ell \mid d_B \\
\rho_p(\alpha^{-1}(p) \mathfrak{D}_p) & \text{if } \ell = p,
\end{cases}$$
where \( (\ell) \) is the prime over \( \ell \) dividing \( a_\theta \), with the convention that \( l(p) = \mathcal{O}_F \) if \( p \nmid d_B \).

It is possible to give a definition of \( \mathcal{O}_\ell(\alpha, A_1, A_2) \) as a sum of characteristic functions, analogous to \([10, (2.11)]\), but we do not need the details of that here. This alternative definition agrees with the one given above by a proof identical to that of \([10, \text{Lemmas 2.19, 2.20}]\), using Propositions 5.2 and 5.7 in place of Lemmas 2.10 and 2.11 of \([10]\).

**Theorem 5.13.** Let \( p \) be a prime number that is nonsplit in \( K_1 \) and \( K_2 \) and suppose \((A_1, A_2)\) is a CM pair over \( \mathbb{F}_p \). For any \( \alpha \in F^\times \) totally positive,

\[
\sum_{(a_1, a_2) \in \Gamma} \# \{ f \in L(a_1 \otimes \mathcal{O}_{K_1}, A_1, a_2 \otimes \mathcal{O}_{K_2}, A_2) : \deg_{\text{CM}}(f) = \alpha \} = \frac{w_1 w_2}{2} \prod_\ell \mathcal{O}_\ell(\alpha, A_1, A_2).
\]

**Proof.** The proof is formally the same as \([10, \text{Proposition 2.18}]\), replacing the definitions there with our analogous definitions, and using the above comment to match up the different definitions of the orbital integral. \( \square \)

**Proposition 5.14.** For any \( \alpha \in F^\times \) we have

\[
\prod_\ell \mathcal{O}_\ell(\alpha, A_1, A_2) = \rho(\alpha a_\theta^{-1} p^{-1} \mathfrak{D}).
\]

**Proof.** This follows from the definition of \( \mathcal{O}_\ell(\alpha, A_1, A_2) \) and the product expansion for \( \rho \). \( \square \)

### 6. Deformation theory

This section is devoted to the calculation of the length of the local ring \( \mathcal{O}^{\text{sh}}_{\mathcal{X}_g,B}(\mathbb{F}_p) \), which relies on the deformation theory of objects \((A_1, A_2, f)\) of \( \mathcal{X}^B_{g,B}(\mathbb{F}_p) \). We continue with the notation of Section 3.3. Fix a prime ideal \( \mathfrak{P} \subset \mathcal{O}_K \) of residue characteristic \( p \) and set \( \mathcal{W} = \mathcal{W}_{\mathfrak{P},p} \) and \( \mathcal{CLN} = \mathcal{CLN}_{\mathfrak{P},p} \). Let \( g \) be the connected \( p \)-divisible group of height 2 and dimension 1 over \( \mathbb{F}_p \).

**Definition 6.1.** Let \((A_1, A_2)\) be a CM pair over \( \mathbb{F}_p \) and \( R \in \mathcal{CLN} \). A deformation of \((A_1, A_2)\) to \( R \) is a CM pair \((\tilde{A}_1, \tilde{A}_2)\) over \( R \) together with an isomorphism of CM pairs \((\tilde{A}_1, \tilde{A}_2)/\mathfrak{P}_p \cong (A_1, A_2)\).

Given a CM pair \((A_1, A_2)\) over \( \mathbb{F}_p \), define \( \text{Def}(A_1, A_2) \) to be the functor \( \mathcal{CLN} \to \text{Sets} \) that assigns to each \( R \in \mathcal{CLN} \) the set of isomorphism classes of deformations of \((A_1, A_2)\) to \( R \). By Proposition 3.6,

\[
\text{Def}(A_1, A_2) \cong \text{Def}_{B}(A_1, \mathcal{O}_{K_1}) \times \text{Def}_{B}(A_2, \mathcal{O}_{K_2})
\]

is represented by \( \mathcal{W} \circ_{\mathcal{W}} \mathcal{W} \cong \mathcal{W} \). Given a nonzero \( f \in L(A_1, A_2) \) define \( \text{Def}(A_1, A_2, f) \) to be the functor \( \mathcal{CLN} \to \text{Sets} \) that assigns to each \( R \in \mathcal{CLN} \) the set of isomorphism classes of deformations of \((A_1, A_2, f)\) to \( R \).

#### 6.1. Deformations of CM pairs

Fix a ring homomorphism \( \theta : \mathcal{O}_K \to \mathcal{O}_B/\mathfrak{m}_\theta \), a CM pair \((A_1, A_2) \in \mathcal{X}^B_{g,B}(\mathbb{F}_p) \), and a nonzero \( f \in L(A_1, A_2) \). Assume \( p \) is nonsplit in \( K_1 \) and \( K_2 \).
Proposition 6.2. Suppose \( p \nmid d_B \).
(a) If \( p \) is inert in \( K_1 \) and \( K_2 \), then the functor \( \text{Def}(A_1, A_2, f) \) is represented by a local Artinian \( \mathcal{W} \)-algebra of length
\[
\frac{\text{ord}_p(\text{deg}_{CM}(f)) + 1}{2}.
\]
(b) If \( p \) is ramified in \( K_1 \) or \( K_2 \), then \( \text{Def}(A_1, A_2, f) \) is represented by a local Artinian \( \mathcal{W} \)-algebra of length
\[
\frac{\text{ord}_p(\text{deg}_{CM}(f)) + \text{ord}_p(\mathcal{O}) + 1}{2}.
\]

Proof. The proofs of (a) and (b) are the same as [10, Lemmas 2.23, 2.24], respectively. □

We will need an analogue for false elliptic curves of a result of Gross ([6, Proposition 3.3]) that gives the structure of the endomorphism ring of the modulo \( m \) reduction of the universal deformation of the \( p \)-divisible group \( \mathfrak{g} \). This is what we prove next.

