

Review for Test 3

Part 1

1. For a given system $\mathbf{x}'(t) = A\mathbf{x}(t)$, solve the initial value problem with the given initial condition. Classify by type and by stability the critical point $(0, 0)$ of the system.

(i) $A = \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix}$, $\mathbf{x}(0) = (1, 1)$

(ii) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{x}(0) = (1, 0)$

(iii) $A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$, $\mathbf{x}(0) = (1, 3)$

(iv) $A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$, $\mathbf{x}(0) = (1, 0)$

(v) $A = \begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}$, $\mathbf{x}(0) = (1, 1)$

(vi) $A = \begin{pmatrix} -10 & 6 \\ 15 & -19 \end{pmatrix}$, $\mathbf{x}(0) = (1, 1)$

(vii) $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\mathbf{x}(0) = (1, 0)$

2. Use the method of variation of parameters to find a general solution of the system $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$.

(i) $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\mathbf{f}(t) = \begin{pmatrix} 3 \\ 3 \end{pmatrix} e^t$

(ii) $A = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$, $\mathbf{f}(t) = \begin{pmatrix} e^t \\ e^{2t} \end{pmatrix}$

(iii) $A = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$, $\mathbf{f}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}$

Answers to part 1:

1. (i) $\mathbf{x}_c(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} -4 \\ 6 \end{pmatrix} e^{2t}$,
 $c_1 = 5, c_2 = 1$, unstable node

(ii) $\mathbf{x}_c(t) = c_1 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$,
 $c_1 = 0, c_2 = 1$, center (stable)

(iii) $\mathbf{x}_c(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$,
 $c_1 = 0, c_2 = 1$, saddle point (unstable)

(iv) $\mathbf{x}_c(t) = e^{-t} \left[c_1 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \right]$,
 $c_1 = 0, c_2 = 1$, stable spiral

(v) $\mathbf{x}_c(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1/5 \end{pmatrix} \right]$, $c_1 = 1, c_2 = 0$, degenerate stable node

(vi) $\mathbf{x}_c(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 2 \\ -5 \end{pmatrix} e^{-25t}$,
 $c_1 = 1, c_2 = 0$, stable node

(vii) $\mathbf{x}_c(t) = e^t \left[c_1 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \right]$,
 $c_1 = 0, c_2 = 1$, unstable spiral point

2. (i) $\mathbf{x}(t) = e^t \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} \times \left[\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + 3 \begin{pmatrix} \cos t + \sin t \\ \sin t - \cos t \end{pmatrix} \right]$

(ii) $\mathbf{x}(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} \times \left[\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 3t/2 - e^t \\ -e^{2t}/4 + e^{3t}/6 \end{pmatrix} \right]$

(iii)

See part 2 on the next page.

Part 2

1. Find the general solution and find the fundamental matrix of the system.

(i) $x' = x + 2y, y' = -2x + y$

(ii) $x' = x + 2y, y' = 3x + y$

2. Find the general solution of the system

$$x' = 2x + 8y + 2, y' = 4y + 16t$$

3. Find the general solution of the system

$$x' = x + 8y + 2e^{-t}, y' = x - y + 2te^{-t}$$

4. Given that the fundamental matrix of the system

$$x' = 3x + y - 2e^{2t}, y' = -x + y + e^{2t}$$
 is

$$\Phi = \begin{pmatrix} e^{2t} & te^{2t} + e^{2t} \\ -e^{2t} & -te^{2t} \end{pmatrix}$$

- (i) Find the general solution of the system.

- (ii) Find the general solution of the system

$$x' = 3x + y + e^{2t}, y' = -x + y$$

5. Given that the fundamental matrix of the system

$$x' = 2x - 2y - e^{-2t}/t, y' = 8x - 6y + 3e^{-2t}/t$$
 is

$$\Phi = \begin{pmatrix} 1 & 2t + 1 \\ 2 & 4t + 1 \end{pmatrix} e^{-2t}$$

- (i) Find the general solution of the system.

- (ii) Find the general solution of the system

$$x' = 2x - 2y + e^{-2t}, y' = 8x - 6y + e^{-2t}$$

6. Consider the system $x' = -x - y + 1, y' = 2x + y - 2$

- (i) Write the system in the matrix form.

- (ii) Find a fundamental matrix of the system.

- (iii) Find a particular solution of the system and write down its general solution.

7. Given that $\lambda = -1$ is a repeated eigenvalue, with eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, of the matrix

$$\begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix},$$

find the general solution of the homogeneous system

$$x' = -6x + 5y, y' = -5x + 4y$$

Answers to part 2:

1. (i) $x(t) = e^t(c_1 \cos 2t + c_2 \sin 2t),$
 $y(t) = e^t(-c_1 \sin 2t + c_2 \cos 2t)$

(ii) $x(t) = c_1 e^{(1+\sqrt{6})t} + c_2 e^{(1-\sqrt{6})t},$
 $y(t) = c_1(\sqrt{6}/2)e^{(1+\sqrt{6})t} - c_2(\sqrt{6}/2)e^{(1-\sqrt{6})t}$

2. $x(t) = c_1 e^{2t} + 4c_2 e^{4t} + 16t + 11,$
 $y(t) = c_2 e^{4t} - 4t - 1$

3. $x(t) = 4c_1 e^{3t} - 2c_2 e^{-3t} + e^{-t}(1/2 - 2t),$
 $y(t) = c_1 e^{3t} + c_2 e^{-3t} + e^{-t}(t/2 - 5/8)$

4. (i) $x(t) = c_1 e^{2t} + c_2(t+1)e^{2t} - e^{2t}(t^2/2 + 2t),$
 $y(t) = -c_1 e^{2t} - c_2 t e^{2t} + e^{2t}(t^2/2 + t)$

(ii) $x(t) = c_1 e^{2t} + c_2(t+1)e^{2t} + (t^2/2 + 1)e^{2t},$
 $y(t) = -c_1 e^{2t} - c_2 t e^{2t} - t^2/2 e^{2t}$

5. (i) $\mathbf{x}_p(t) = e^{-2t} \begin{pmatrix} 10t - \ln|t|(10t+1) \\ 20t - \ln|t|(20t-3) \end{pmatrix}$

(ii) $\mathbf{x}_p(t) = e^{-2t} \begin{pmatrix} t^2 + t \\ 2t^2 + t \end{pmatrix}$

6. (i) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(ii) $\begin{pmatrix} -\cos t & -\sin t \\ \cos t - \sin t & \sin t + \cos t \end{pmatrix}$

(iii) $\mathbf{x}_p(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

7. $x(t) = c_1 e^{-t} + c_2 e^{-t}t,$
 $y(t) = c_1 e^{-t} + c_2 e^{-t}(t + 1/5)$