Test 2

Problem	1	2	3	4	5	6	Grade
Points	/10	/10	/10	/10	/10	/10	/60

NAME: Solution Key

Show all your work for the full credit. Crib sheets and notes are not allowed.

Problem 1. Solve the linear system $Ax = b$ using Cramer's rule, where

1. $$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 7 \\ 8 \\ 1 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 3 & 0 \\ 0 & -1 & 1 \end{vmatrix} \overset{R_2 \leftarrow R_2 - 2R_1}{=} \begin{vmatrix} 1 & 0 & 2 \\ 0 & 3 & -4 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ -1 & 1 \end{vmatrix} = 3 - 4 = -1$$

$$|A_1| = \begin{vmatrix} 7 & 0 & 2 \\ 8 & 3 & 0 \\ 1 & -1 & 1 \end{vmatrix} \underset{R_1 \leftarrow R_1 - 2R_2}{=} \begin{vmatrix} 5 & 2 & 0 \\ 8 & 3 & 0 \\ 1 & -1 & 1 \end{vmatrix} = (-1)^6 \cdot 1 \begin{vmatrix} 5 & 2 \\ 8 & 3 \end{vmatrix} = 15 - 16 = -1$$

$$|A_2| = \begin{vmatrix} 1 & 7 & 2 \\ 2 & 8 & 0 \\ 0 & 1 & 1 \end{vmatrix} \underset{R_2 \leftarrow R_2 - 2R_1}{=} \begin{vmatrix} 1 & 7 & 2 \\ 0 & -6 & -4 \\ 0 & 1 & 1 \end{vmatrix} = (-1)^2 \cdot 1 \begin{vmatrix} -6 & -4 \\ 1 & 1 \end{vmatrix} = -6 + 4 = -2$$

$$|A_3| = \begin{vmatrix} 1 & 0 & 7 \\ 2 & 3 & 8 \\ 0 & -1 & 1 \end{vmatrix} \underset{R_2 \leftarrow R_2 - 2R_1}{=} \begin{vmatrix} 1 & 0 & 7 \\ 0 & 3 & -6 \\ 0 & -1 & 1 \end{vmatrix} = (-1)^2 \cdot 1 \begin{vmatrix} 3 & -6 \\ -1 & 1 \end{vmatrix} = 3 - 6 = -3.$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{-1}{-1} = 1, \quad x_2 = \frac{|A_2|}{|A|} = \frac{-2}{-1} = 2, \quad x_3 = \frac{|A_3|}{|A|} = \frac{-3}{-1} = 3$$

$$\boxed{x = (1, 2, 3).}$$

Problem 2. Determine if the following linear transformation is invertible, and, if it is invertible, then find the formula for the inverse transformation:

$$T(x, y, z) = (x + 3y + 3z,\ x + 4y + 3z,\ x + 3y + 4z).$$

Find the standard matrix of T: $[T] = [T(e_1), T(e_2), T(e_3)]$

$$[T] = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$$

Find $[T]^{-1}$:

$$([T] \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1 \end{array}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 4 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) R_1 - 3R_3 \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) R_1 - 3R_2$$

T is invertible since $[T]$ is invertible.

$$[T]^{-1} = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$T^{-1}(x, y, z) = [T]^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7x - 3y - 3z \\ -x + y \\ -x + z \end{pmatrix}.$$

$$\boxed{T^{-1}(x, y, z) = (7x - 3y - 3z,\ -x + y,\ -x + z)}.$$

Problem 3. Let

$$A = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & +1 \\ 5 & 2 \end{pmatrix}.$$