Lemma 6.3. Take \( k = K_j \) for \( j \in \{1, 2\} \) and \( L = K \). Let \( (A, i, \kappa) \in \mathcal{W}(\mathbb{F}_p) \) for \( p \mid d_B \).
Set
\[
R = \text{End}_{\mathcal{O}_B}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \text{End}_{\mathcal{O}_B}(A[p^{\infty}]),
\]
let \( \mathcal{A} \) be the universal deformation of \( A \) to \( \mathcal{W} = W \), and for each integer \( m \geq 1 \) set
\[
R_m = \text{End}_{\mathcal{O}_B \otimes_{\mathcal{W}} W_m}(\mathcal{A} \otimes_{\mathcal{W}} W_m) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \text{End}_{\mathcal{O}_B \otimes_{\mathcal{W}} W_m}(\mathcal{A}[p^{\infty}] \otimes_{\mathcal{W}} W_m),
\]
where \( W_m = W/(p^m) \). Then the reduction map \( R_m \to R \) induces an isomorphism
\[
R_m \cong \mathcal{O}_K + p^{m-1}R,
\]
where \( \mathcal{O}_K = \kappa(\mathcal{O}_{k,p}) \).

Proof. We will use Grothendieck-Messing deformation theory. Let \( D = D(A) \) be the co-
variant 
Dieudonné module of \( A \) as above and set \( \mathcal{O} = \mathcal{O}_B \otimes_{\mathcal{W}} \mathcal{O}_K \). For any \( m \geq 1 \) there are \( \mathcal{O} \)-linear isomorphisms of \( W_m \)-modules
\[
H^1_{dR}(\mathcal{A} \otimes_{\mathcal{W}} W_m) \cong D \otimes_{\mathcal{W}} W_m \cong D/p^m D.
\]

For any \( m \geq 1 \) the surjection \( W_m \to \mathbb{F}_p \) has kernel \( pW/p^mW \), which has the canonical divided power structure. By Proposition 3.6, \( (A, i, \kappa) \) has a unique deformation to \( W_m \), namely \( \mathcal{A}_m = \mathcal{A} \otimes_{\mathcal{W}} W_m \). Therefore there is a unique direct summand \( M_m \subset H^1_{dR}(A) \), where \( H^1_{dR}(A) = H^1_{dR}(\tilde{A}) \) for any deformation \( \tilde{A} \) of \( A \) to \( W_m \), stable under the action of \( \mathcal{O} \) on \( H^1_{dR}(A) \), that reduces to \( \text{Fil}(A) \) (the Hodge filtration of \( A \)), and such that the diagram
\[
\begin{array}{ccc}
\mathcal{O}_k & \xrightarrow{\text{End}_{\mathcal{O}_B \otimes_{\mathcal{W}} W_m}(\tilde{H}^1_{dR}(A)/M_m)} & \text{End}_{\mathcal{O}_B \otimes_{\mathcal{W}} W_m}(\tilde{H}^1_{dR}(A)/M_m) \\
W_m & \downarrow & \\
& &
\end{array}
\]
commutes, namely \( M_m = \text{Fil}(\mathcal{A}_m) \). The Hodge sequence for \( A \) takes the form
\[
0 \to \text{Fil}(A) \to D/pD \to \text{Lie}(A) \to 0.
\]

Using a \( W \)-basis \( \{e_1, e_2, e_3, e_4\} \) for \( D \) as in Proposition 3.16, it also defines an \( \mathbb{F}_p \)-basis for \( D/pD \), and \( \text{Fil}(A) = \ker(D/pD \to D/\mathcal{W}D) \) has \( \{e_2, e_4\} \) as an \( \mathbb{F}_p \)-basis.
Any $f \in R$ induces a map $H_1^{dR}(A) \to H_1^{dR}(A)$ which lifts to a map $\tilde{f} : \tilde{H}_1^{dR}(A) \to H_1^{dR}(A)$, and $f$ lifts to an element of $R_m$ if and only if $\tilde{f}(M_m) \subset M_m$. The map 

$$\tilde{f} : \tilde{H}_1^{dR}(A) \cong D/p^m D \to D/p^m D \cong \tilde{H}_1^{dR}(A)$$

corresponds to the reduction modulo $p^m$ of $f : D \to D$. We have $M_m \cong N = \text{Span}_{W_m}(e_2, e_4)$ under the isomorphism $\tilde{H}_1^{dR}(A) \cong D/p^m D$. Expressing

$$f = \begin{bmatrix} x \\ py \Pi \\ y \Pi \end{bmatrix} \in R$$
as an element of $M_4(W)$ as in (3.6), we have

$$f \text{ lifts to an element of } R_m \iff \tilde{f}(N) \subset N$$

$$\iff f(e_2), f(e_4) \in W e_2 + W e_4 + p^m D$$

$$\iff y \in p^{m-1}O_{K,p}$$

$$\iff f \in O_K + p^{m-1}R.$$ 

Proposition 6.4. If $p \mid d_B$ and $\mathfrak{P}$ divides $\ker(\theta)$, then $\text{Def}(A_1, A_2, f)$ is represented by a local Artinian $\mathfrak{W}$-algebra of length $\frac{1}{2} \text{ord}_p(\text{deg}_{CM}(f))$.

