Test 4

Problem	1	2	3	4	5	Grade
Points	/20	/20	/20	/10	/20	/90

NAME: Solution Key

Show all your work for full credit. You may use your note card or a calculator.

Problem 1. Let $y = (1, 0, 1, 0)$, $S = \{(-1, 0, 2, 1),\ (-1, 3, -1, 1)\}$, and $W = \text{span}(S)$.

1. Find the orthogonal projection of vector y on the subspace W of R^4.

Let $v_1 = (-1,0,2,1)$ and $v_2 = (-1,3,-1,1)$.

Check orthogonality: $v_1 \cdot v_2 = 1-2+1=0 \Rightarrow S$ is an orthogonal set.

$$\hat{y} = \text{proj}_W y = \frac{y\cdot v_1}{\|v_1\|^2} v_1 + \frac{y \cdot v_2}{\|v_2\|^2} v_2$$

$$= \frac{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\2\\1\end{pmatrix}}{1^2+2^2+1^2}\cdot\begin{pmatrix}-1\\0\\2\\1\end{pmatrix} + \frac{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\-1\\1\end{pmatrix}}{1+9+1+1}\begin{pmatrix}-1\\3\\-1\\1\end{pmatrix} = \frac{-1+2}{6}\begin{pmatrix}-1\\0\\2\\1\end{pmatrix} + \frac{-1-1}{12}\begin{pmatrix}-1\\3\\-1\\1\end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix}-1-(-1)\\0-3\\2-(-1)\\1-1\end{pmatrix} = \frac{1}{6}\begin{pmatrix}0\\-3\\3\\0\end{pmatrix} = \begin{pmatrix}0\\-1/2\\1/2\\0\end{pmatrix}.$$

2. Find the distance from vector y to subspace W.

$$z = y - \hat{y} = \begin{pmatrix}1\\0\\1\\0\end{pmatrix} - \begin{pmatrix}0\\-1/2\\1/2\\0\end{pmatrix} = \begin{pmatrix}1\\1/2\\1/2\\0\end{pmatrix}$$

$$\|z\|^2 = 1 + \frac{1}{4} + \frac{1}{4} = \frac{6}{4} \Rightarrow \|y-\hat{y}\| = \sqrt{\frac{6}{4}} = \frac{\sqrt{6}}{2}.$$

Problem 2. The set $\beta = \{\underset{v_1}{(1,0,1)},\ \underset{v_2}{(1,2,1)},\ \underset{v_3}{(0,1,3)}\}$ is a basis for R^3.

1. Find an orthonormal basis γ for R^3 by orthonormalizing set β. Hint: first, apply the Gram–Schmidt process, then normalize the vectors.

$$q_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$q_2 = v_2 - \frac{v_2 \cdot q_1}{\|q_1\|^2} q_1 = \begin{pmatrix}1\\2\\1\end{pmatrix} - \frac{\begin{pmatrix}1\\2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}}{1^2+1^2}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix}.$$

Let $q_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ – (scale).

$$q_3 = v_3 - \frac{v_3 \cdot q_1}{\|q_1\|^2} q_1 - \frac{v_3 \cdot q_2}{\|q_2\|^2} q_2 = \begin{pmatrix}0\\1\\3\end{pmatrix} - \frac{\begin{pmatrix}0\\1\\3\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{\begin{pmatrix}0\\1\\3\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}}{1}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\\3\end{pmatrix} - \frac{3}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\3\end{pmatrix} - \frac{3}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}-3/2\\0\\3/2\end{pmatrix}.$$

Let $q_3 = \begin{pmatrix}-1\\0\\1\end{pmatrix}$ – (scale).

<u>Normalize:</u>

$\|q_1\| = \sqrt{1^2+1^2} = \sqrt{2} \Rightarrow \hat{q}_1 = \begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}$

$\|q_2\| = 1 \Rightarrow \hat{q}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$\|q_3\| = \sqrt{1^2+1^2} = \sqrt{2} \Rightarrow \hat{q}_3 = \begin{pmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}$

$$\Rightarrow \gamma = \left\{\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}-\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix}\right\}.$$

2. Find the change of coordinate matrix from γ-basis to the standard basis for R^3.

$$P = P_{\text{standard} \leftarrow \gamma} = \begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix} \text{ – orthogonal.}$$

$\varepsilon = \{e_1, e_2, e_3\}$ – the standard basis for R^3

$$P^{-1} = P^T = \begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix} = P_{\gamma \leftarrow \varepsilon}$$

3. Find the γ-coordinate vector of $y = (\sqrt{8}, 2, -\sqrt{8})$. $= [y]_\varepsilon$

$$[y]_\gamma = P_{\gamma \leftarrow \varepsilon}[y]_\varepsilon$$

$$= \begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\sqrt{8}\\2\\-\sqrt{8}\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{8}}{\sqrt{2}} + 0 - \frac{\sqrt{8}}{\sqrt{2}}\\ 0 + 2 + 0\\ -\frac{\sqrt{8}}{\sqrt{2}} + 0 - \frac{\sqrt{8}}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}2-2\\2\\-2-2\end{pmatrix} = \begin{pmatrix}0\\2\\-4\end{pmatrix}.$$

Problem 3. Let $S = \{(1,0,3,-4),\ (2,1,4,-6)\}$.

