MACS 315
Review Problems for Midterm Test 2

1. (a) Write $f(t)$ in terms of the unit step function if:

$$f(t) = \begin{cases} t, & 0 \le t < \pi, \\ \pi, & \pi \le t < 2\pi, \\ 4\pi - t, & 2\pi \le t < 4\pi, \\ 0, & 4\pi \le t \end{cases}$$

   (b) Find the Laplace transform of $f(t)$

2. A spring, with a 16 lbs weight attached to its lower end, is suspended from a support. The weight, at rest, stretches the spring 0.2 ft. The entire spring-weight system is immersed in a liquid that imparts a damping force equal to 20 times the instantaneous velocity of the weight. At time $t > 0$, a force equal to $5 \sin 2t$ lbs is applied to the support. SET UP the initial value problem for the displacement if, at the initial time, the weight was released 0.3 ft below its equilibrium position with the upward velocity of $\frac{2\sqrt{10}}{15}$ft/s. **(Do not solve the IVP)**

3. Use Laplace transforms to solve the initial value problem

$$x'' + 2x' + 2x = 2\delta(t - \pi), \quad x(0) = 0, \quad x'(0) = 0.$$

4. Find the Laplace transform of each of the following functions.

   a)
$$e^t \int_0^t e^{-\tau} \sin \tau \, d\tau.$$

   b) $f(t) = te^{-t} \sin t$

5. Find the inverse Laplace transform for

$$F(s) = \frac{2s + 1}{s(s^2 + 9)}.$$

6. A 6-pound weight stretches a spring 1 foot. The weight is released from a point 1 foot below the equilibrium position with a downward velocity of 1 ft/s.

   i. Find the equation of motion of the weight.
   ii. Determine the amplitude, period and frequency of motion.
   iii. At what times does the weight return to the point where it was released?

7. A 20 pound weight stretches a spring 6 inches. The weight is released from rest 6 inches below the equlibrium position with an upward velocity of 1 ft/s.

   i. Set up the initial value problem for the displacement of the spring.
   ii. Assume now that the entire system is immersed in a liquid that imparts a damping force numerically equal to 5 times the instantaneous velocity. Set up the initial value problem for the displacement of the spring.
   iii. Assume now that in addition to the damping force in (ii), at $t > 0$ a force equal to $10 \cos(3t)$ is applied to to support. Set up the initial value problem for the displacement of the spring.

8. Use the Laplace transfrom to solve each of the following IVP

   i. $y'' + 6y' + 5y = t - \mathcal{U}(t - 2)$, $y(0) = 1$, $y'(0) = 0$.
   ii. $y'' - 2y' = 1 + \delta(t - 2)$, $y(0) = 0$, $y'(0) = 1$.

9. Find the inverse Laplace transform of the the following functions

   i.
$$F(s) = \frac{1}{(s+1)(s-2)}.$$

   ii.
$$F(s) = \frac{3s - 2}{s^2(s+1)^2}.$$

10. Find the Laplace transform of the following functions

   i.
$$f(t) = \int_0^t \cos \tau \sin(t - \tau) \, d\tau.$$

   ii.
$$f(t) = \int_0^t \tau^2 \, (t - \tau) \mathrm{e}^{(t-\tau)} \, d\tau.$$

11. Write the following function in terms of unit step functions and compute its Laplace transform.

$$f(t) = \begin{cases} 1 & 0 \le t < 4, \\ 2 & 4 \le t < 5, \\ 0 & 5 \le t \end{cases}$$

12. A 10-pound weight stretches a spring 3 inches. The weight is released from rest 3 inches below the equilibrium position.

   i. Find the equation of motion.
   ii. Suppose now that there is a damping force numerically equal to twice the instanteneous velocity. Set up the initial value problem (**do not solve it**).

ii. Set up the initial value problem if in addition to the damping force in (ii) the mass is driven by an external force equal to $f(t) = 2\cos t$. (**do not solve IVP**)