Proof. As usual $A_j \cong M_j \otimes_{O_{K_j}} E_j$ for some supersingular elliptic curve $E_j$. Isomorphisms $E_j[p^\infty] \cong \mathfrak{P}$ may be chosen so that the CM actions $O_{K_1,p} \to \Delta$ and $O_{K_2,p} \to \Delta$ on $E_1$ and $E_2$ have the same image $O_K \cong \mathbb{Z}_p$. Fix a uniformizer $\Pi \in \Delta$ satisfying $x \Pi = \Pi x$ for all $x \in O_K \subset \Delta$. There is an isomorphism of $\mathbb{Z}_p$-modules $L_p(A_1, A_2) \cong R$, where

$$R = \left\{ \begin{bmatrix} x \\ py \Pi \\ y \Pi \\ x \end{bmatrix} : x, y \in O_K \right\},$$

and the CM actions $\kappa_1$ and $\kappa_2$ are identified with a single action $O_K \to R$ given by $x \mapsto \text{diag}(x, x)$. Under the isomorphism $L_p(A_1, A_2) \cong R$ the quadratic form $\text{deg}^*$ on $L_p(A_1, A_2)$ is identified with the quadratic form $Q$ on $R$ defined in Proposition 5.4. There is a decomposition of left $O_K$-modules $R = R_+ \oplus R_-$, with $R_+ = O_K$, embedded diagonally in $R$, and $R_- = O_K \Pi$, where

$$P = \begin{bmatrix} 0 & \Pi \\ \Pi & 0 \end{bmatrix}.$$

and this decomposition is orthogonal with respect to the quadratic form $\text{deg}^*$. Define $\varphi_\pm : O_{K,p} \to O_K \subset R$ by

$$\varphi_+(x_1 \otimes x_2) = \kappa_2(x_2)\kappa_1(\mathfrak{P}_1)$$

$$\varphi_-(x_1 \otimes x_2) = \kappa_2(x_2)\kappa_1(x_1),$$

and let $\Phi$ be the isomorphism $\varphi_+ \times \varphi_- : O_{K,p} \to O_K \times O_K$. Then the usual action of $O_K$ on $R$ is given by

$$x \bullet f = \varphi_+(x)f_+ + \varphi_-(x)f_-$$

for $f = f_+ + f_- \in R$. It follows that $\Phi(\text{deg}_{CM}(f)) = (\text{deg}^*(f_+), \text{deg}^*(f_-))$ and thus

$$\text{ord}_{p_+}(\text{deg}_{CM}(f)) = \text{ord}_p(\text{deg}^*(f_+))$$

$$\text{ord}_{p_-}(\text{deg}_{CM}(f)) = \text{ord}_p(\text{deg}^*(f_-)).$$
where \( p_\pm = p \) and \( p_\pm = \overline{p} \) (see the proof of Proposition 5.7). Since \( \deg^+(P) = Q(P) = -p^2 \), for any integer \( m \geq 1 \) and any \( f \in R \) we have

\[
f \in \mathcal{O}_K + p^{m-1}R \iff f_- \in p^{m-1}\mathcal{O}_K P \\
\iff \ord_p(\deg^+(f_-)) \geq 2m \\
\iff \frac{1}{2}\ord_p(\deg_{CM}(f)) \geq m.
\]

The functor

\[
\text{Def}(\mathbf{A}_1, \mathbf{A}_2) \cong \text{Def}_{\mathcal{O}_B}(A_1[p^\infty], \mathcal{O}_K) \times \text{Def}_{\mathcal{O}_B}(A_2[p^\infty], \mathcal{O}_K)
\]

is represented by \( \mathcal{W} \circledast \mathcal{W} \cong \mathcal{W} \). Let \((\widetilde{A}_1, \widetilde{A}_2)\) be the universal deformation of \((\mathbf{A}_1, \mathbf{A}_2)\) to \( \mathcal{W} = W \). It follows from [15, Proposition 2.9] that the functor \( \text{Def}(\mathbf{A}_1, \mathbf{A}_2, f) \) is represented by

\[
\text{Hom}_{\mathcal{O}_B \otimes \mathcal{O}_W}(\widetilde{A}_1[p^\infty] \otimes_W W_m, \widetilde{A}_2[p^\infty] \otimes_W W_m).
\]

Since there are \( \mathcal{O}_B \otimes \mathcal{O}_K \)-linear isomorphisms \( A_1[p^\infty] \cong A_2[p^\infty] \) and \( \widetilde{A}_j \otimes_W \mathbb{F}_p \cong A_j \), there is an \( \mathcal{O}_B \otimes \mathcal{O}_K \)-linear isomorphism \( A_1[p^\infty] \cong A_2[p^\infty] \) by the uniqueness of the universal deformation. Hence

\[
\text{Hom}_{\mathcal{O}_B \otimes \mathcal{O}_W}(\widetilde{A}_1[p^\infty] \otimes_W W_m, \widetilde{A}_2[p^\infty] \otimes_W W_m) \cong R_m \cong \mathcal{O}_K + p^{m-1}R
\]
in the notation of Lemma 6.3, and then \( m = \frac{1}{2}\ord_p(\deg_{CM}(f)) \) by the above calculation. \( \square \)

With \((\mathbf{A}_1, \mathbf{A}_2)\) as above, suppose \( p \mid d_B \) and \( \mathfrak{P} \) does not divide \( \ker(\theta) \). As usual \( \mathfrak{A}_j \cong M_j \otimes_{\mathcal{O}_K} E_j \) for some supersingular \( E_j \). Choose isomorphisms \( E_j[p^\infty] \cong \mathfrak{g} \) so that the CM actions \( g_1 : \mathcal{O}_{K_1, p} \to \Delta \) and \( g_2 : \mathcal{O}_{K_2, p} \to \Delta \) on \( E_1 \) and \( E_2 \) have the same image \( \mathcal{O}_K \cong \mathbb{Z}_p^2 \). Fix a uniformizer \( \Pi \in \Delta \) satisfying \( \Pi g_1(x) = g_1(\overline{x})\Pi \) for all \( x \in \mathcal{O}_{K_1, p} \).