1. Find a basis for $S^\perp$. Hint: use the relation $\mathrm{col}(A)^\perp = \mathrm{Null}(A^T)$ for $A \in R^{m\times n}$.

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \\ -4 & -6 \end{pmatrix}$. $\mathrm{span}(S) = \mathrm{Col}(A)$

$S^\perp = \mathrm{span}(S)^\perp = \mathrm{Col}(A)^\perp = \mathrm{Null}(A^T)$.

Solve $A^T x = 0$

$\begin{pmatrix} 1 & 0 & 3 & -4 \\ 2 & 1 & 4 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & -4 \\ 0 & 1 & -2 & 2 \end{pmatrix} \Rightarrow$ x_1, x_2 – basic vars, x_3, x_4 – free vars.

$x_4 = t$

$x_3 = s$

$x_2 = 2x_3 - 2x_4 = 2s - 2t$

$x_1 = -3x_3 + 4x_4 = -3s + 4t$

$\Rightarrow \bar{x} = s\begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 4 \\ -2 \\ 0 \\ 1 \end{pmatrix}$.

Basis for $S^\perp$ is $\{(-3,2,1,0), (4,-2,0,1)\}$.

2. Show that the basis vectors are orthogonal to vectors in set S.

Let $v_1 = (1,0,3,-4)$, $v_2 = (2,1,4,-6)$,

$w_1 = (-3,2,1,0)$, $w_2 = (4,-2,0,1)$.

$w_1 \cdot v_1 = -3+3 = 0$ $\quad w_2 \cdot v_1 = 4-4 = 0$

$w_1 \cdot v_2 = -6+2+4 = 0$ $\quad w_2 \cdot v_2 = 8-2-6 = 0$.

Problem 4. Determine if the matrix is orthogonally diagonalizable? Explain why.

1. $$A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 4 & 7 \\ -2 & 7 & 0 \end{pmatrix}$$

Matrix $A \in R^{n\times n}$ is orthogonally diagonalizable if and only if A is symmetric; that is, $A = A^T$.

$A^T = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 4 & 7 \\ 2 & 7 & 0 \end{pmatrix} \neq A \Rightarrow A$ is NOT Diagonalizable orthogonally.

2. $$B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = B^T \Rightarrow B \text{ is orthogonally diagonalizable.}$$

Problem 5. Orthogonally diagonalize the following matrix whose eigenvalues are $\lambda_1 = -6$ and $\lambda_2 = 3$:

$$A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{pmatrix}.$$

Find bases for eigenspaces.

$\lambda_1 = -6.$

$$A + 6I = \begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix} \to \begin{pmatrix} 2 & 5 & 4 \\ 8 & 2 & -2 \\ -2 & 4 & 5 \end{pmatrix} \to \begin{pmatrix} 2 & 5 & 4 \\ 0 & -18 & -18 \\ 0 & 9 & 9 \end{pmatrix} \to \begin{pmatrix} 2 & 5 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} x_3 &= t \\ x_2 &= -t \\ x_1 &= t/2 \end{aligned} \implies x = t\begin{pmatrix} 1/2 \\ -1 \\ 1 \end{pmatrix}. \quad \text{let } t = 2 \implies \bar{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$$\|v_1\| = \sqrt{1+4+4} = 3. \quad \hat{v}_1 = \frac{1}{\|v_1\|} v_1 = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}.$$

$\lambda_2 = 3$

$$A - 3I = \begin{pmatrix} -1 & 2 & -2 \\ 2 & -4 & 4 \\ -2 & 4 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \implies x_1 = 2x_2 - 2x_3 \qquad \begin{aligned} x_3 &= 0 \\ x_2 &= 1 \\ x_1 &= 2 \end{aligned} \implies v_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{aligned} x_2 &= 0 \\ x_3 &= 1 \\ x_1 &= -2 \end{aligned} \implies v_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

Use Gram-Schmidt to orthogonalize $\{\bar{v}_2, \bar{v}_3\}$.

$q_2 = v_2$

$$q_3 = v_3 - \frac{v_3 \cdot q_2}{\|q_2\|^2} q_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}}{5} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \frac{4}{5}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}.$$

let $q_3 = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}.$

Normalize: $\hat{q}_2 = \frac{1}{\|q_2\|} q_2 = \frac{1}{\sqrt{4+1}} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{pmatrix}$

$$\hat{q}_3 = \frac{1}{\|q_3\|} q_3 = \frac{1}{\sqrt{4+16+25}} \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} = \frac{1}{3\sqrt{5}} \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}.$$

Thus,

$$A = QDQ^T, \quad D = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad Q = \begin{pmatrix} 1/3 & 2/\sqrt{5} & -\frac{2}{3\sqrt{5}} \\ -2/3 & 1/\sqrt{5} & \frac{4}{3\sqrt{5}} \\ 2/3 & 0 & \frac{\sqrt{5}}{3} \end{pmatrix}.$$