13. Use the Laplace transform to solve the following initial value problem:

$$y'' + 4y = \delta(t - 2\pi), \; y(0) = 1, \; y'(0) = 1.$$

14. Find the Inverse Laplace tranform of

$$F(s) = \frac{e^{-\pi s}}{(s+1)(s-2)}$$

15. Find the Laplace transform of the following functions:

i. $f(t) = t\int_0^t \tau^3 e^\tau \, d\tau$

ii.

$$f(t) = \begin{cases} t^2, & 0 \le t < 1, \\ \cos(2\pi t), & 1 \le t < \frac{3}{2}, \\ 0, & \frac{3}{2} \le t < \infty, \end{cases}$$

16. Find the inverse Laplace transform of

$$\frac{s+3}{s^2 - 2s + 6}$$

Answers:

1(a): $t + (\pi - t)U(t-\pi) + (3\pi - t)U(t - 2\pi) - (4\pi - t)U(t - 4\pi)$
1(b): $1/s^2 - e^{-\pi s}/s^2 + e^{-2\pi s}(\pi/s - 1/s^2) + e^{-4\pi s}/s^2$
2: $x'' + 40x' + 160x = 10\sin 2t$, $x(0) = 0.3$, $x'(0) = -2\sqrt{10}/15$
3: $x(t) = 2e^{\pi - t}\sin(t - \pi)U(t - \pi)$
4(a): $1/[(s-1)(s^2+1)]$
4(b): $2(s+1)/((s+1)^2+1)^2$
5: $\frac{1}{9}(1 - \cos 3t + 6\sin 3t)$
6(i): $x(t) = \cos 4\sqrt{2}t + (1/[4\sqrt{2}])\sin 4\sqrt{2}t$
6(ii): amp.$=\sqrt{33/32}$, per.$=\pi/(2\sqrt{2})$, freq.$=2\sqrt{2}/\pi$
6(iii): $n\pi/(2\sqrt{2})$ or $(\pi - 2\phi + 2n\pi)/(4\sqrt{2})$, $n = 0, 1, \ldots$, and $\tan\phi = 4\sqrt{2}$
7(i): $x'' + 64x = 0$, $x(0) = 1/2$, $x'(0) = -1$
7(ii): $x'' + 8x' + 64x = 0$, $x(0) = 1/2$, $x'(0) = -1$
7(iii): $x'' + 8x' + 64x = 16\cos 3t$, $x(0) = 1/2$, $x'(0) = -1$
8(i): $\frac{3}{4}e^{2t} - \frac{t}{2} - \frac{3}{4} + \frac{1}{2}(e^{2(t-2)} - 1)U(t-2)$
8(ii): $0.5(e^{2t} - 1)$
9(i): $\frac{1}{3}(e^{2t} - e^{-t})$
9(ii): $7 - 2t - 7e^{-t} - 5te^{-t}$
10(i): $s/(s^2+1)^2$
10(ii): $2/[s^3(s-1)]$
11: $1 + U(t-4) - 2U(t-5)$, $(1 + e^{-4s})/s - 2e^{-5s}$
12(i): $\frac{5}{16}x'' + 40x = 0$, $x(0) = 1/4$, $x'(0) = 0$
12(ii): $\frac{5}{16}x'' + 2x' + 40x = 0$, $x(0) = 1/4$, $x'(0) = 0$
12(iii): $\frac{5}{16}x'' + 2x' + 40x = 2\cos t$, $x(0) = 1/4$, $x'(0) = 0$
13: $0.5\sin 2t[1 + U(t - 2\pi)] + \cos 2t$
14: $-\frac{1}{3}e^{\pi - t}U(t-\pi) + \frac{1}{3}e^{2(t-\pi)}U(t-\pi)$
15(i): $-\frac{d}{ds}\left(\frac{6}{s(s-1)^4}\right)$
15(ii): $\frac{2}{s^3} + e^{-s}\left(\frac{s}{s^2+4\pi^2} + \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) + e^{-3s/2}\frac{s}{s^2+4\pi^2}$
16: $e^t\cos\sqrt{5}t + \frac{4}{\sqrt{5}}e^t\sin\sqrt{5}t$