There is an isomorphism of \( \mathbb{Z}_p \)-modules \( L_p(\mathbf{A}_1, \mathbf{A}_2) \cong R' \), where

\[
R' = \left\{ \begin{bmatrix} px \\ y \Pi \\ x \end{bmatrix} : x, y \in \mathcal{O}_K \right\},
\]

and the quadratic form \( \deg^+ \) on \( L_p(\mathbf{A}_1, \mathbf{A}_2) \) is identified with the quadratic form \( uQ' \) on \( R' \) defined in Proposition 5.6. There is a decomposition of left \( \mathcal{O}_K \)-modules \( R' = R'_+ \oplus R'_- \), where \( R'_+ = \mathcal{O}_K P_1 \) and \( R'_- = \mathcal{O}_K P_2 \), with

\[
P_1 = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & \Pi \\ \Pi & 0 \end{bmatrix}.
\]

**Lemma 6.5.** With notation as above, let \( \mathfrak{A}_j \) be the universal deformation of \( \mathbf{A}_j \) to \( \mathcal{W} = W \), and for each integer \( m \geq 1 \) set

\[
R'_m = \text{Hom}_{\mathcal{O}_B \otimes \mathcal{O}_W}(\mathfrak{A}_1 \otimes_W W_m, \mathfrak{A}_2 \otimes_W W_m) \otimes_{\mathbb{Z}} \mathbb{Z}_p.
\]

Then the reduction map \( R'_m \twoheadrightarrow R' \) induces an isomorphism

\[
R'_m \cong \mathcal{O}_K P_1 + p^{m-1}\mathcal{O}_K P_2.
\]

**Proof.** This is very similar to the proof of Lemma 6.3. \( \square \)
Proposition 6.6. If \( p \mid d_B \) and \( \mathfrak{P} \) does not divide \( \ker(\theta) \), then \( \text{Def}(A_1, A_2, f) \) is represented by a local Artinian \( \mathcal{W} \)-algebra of length
\[
\frac{\text{ord}_p(\deg_{CM}(f)) + 1}{2}.
\]

Proof. The proof is the same as in Proposition 6.4, using Lemma 6.5, the key difference being \( \deg^* (P_2) = uQ'(P_2) = -up \).

6.2. The étale local ring. Let \( \mathcal{Z} \) be a stack over \( \text{Spec}(O_K) \) and let \( z \in \mathcal{Z}(\mathbb{F}_P) \) be a geometric point. An étale neighborhood of \( z \) is a commutative diagram in the 2-category of stacks over \( \text{Spec}(O_K) \)
\[
\begin{array}{ccc}
\mathcal{Z} & \xrightarrow{\theta} & \mathcal{Z} \\
\downarrow & & \downarrow \\
\text{Spec}(\mathbb{F}_P) & \xrightarrow{\pi} & \mathcal{Z} \\
\end{array}
\]
where \( U \) is an \( O_K \)-scheme and \( U \to Z \) is an étale morphism. The strictly Henselian local ring of \( \mathcal{Z} \) at \( z \) is the direct limit
\[
O_{\mathcal{Z}, z} = \lim_{\leftarrow} (U, \pi(z))
\]
over all étale neighborhoods of \( z \), where \( O_{U, \pi(z)} \) is the local ring of the scheme \( U \) at the image of \( \pi(z) \). The ring \( O_{\mathcal{Z}, z} \) is a strictly Henselian local ring with residue field \( \mathbb{F}_P \) and the completion \( \hat{O}_{\mathcal{Z}, z} \) is a \( \mathcal{W} \)-algebra.

Theorem 6.7. Let \( \alpha \in F^\times \), let \( \theta : O_K \to O_B / m_B \) be a ring homomorphism, and suppose \( \mathfrak{P} \subset O_K \) is a prime ideal lying over a prime \( p \). Set
\[
\nu_p(\alpha) = \frac{1}{2} \text{ord}_p(a \theta \mathcal{D}), \quad \nu'_p(\alpha) = \frac{1}{2} \text{ord}_p(\alpha),
\]
where \( p = \mathfrak{P} \cap O_F \). For any \( x = (A_1, A_2, f) \in \mathcal{Z}\theta^B(\mathbb{F}_P) \), the strictly Henselian local ring \( O_{\mathcal{Z}, x}^{sh} \) is Artinian of length \( \nu_p(\alpha) \) if \( p \nmid d_B \) or \( p \mid d_B \) and \( \mathfrak{P} \nmid \ker(\theta) \), and is Artinian of length \( \nu'_p(\alpha) \) if \( p \mid d_B \) and \( \mathfrak{P} \mid \ker(\theta) \).

By length we mean the length of the ring as a module over itself.

Proof. Using Corollary 3.14, the same proof as in [10, Proposition 2.25] shows the functor \( \text{Def}(A_1, A_2, f) \) is represented by the ring \( \hat{O}_{\mathcal{Z}, x}^{sh} \). The result then follows from Propositions 6.2, 6.4, 6.6, and the fact that \( \text{length}(\hat{O}_{\mathcal{Z}, x}^{sh}) = \text{length}(O_{\mathcal{Z}, x}^{sh}) \).

