Test 3

Problem	1	2	3	4	5	Grade
Points	/20	/20	/20	/20	/20	/100

NAME: Solution key

Show all work for full credit. Textbook, crib sheets, or notes are not allowed.

Problem 1. Diagonalize the following matrix if possible:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 5 & -6 \\ 0 & 3 & -4 \end{pmatrix}. \quad p_A(t) = \begin{vmatrix} 1-t & 1 & -1 \\ 0 & 5-t & -6 \\ 0 & 3 & -4-t \end{vmatrix}$$

$$= (1-t)\begin{vmatrix} 5-t & -6 \\ 3 & -4-t \end{vmatrix} = (1-t)\left[(t-5)(t+4)+18\right]$$

$$= (1-t)(t^2-t-2) = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 2$$

3×3 matrix A has 3 distinct eigenvalues; A is diagonalizable.

Find eigenvectors.

$\lambda_1 = -1 \Rightarrow A + I = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 6 & -6 \\ 0 & 3 & -3 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{cases} x_3 = t \\ x_2 = t \\ x_1 = 0 \end{cases} \Rightarrow v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$

$\lambda_2 = 1 \Rightarrow A - I = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 4 & -6 \\ 0 & 3 & -5 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$x_3 = x_2 = 0,\ x_1 = 1 \Rightarrow v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$

$\lambda_3 = 2$ $\quad A - 2I = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 3 & -6 \\ 0 & 3 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{cases} x_3 = t \\ x_2 = 2t \\ x_1 = t \end{cases} \Rightarrow v_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$

Let $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}.$

Then $A = VDV^{-1}.$

Problem 2. Diagonalize the following matrix if possible:

$$A = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}. \qquad p_A(t) = \begin{vmatrix} 2-t & 1 & -1 \\ -1 & 3-t & -1 \\ -1 & 1 & 1-t \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 2-t+1 & (1-t)(2-t)+(-1) \\ 0 & 2-t & t-2 \\ -1 & 1 & 1-t \end{vmatrix} = \begin{vmatrix} 3-t & t^2-3t+1 \\ 2-t & t-2 \end{vmatrix}$$

$$= (t-2)\begin{vmatrix} 3-t & t^2-3t+1 \\ -1 & 1 \end{vmatrix} = (t-2)\left[(3-t)+t^2-3t+1\right]$$

$$= (t-2)(t^2-4t+4) = (t-2)^3, \implies \lambda = 2, \text{ multiplicity } 3.$$

$$A - 2I = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{Only 1 free variable.}$$

$$\Rightarrow \dim(E_{\lambda=2}) = 1 < 3.$$

Matrix A is <u>NOT diagonalizable</u>.
The geometric multiplicity of $\lambda=2$ is less that its algebraic multiplicity.

Problem 3. Let $T : P^2 \to P^3$ be a transformation defined by

$$T(p(t)) = \int_0^t (p(t) + 2p'(t))dt.$$

1. Prove that T is a linear transformation.

1. $T(cp(t) + q(t)) = \int [cp(t) + q(t) + 2(cp'(t) + q'(t))] dt$

$= \int [cp(t) + c2p'(t) + q(t) + 2q'(t)] dt$

$= c \int (p(t) + 2p'(t))dt + \int (q(t) + 2q'(t))dt$

$= c\, T(p(t)) + T(q(t))$

$\Rightarrow T$ is a linear transformation.

2. Find the matrix of T relative to the standard bases both in the domain and the co-domain.

Let $\beta = \{1, t, t^2\}$ and $\gamma = \{1, t, t^2, t^3\}$ be bases (standard) for P_2 and P_3, respectively.

$[T]_\beta^\gamma = [[T(1)]_\gamma, [T(t)]_\gamma, [T(t^2)]_\gamma]$

$T(1) = \int dt = t + \cancel{c}^{\,0} = t.$

$T(t) = \int (t+2)dt = \frac{t^2}{2} + 2t + \cancel{c_1}^{\,0} = \frac{t^2}{2} + 2t$

$T(t^2) = \int (t^2 + 2 \cdot 2t)dt = \frac{t^3}{3} + 4\frac{t^2}{2} + \cancel{c_2}^{\,0} = \frac{t^3}{3} + 2t^2.$

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1/2 & 2 \\ 0 & 0 & 1/3 \end{pmatrix}.$$

Problem 4. Let $A = \begin{pmatrix} 3 & -2 \\ 5 & 1 \end{pmatrix}$.

1. Diagonalize matrix A if possible.

$$P_A(t) = \begin{vmatrix} 3-t & -2 \\ 5 & 1-t \end{vmatrix} = (3-t)(1-t)+10 = t^2-4t+13=0$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16-52}}{2} = 2 \pm \frac{\sqrt{-36}}{2} = 2 \pm 3i \Rightarrow \text{distinct eigenvalues}$$
matrix is diagonalizable.

$\lambda_1 = 2+3i$

$$A - \lambda I = \begin{pmatrix} 3-2-3i & -2 \\ 5 & 1-2-3i \end{pmatrix} = \begin{pmatrix} 1-3i & -2 \\ 5 & -1-3i \end{pmatrix} \Rightarrow x_1 = \frac{(1+3i)x_2}{5}$$

Let $x_2 = 5$. Then $x_1 = 1+3i$. $\Rightarrow v_1 = \begin{pmatrix} 1+3i \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + i \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

$\Rightarrow v_2 = \begin{pmatrix} 1-3i \\ 5 \end{pmatrix} \Rightarrow A = VDV^{-1}$, where

$$D = \begin{pmatrix} 2+3i & 0 \\ 0 & 2-3i \end{pmatrix}, \quad V = \begin{pmatrix} 1+3i & 1-3i \\ 5 & 5 \end{pmatrix}.$$

2. Factor matrix A in the form VSV^{-1}, where V is invertible and matrix S has the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Theorem: If $\lambda = a - bi$ is an eigenvalue with an eig. vector v, then $A = VSV^{-1}$, where

$$V = [\mathrm{Re}(v), \mathrm{Im}(v)], \quad S = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

$$\Rightarrow S = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & -3 \\ 5 & 0 \end{pmatrix}$$ since $\mathrm{Re}\begin{pmatrix} 1-3i \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

$\mathrm{Im}\begin{pmatrix} 1-3i \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$.

and $A = VSV^{-1}$.

Problem 5. Let $S = \{(4,3,2,1),\ (1,2,3,4)\} \subset R^4$.

1. Find a basis for the orthogonal complement of the set S. $(Col(A))^{\perp} = Nul(A^T)$.

Let $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \\ 2 & 3 \\ 1 & 4 \end{pmatrix}$. Solve $A^T x = 0$.

$Span(S) = Col(A)$.

$$A^T = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -5 & -10 & -15 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$\Rightarrow x_3, x_4$ - free, x_1, x_2 - basic.

$$\begin{cases} x_4 = t \\ x_3 = s \\ x_2 = -2s - 3t \\ x_1 = s + 2t \end{cases} \Rightarrow x = s\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

Basis for $S^{\perp}$ is $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$.

2. Specify the dimensions of Span(S) and $S^{\perp}$. Explain your answer.

Vectors $(4,3,2,1)$ and $(1,2,3,4)$ are linearly independent

$\Rightarrow \dim(Span(S)) = 2$.

$\dim(S^{\perp}) = 2 = \#$ of vectors in the basis found in part 1.