Final Exam Review

1. Solve the following

   i. $xy' = (x+1)y + x^2$, $y(0) = 1$

   ii. $y'' - 16y = 2e^{4x}$

   iii. $y'' - y - x = \sin x$, $y(0) = 2$, $y'(0) = 3$

2. When a mass of 2 kg is attached to a spring whose constant is 32 N/m, it comes to rest in the equilibrium position. Starting at $t = 0$, the force $f(t) = 68e^{-2t}\cos 4t$ is applied to the system.

   i. Find the equation of motion in the absence of damping.

   ii. Identify the transient and steady states.

3. Given that $y_1(x) = x^2$ and $y_2(x) = x^3$ are solutions of $x^2y'' - 4xy' + 6y = 0$. Use variation of parameters to find a particular solution of

$$x^2y'' - 4xy' + 6y = 2x^3 + x^2.$$

4. A 20 pound weight stretches a spring 6 inches. The weight is released from rest 6 inches below the equilibrium position with an upward velocity of 1 ft/s.

   i. Set up the initial value problem for the displacement of the spring.

   ii. Assume now that the entire system is immersed in a liquid that imparts a damping force numerically equal to 5 times the instantaneous velocity. Set up the initial value problem for the displacement of the spring.

   iii. Assume now that in addition to the damping force in (ii), at $t > 0$ an upward force equal to $10\cos(3t)$ is applied to to support. Set up the initial value problem for the displacement of the spring.

5. Use the Laplace transform to solve each of the following IVP: $y'' + y = \delta(t - 2\pi)$, $y(0) = 1$, $y'(0) = 0$.

6. Find the inverse Laplace transform of the the following functions

   i. $$F(s) = \frac{1}{s^2 + s - 20}$$

   ii. $$F(s) = \frac{s-1}{s^2(s^2+1)}$$

   iii. $$F(s) = \frac{6s+3}{(s^2+4)(s^2+1)}$$

7. Find the Laplace transform of the following functions $f(t) = t\,e^t\,\mathcal{U}(t-2)$

8. Solve $(1 + x^2)y' + 4xy = 6x$, $y(1) = 1$

9. The values of mass m, spring constant k, dashpot resistance c, and force $f(t)$ are given in a mass-spring-dashpot system with an external forcing function. Solve the initial value problem:

$$mx'' + cx' + kx = f(t), \quad x(0) = 0 = x'(0)$$

given that: $m = 1, k = 4, c = 4, \quad f(t) = t$ if $0 \le t < 0$, $\; f(t) = 0$ if $t \ge 2$.

10. (a) Find the Laplace transform for the following: $e^{-2t}\sin 3t$

    (b) Find the inverse Laplace transform of each of the following:

    i. $$\frac{s}{s^2 + 2s + 10}$$

    ii. $$e^{-\pi s}\frac{1}{(s-2)^2}$$

11. Using the method of undetermined coefficients, find the general solution of $y'' + 4y' + 4y = 10e^{-2x}$.

12. Solve the initial value problem

$$xy' + 2y = e^x, \quad y(1) = 2.$$

13. Let $P(t)$ denote the population (in millions) of ladybugs in a forest at any time t (in years). The population changes according to the initial value problem

$$P' = (1/40)P(50 - P), \quad P(0) = 10.$$

Determine how many ladybugs there will be in the forest after 4 years, and find $\lim_{t\to\infty} P(t)$.

14. An 8 pound weight attached to a spring stretches it 16 feet. The weight is released from its equilibrium position with an upward velocity of 2 ft/s. The medium through which the weight moves offers a resistance numerically equal to 0.5 times the instantaneous velocity. Determine when the weight first attains its extreme displacement from its equilibrium position.

15. Use the Laplace transform method to solve the initial value problem

$$y'' + 9y = 3\delta(t - \pi), \quad y(0) = 1, \quad y'(0) = 0.$$

16. (a) Find the Laplace transform for the following: $te^{-t}\sin(2t)$

    (b) Find the inverse Laplace transform of each of the following:

    i. $$\frac{3s+2}{s^2+2s+5}$$

    ii. $$\frac{e^{-\pi s/2} + 1}{s(s+2)}$$

17. Find the general solution of $y'' + 3y' + 2y = 6xe^x$.

Answers:

1(i): $y = x(e^{x-1} - 1)$

1(ii): $y = c_1e^{4x} + c_2e^{-4x} + (1/4)xe^{4x}$

1(iii): $y = (13/4)e^x - (5/4)e^{-x} - x - (1/2)\sin x$

2(i): $x = c_1\cos 4t + c_2\sin 4t + e^{-2t}((1/2)\cos 4t - 2\sin 4t)$

2(ii): Steady state is $c_1\cos 4t + c_2\sin 4t$, transient is $e^{-2t}((1/2)\cos 4t - 2\sin 4t)$

3: $y = c_1x^2 + c_2x^3 - 2x^3 - x^2 + \ln|x|(2x^3 - x^2)$

4(i): $x'' + 64x = 0$, $x(0) = 1/2$, $x'(0) = 1$

4(ii): $x'' + 8x' + 64x = 0$, $x(0) = 1/2$, $x'(0) = 1$

4(iii): $x'' + 8x' + 64x = 16\cos 3t,\ x(0) = 1/2,\ x'(0) = 1$
5: $y = \cos t - U(t-\pi)\ \sin t$
6(i): $f = (e^{4t} - e^{-5t})/9$
6(ii): $f = 1 - t - \cos t + \sin t$
6(ii): $f = -2\cos 2t - (1/2)\sin 2t + 2\cos t + \sin t$
7: $F = e^{-2(s+1)}((s+1)^{-2} + 2(s+1)^{-1})$
8: $y = (3x^2 + 1.5x^4 - 0.5)/(x^2+1)^2$
9: $x = (t - 1 + e^{-2t}[1+t])/4$
$\quad +U(t-2)(t - 1 - e^{-2(t-2)}[3t-7])/4$
10(a): $3/([s+2]^2 + 9)$
10(b,i): $e^{-t}[\cos 3t - (1/3)\sin 3t]$
10(b,ii): $(t-\pi)e^{2(t-\pi)}U(t-\pi)$
11: $y = c_1 e^{-2x} + c_2 x e^{-2x} + (5/4)(x-1) + 5x^2 e^{-2x}$
12: $y = (xe^x - e^x + 2)/x^2$
13: $p(4) = 50/(4e^{-5}+1),\ p(\infty) = 50$
14: $t = \pi/4$
15:
16(a): $4(s+1)/[(s+1)^2+4]^2$
16(b,i): $3e^{-t}\cos 2t - 0.5e^{-t}\sin 2t$
16(b,ii): $(1-e^{-2t})/2$
17: $y = c_1 e^{-x} + c_2 e^{-2x} + e^x(x - 5/6)$