7. Final formula

As in the introduction, let \( \chi \) be the quadratic Hecke character associated with the extension \( K/F \). For any \( \alpha \in F^\times \) totally positive and any ring homomorphism \( \theta : O_K \to O_B / m_B \), define a finite set of prime ideals
\[
\text{Diff}_\theta(\alpha) = \{ p \subset O_F : \chi_p(a \theta \mathcal{D}) = -1 \},
\]
where \( a_\theta = \ker(\theta) \cap O_F \). It follows from the product formula \( \prod_p \chi_p(x) = 1 \) that \( \text{Diff}_\theta(\alpha) \) has odd cardinality, and in particular is nonempty. Note that any prime in \( \text{Diff}_\theta(\alpha) \) is inert in \( K \). Recall \( \Gamma = \text{Cl}(O_{K_1}) \times \text{Cl}(O_{K_2}) \).
Lemma 7.1. For any prime $\mathfrak{P} \subset \mathcal{O}_K$ and any ring homomorphism $\theta: \mathcal{O}_K \rightarrow \mathcal{O}_B/\mathfrak{m}_B$, we have $\#(\mathcal{X}^B_\theta(\mathbb{F}_\mathfrak{P})) = |\Gamma|$.

Proof. Let $\theta_\mathfrak{P} = \theta|_{\mathcal{O}_K}$. By definition, an object of $\mathcal{X}^B_\theta(\mathbb{F}_\mathfrak{P})$ is a pair $(\mathbf{A}_1, \mathbf{A}_2)$ with $\mathbf{A}_j$ an object of $\mathcal{Y}^B_j(\theta_j)(\mathbb{F}_\mathfrak{P})$, so by what we proved in Section 3.3,

$$\#(\mathcal{X}^B_\theta(\mathbb{F}_\mathfrak{P})) = \#(\mathcal{Y}^B_1(\theta_1)(\mathbb{F}_\mathfrak{P})) \cdot \#(\mathcal{Y}^B_2(\theta_2)(\mathbb{F}_\mathfrak{P})) = |\text{Cl}(\mathcal{O}_{K_1})| \cdot |\text{Cl}(\mathcal{O}_{K_2})| = |\Gamma|.$$  

\hfill $\square$

Proposition 7.2. Suppose $\alpha \in F^\times$ and $\theta: \mathcal{O}_K \rightarrow \mathcal{O}_B/\mathfrak{m}_B$ is a ring homomorphism. If $\#\text{Diff}_\theta(\alpha) > 1$ then $\mathcal{X}^B_{\theta,\alpha} = \emptyset$. Suppose $\text{Diff}_\theta(\alpha) = \{\mathfrak{p}\}$, let $\mathfrak{P} \subset \mathcal{O}_K$ be the prime over $\mathfrak{p}$, and let $p\mathbb{Z} = \mathfrak{p} \cap \mathbb{Z}$. Then the stack $\mathcal{X}^B_{\theta,\alpha}$ is supported in characteristic $p$. More specifically, it only has geometric points over the field $\mathbb{F}_p$ (if it has any at all).

Proof. By Proposition 4.6 the stack $\mathcal{X}^B_{\theta,\alpha}$ has no geometric points in characteristic 0. Suppose $\mathcal{X}^B_{\theta,\alpha}(\mathbb{F}_\mathfrak{P}) \neq \emptyset$ for some prime ideal $\mathfrak{P} \subset \mathcal{O}_K$. Fix $(\mathbf{A}_1, \mathbf{A}_2, f) \in \mathcal{X}^B_{\theta,\alpha}(\mathbb{F}_\mathfrak{P})$, and let $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_F$ and $p\mathbb{Z} = \mathfrak{p} \cap \mathbb{Z}$. Any prime ideal $\mathfrak{q}$ of $\mathcal{O}_F$ lying over $p$ or lying over any divisor of $d_B$ is inert in $\mathbb{K}$ (by Proposition 4.6(d) and our assumption about the primes dividing $d_B$), so for such a $\mathfrak{q}$,

$$\chi_l(\mathfrak{q}) = \begin{cases} -1 & \text{if } l = \mathfrak{q} \\ 1 & \text{if } l \neq \mathfrak{q} \end{cases}$$

for any prime $l \subset \mathcal{O}_F$. By Theorem 5.11, the quadratic space $(\mathbb{K}, \beta \cdot N_{\mathbb{K}/\mathbb{F}})$ represents $\alpha$ for any $\beta \in \mathbb{K}^\times$ satisfying $\beta \mathcal{O}_F = a_\mathfrak{p}\mathcal{D}^{-1}\mathcal{O}_F$. It follows that $\chi_l(\alpha) = \chi_l((a_\mathfrak{p}\mathcal{D}^{-1}))$ for every prime $l \subset \mathcal{O}_F$, so $\text{Diff}_\theta(\alpha) = \{\mathfrak{p}\}$. This shows that if $\mathcal{X}^B_{\theta,\alpha}(\mathbb{F}_\mathfrak{P}) \neq \emptyset$ then $\text{Diff}_\theta(\alpha) = \{\mathfrak{p}\}$, where $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_F$.  

Recall the definition of the arithmetic degree of $\mathcal{X}^B_{\theta,\alpha}$ from the introduction:

$$\deg(\mathcal{X}^B_{\theta,\alpha}) = \sum_{\mathfrak{P} \subset \mathcal{O}_K} \log(|\mathfrak{P}|) \sum_{x \in \mathcal{X}^B_{\theta,\alpha}(\mathbb{F}_\mathfrak{P})} \text{length}(\mathfrak{P}^\mathfrak{b}_{\mathfrak{P}_X,\alpha}) / |\text{Aut}(x)|.$$

Theorem 7.3. Let $\alpha \in F^\times$ be totally positive and suppose $\alpha \in \mathcal{D}^{-1}$. Let $\theta: \mathcal{O}_K \rightarrow \mathcal{O}_B/\mathfrak{m}_B$ be a ring homomorphism with $a_\theta = \ker(\theta) \cap \mathcal{O}_F$, suppose $\text{Diff}_\theta(\alpha) = \{\mathfrak{p}\}$, and let $p\mathbb{Z} = \mathfrak{p} \cap \mathbb{Z}$.

(a) If $\mathfrak{p} \nmid d_B$ then

$$\deg(\mathcal{X}^B_{\theta,\alpha}) = \frac{1}{2} \log(p) \cdot \text{ord}_\mathfrak{p}(\alpha \mathcal{D}) \cdot \rho(\alpha a_\mathfrak{p}^{-1} \mathfrak{P}^{-1} \mathcal{D}).$$

(b) Suppose $\mathfrak{p} \mid d_B$ and let $\mathfrak{P} \subset \mathcal{O}_K$ be the prime over $\mathfrak{P}$. If $\mathfrak{P}$ divides $\ker(\theta)$ then

$$\deg(\mathcal{X}^B_{\theta,\alpha}) = \frac{1}{2} \log(p) \cdot \text{ord}_\mathfrak{p}(\alpha) \cdot \rho(\alpha a_\mathfrak{p}^{-1} \mathfrak{P}^{-1} \mathcal{D}).$$

If $\mathfrak{P}$ does not divide $\ker(\theta)$ then

$$\deg(\mathcal{X}^B_{\theta,\alpha}) = \frac{1}{2} \log(p) \cdot \text{ord}_\mathfrak{p}(\alpha \mathcal{D}) \cdot \rho(\alpha a_\mathfrak{p}^{-1} \mathfrak{P}^{-1} \mathcal{D}).$$

If $\alpha \notin \mathcal{D}^{-1}$ or if $\#\text{Diff}_\theta(\alpha) > 1$, then $\deg(\mathcal{X}^B_{\theta,\alpha}) = 0$. 
Proof. (a) Using Theorem 6.7, Proposition 7.2, Lemma 4.7, and \(|F_\mathfrak{P}| = N_{K/Q}(\mathfrak{P}) = p^2\),

\[
\deg(\mathcal{F}_{\theta,\alpha}^B) = \log(|F_\mathfrak{P}|) \sum_{x \in \mathcal{F}_{\theta,\alpha}(\mathfrak{F}_\mathfrak{P})} \frac{\text{length}(\mathcal{O}^{|\mathfrak{P}|,x})}{|\text{Aut}(x)|} = 2 \log(p) \nu_\mathfrak{p}(\alpha) \sum_{(A_1,A_2,f) \in \mathcal{F}_{\theta,\alpha}(\mathfrak{F}_\mathfrak{P})} \frac{1}{|\text{Aut}(A_1,A_2,f)|} = 2 \log(p) \nu_\mathfrak{p}(\alpha) \sum_{(A_1,A_2) \in \mathcal{F}_{\theta,\alpha}(\mathfrak{F}_\mathfrak{P})} \sum_{f \in L(A_1,A_2)} \frac{1}{w_1w_2}.
\]

Now using Theorem 5.13, Proposition 5.14, and Lemma 7.1, we have

\[
\deg(\mathcal{F}_{\theta,\alpha}^B) = \log(p) \frac{\nu_\mathfrak{p}(\alpha)}{|\Gamma|} \sum_{(A_1,A_2) \in \mathcal{F}_{\theta,\alpha}(\mathfrak{F}_\mathfrak{P})} \prod_{\ell} O_\ell(\alpha,A_1,A_2) = \log(p) \frac{\nu_\mathfrak{p}(\alpha)}{|\Gamma|} \sum_{(A_1,A_2) \in \mathcal{F}_{\theta,\alpha}(\mathfrak{F}_\mathfrak{P})} \rho(\alpha a_\mathfrak{p}^{-1}p^{-1}D) = \frac{1}{2} \log(p) \cdot \text{ord}_p(\alpha \mathfrak{p}D) \cdot \rho(\alpha a_\mathfrak{p}^{-1}p^{-1}D).
\]

(b) Suppose \(p \mid d_B\). If \(\mathfrak{P}\) divides \(\ker(\theta)\) then a similar calculation as in (a), replacing \(\nu_\mathfrak{p}(\alpha)\) with \(\nu'_\mathfrak{p}(\alpha)\), gives the desired result. If \(\mathfrak{P}\) does not divide \(\ker(\theta)\) then the exact same calculation as in (a) gives the desired formula, noting that \(\nu_\mathfrak{p}(\alpha) = \frac{1}{2} \text{ord}_p(\alpha \mathfrak{p})\) for \(p \mid d_B\). The final claim follows from Proposition 7.2 and the fact that \(\deg_{\text{CM}}\) takes values in \(D^{-1}\).  

\[\square\]

Appendix A. Hecke correspondences

In this section we will define the Hecke correspondences \(T_m\) on \(\mathcal{M}\) and \(\mathcal{M}^B\), and prove the equalities (1.2) and (1.4) in the introduction (we continue with the same notation as in Sections 1.1 and 1.2). For any ring \(R\) we write \(\text{length}(R)\) for \(\text{length}_R(R)\).

Fix a positive integer \(m\). Let \(\mathcal{M}(m)\) be the category fibered in groupoids over \(\text{Spec}(O_K)\) with \(\mathcal{M}(m)(S)\) the category of triples \((E_1,E_2,\varphi)\) with \(E_i\) an object of \(\mathcal{M}(S)\) and \(\varphi \in \text{Hom}_S(E_1,E_2)\) satisfying \(\deg(\varphi) = m\) on every connected component of \(S\). The category \(\mathcal{M}(m)\) is a stack, flat of relative dimension 1 over \(\text{Spec}(O_K)\), and there are two finite flat morphisms

\[
\mathcal{M}(m) \xrightarrow{\pi_1} \mathcal{M} \xrightarrow{\pi_2}
\]

given by \(\pi_1(E_1,E_2,\varphi) = E_i\). Define \(T_m : \text{Div}(\mathcal{M}) \rightarrow \text{Div}(\mathcal{M})\) by \(T_m = (\pi_2)_* \circ (\pi_1)^*\).

For \(i \in \{1,2\}\) let \(f_i : \mathfrak{K}_i \rightarrow \mathcal{M}\) be the finite morphism defined by forgetting the complex multiplication structure. Consider \(\mathfrak{D}_1 = \mathfrak{K}_1 \times f_1^* \mathcal{M} \times \mathfrak{K}_1 \mathcal{M}(m)\). Up to the obvious isomorphism of stacks, the objects of \(\mathfrak{D}_1\) can be described as triples \((E_1,E_2,\varphi)\) with \(E_1 \in \mathfrak{K}_1, E_2 \in \mathcal{M}\), and \(\varphi : E_1 \rightarrow E_2\) a degree \(m\) isogeny. Now let \(g\) be the composition \(\mathfrak{D}_1 \rightarrow \mathcal{M}(m) \xrightarrow{\pi_2} \mathcal{M}\).

The fiber product \(\mathfrak{D}_1 \times_{g,\mathcal{M},f_2} \mathfrak{K}_2\) is easily seen to be isomorphic to \(\mathfrak{D}_m\).

Viewing \(\mathfrak{D}_1\) as a closed substack of \(\mathcal{M}(m)\) through the image of \(\mathfrak{D}_1 \rightarrow \mathcal{M}(m)\), the divisor \(T_m \mathfrak{D}_1\) on \(\mathcal{M}\) is \((\pi_2)_*[\mathfrak{D}_1]\), where \([\mathfrak{D}_1]\) is the divisor associated with \(\mathfrak{D}_1\) (see [20, Definition
so to prove \( \deg(T_m) = I(T_m, Z_1, Z_2) \), we need to show
\[
(A.1) \quad \deg(D_1 \times_{\mathcal{M}} f_2 Y_2) = I((\pi_2)_*[Z_1], [Y_2]),
\]
where we are writing \([Y_2]\) for the divisor on \(\mathcal{M}\) determined by the image of \(f_2\).

Let \(k = \mathbb{F}_q\) for \(\mathcal{R} \subset \mathcal{O}_K\) a prime ideal and let \(x \in \mathcal{M}(k)\) be a geometric point. For any two prime divisors \(Z\) and \(Z'\) on \(\mathcal{M}\) intersecting properly, define the Serre intersection multiplicity at \(x\) by
\[
I^\mathcal{M}_x(Z, Z') = \sum_{i \geq 0} (-1)^i \text{length}_{\mathcal{O}_{\mathcal{M},x}} \text{Tor}_i^\mathcal{O}_{\mathcal{M},x}(\mathcal{O}_{\mathcal{M},x}^{\text{sh}}(Z), \mathcal{O}_{\mathcal{M},x}^{\text{sh}}(Z'))
\]
if \(x \in (Z \cap Z')(k)\) and set \(I^\mathcal{M}_x(Z, Z') = 0\) otherwise. Extend this definition bilinearly to all divisors on \(\mathcal{M}\). Again, if \(Z\) and \(Z'\) are prime divisors on \(\mathcal{M}\) intersecting properly, there is a way of defining a 0-cycle \(Z \cdot Z'\) on \(\mathcal{M}\) in such a way that
\[
\text{Coeff}_x(Z \cdot Z') = I^\mathcal{M}_x(Z, Z'),
\]
where \(\text{Coeff}_x(Z \cdot Z')\) is the coefficient in the 0-cycle \(Z \cdot Z'\) of the 0-dimensional closed substack determined by the image of \(x : \text{Spec}(k) \to \mathcal{M}\) (see [18, Chapter V] and [19, Chapter I]).

With notation as above, let \(D_2 = \mathcal{M}(m) \times_{\pi_2, \mathcal{M}} f_2 Y_2\), so \([D_2] = (\pi_2)^*[Y_2]\). Also, let \(x \in \mathcal{M}(m)(k)\) with \(x = (E_1, E_2, \varphi)\) where \(E_i \in \mathcal{N}\). We claim
\[
(A.2) \quad \text{Tor}_i^{\mathcal{O}_{\mathcal{M},x}}(\mathcal{O}_{\mathcal{M},x}^{\text{sh}}(D_1), \mathcal{O}_{\mathcal{M},x}^{\text{sh}}(D_2)) = 0
\]
for all \(i > 0\). To prove this, first consider the stack \(D'_1 = \mathcal{N} \times_{\pi_1, \mathcal{M}} f_1, \mathcal{M}(m)\). This category has objects \((E_1, E_2, \varphi)\) with \(E_1 \in \mathcal{M}, E_2 \in \mathcal{N}\), and \(\varphi : E_1 \to E_2\) a degree \(m\) isogeny. It follows that there is an isomorphism of stacks \(D'_1 = D_1\) and
\[
\mathcal{O}_{\mathcal{D}_1,x}^{\text{sh}} \cong \mathcal{O}_{\mathcal{D}_2,x}^{\text{sh}} \cong \mathcal{O}_{\mathcal{M},x}^{\text{sh}}(m) \otimes \mathcal{O}_{\mathcal{M},x}^{\text{sh}}(\pi_2(x)) \mathcal{O}_{\mathcal{N},x}(\varphi_1(x))
\]
We already have
\[
\mathcal{O}_{\mathcal{D}_2,x}^{\text{sh}} \cong \mathcal{O}_{\mathcal{M},x}^{\text{sh}}(m) \otimes \mathcal{O}_{\mathcal{M},x}^{\text{sh}}(\pi_2(x)) \mathcal{O}_{\mathcal{N},x}(\varphi_2(x))
\]
so from \(\pi_2\) being flat,
\[
\text{Tor}_i^{\mathcal{O}_{\mathcal{M},x}}(\mathcal{O}_{\mathcal{M},x}^{\text{sh}}(D_1), \mathcal{O}_{\mathcal{M},x}^{\text{sh}}(D_2)) \cong \mathcal{O}_{\mathcal{M},x}^{\text{sh}}(m) \otimes \mathcal{O}_{\mathcal{M},x}^{\text{sh}}(\pi_2(x)) \text{Tor}_i^{\mathcal{O}_{\mathcal{M},x}}(\mathcal{O}_{\mathcal{M},x}^{\text{sh}}(\pi_2(x)), \mathcal{O}_{\mathcal{N},x}(\varphi_2(x))).
\]
As \(\mathcal{O}_{\mathcal{M},x}(\pi_2(x))\) and \(\mathcal{O}_{\mathcal{N},x}(\varphi_2(x))\) are regular local rings of dimension 2 and 1, respectively, \(\mathcal{O}_{\mathcal{M},x}^{\text{sh}}(\pi_2(x))\) is a Cohen-Macaulay \(\mathcal{O}_{\mathcal{N},x}(\varphi_2(x))\)-module, and thus \((A.2)\) holds for all \(i > 0\) by [18, p. 111].

There is a projection formula
\[
((\pi_2)_*[Z_1]) \cdot [Y_2] = (\pi_2)_*[Z_1] \cdot ((\pi_2)^*[Y_2]),
\]
This is a special case of a more general formula, but it takes this form in our case since \((A.2)\) holds (see [18, p. 118, formulas (10), (11)]). It follows that for any \(y \in \mathcal{M}(k)\),
\[
I^\mathcal{M}_y((\pi_2)_*[Z_1], [Y_2]) = \text{Coeff}_y((\pi_2)_*[Z_1] \cdot [Y_2])
\]
\[
= \sum_{x \in \pi_2^{-1}(y)} \text{Coeff}_x([Z_1] \cdot ((\pi_2)^*[Y_2]))
\]
\[
= \sum_{x \in \pi_2^{-1}(y)} I^\mathcal{M}_x([Z_1], [Y_2]).
\]
Letting \( h_i : \mathcal{D}_i \to \mathcal{M}(m) \) be the natural projection, there is an isomorphism of stacks
\[
\mathcal{D}_1 \times_{h_1, \mathcal{M}(m), h_2} \mathcal{D}_2 \cong \mathcal{D}_1 \times_{g, \mathcal{M}, f_2} \mathcal{D}_2.
\]
Also, by (A.2) we have
\[
I_y^{\mathcal{M}(m)}([\mathcal{D}_1], [\mathcal{D}_2]) = \text{length}(\mathcal{O}^{\mathcal{M}(m)}_{\mathcal{D}_1, x} \otimes \mathcal{O}^{\mathcal{M}(m)}_{\mathcal{D}_2, x}).
\]
Therefore, for any \( y \in \mathcal{M}(k) \),
\[
\sum_{\pi \in \pi_y^{-1}([y])} \text{length}(\mathcal{O}^{\mathcal{M}(m)}_{\mathcal{D}_1, x} \otimes \mathcal{O}^{\mathcal{M}(m)}_{\mathcal{D}_2, x}) = \sum_{\pi \in \pi_y^{-1}([y])} \text{length}(\mathcal{O}^{\mathcal{M}(m)}_{\mathcal{D}_1 \times_{h_1, \mathcal{M}(m), h_2} \mathcal{D}_2, x})
\]
\[
= \sum_{\pi \in \pi_y^{-1}([y])} I_y^{\mathcal{M}(m)}([\mathcal{D}_1], [\mathcal{D}_2])
\]
\[
= I_y^{\mathcal{M}(m)}((\pi), [\mathcal{D}_1], [\mathcal{D}_2]).
\]
Since \( \mathcal{D}_2 \) is regular and the local ring at \( y \) of any prime divisor appearing in \( (\pi), [\mathcal{D}_1] \) is a 1-dimensional domain, hence Cohen-Macaulay, the Tor terms appearing in the sum
\[
I_y^{\mathcal{M}(m)}((\pi), [\mathcal{D}_1], [\mathcal{D}_2])
\]
are zero for all \( i > 0 \). Multiplying both sides of the above equality by \( \log(|\mathcal{F}_Y|)/|\text{Aut}(y)| \) and summing over all \( y \) and over all \( \mathcal{F} \) then gives the equality (A.1).

The definition of \( T_m : \text{Div}(\mathcal{M}^B) \to \text{Div}(\mathcal{M}^B) \) and the proof of the equality \( \text{deg}(T_m^B) = I(T_m^B, \mathcal{D}_1^B, \mathcal{D}_2^B) \) is exactly the same as the elliptic curve case. The equality (1.4) then follows from the decomposition (4.1).

**Acknowledgment**

This research forms part of my Boston College Ph.D. thesis. I would like to thank my advisor Ben Howard.

**References**