Compute

1. $A^{-1} + 2B^{-1}$

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} = \frac{1}{-3+4}\begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}$$

$$B^{-1} = \frac{1}{6-5}\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$$

$$A^{-1} + 2B^{-1} = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} + 2\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} + \begin{pmatrix} 4 & -2 \\ -10 & 6 \end{pmatrix}$$

$$\boxed{A^{-1} + 2B^{-1} = \begin{pmatrix} 5 & -4 \\ -8 & 3 \end{pmatrix}}$$

2. $(AB)^{-1}$ without computing AB

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 2\cdot 1 - 1\cdot 2 & 2(-2) - 1\cdot(-3) \\ -5\cdot 1 + 3\cdot 2 & -5(-2) + 3(-3) \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}}$$

Problem 4. Compute the determinant of

$$A = \begin{pmatrix} 0 & 2 & 0 & 4 \\ -1 & 4 & 3 & 3 \\ 0 & 5 & 0 & -2 \\ 6 & -3 & 0 & 1 \end{pmatrix}$$

using:

1. Cofactor expansion.

$$\begin{vmatrix} 0 & 2 & 0 & 4 \\ -1 & 4 & 3 & 3 \\ 0 & 5 & 0 & -2 \\ 6 & -3 & 0 & 1 \end{vmatrix} = (-1)^{2+3} \cdot 3 \begin{vmatrix} 0 & 2 & 4 \\ 0 & 5 & -2 \\ 6 & -3 & 1 \end{vmatrix} = -3 \cdot (-1)^{3+1} \cdot 6 \begin{vmatrix} 2 & 4 \\ 5 & -2 \end{vmatrix}$$

$$= -18(-4-20) = 18 \cdot 24 = 432.$$

2. Row-reduction.

$$\begin{vmatrix} 0 & 2 & 0 & 4 \\ -1 & 4 & 3 & 3 \\ 0 & 5 & 0 & -2 \\ 6 & -3 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & 4 & 3 & 3 \\ 0 & 2 & 0 & 4 \\ 0 & 5 & 0 & -2 \\ 6 & -3 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & 4 & 3 & 3 \\ 0 & 2 & 0 & 4 \\ 0 & 5 & 0 & -2 \\ 0 & 21 & 18 & 19 \end{vmatrix}$$

$$= -2 \begin{vmatrix} -1 & 4 & 3 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 5 & 0 & -2 \\ 0 & 21 & 18 & 19 \end{vmatrix} = -2 \begin{vmatrix} -1 & 4 & 3 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -12 \\ 0 & 0 & 18 & -23 \end{vmatrix} = +2 \begin{vmatrix} -1 & 4 & 3 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 18 & -23 \\ 0 & 0 & 0 & -12 \end{vmatrix}$$

$$= +2\,(-1) \cdot 1 \cdot 18 \cdot (-12) = 36 \cdot 12 = 432.$$

Problem 5. Let G be the parallelepiped with one vertex at the origin $O = (0, 0, 0)$, and with other three vertices that are connected to the origin by an edge at the following points: $(1, 2, 3)$, $(0, -1, 3)$, $(2, -2, 0)$.

1. Find the volume of G.

$$\bar{a}_1 = \begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \quad \bar{a}_2 = \begin{pmatrix}0\\-1\\3\end{pmatrix} - \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\-1\\3\end{pmatrix}, \quad \bar{a}_3 = \begin{pmatrix}2\\-2\\0\end{pmatrix} - \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}2\\-2\\0\end{pmatrix}$$

$$\det(\bar{a}_1, \bar{a}_2, \bar{a}_3) = \begin{vmatrix}1 & 0 & 2\\ 2 & -1 & -2\\ 3 & 3 & 0\end{vmatrix} \leftarrow = \begin{vmatrix}-1 & -2\\ 3 & 0\end{vmatrix} + 2\begin{vmatrix}2 & -1\\ 3 & 3\end{vmatrix} = 6 + 2(6+3)$$

$$= 6 + 2 \cdot 9 = 6 + 18 = 24.$$

$$\text{Volume}(G) = 24.$$

2. Find the volume of the image of parallelepiped G under the transformation $T(x) = Ax$, where

$$A = \begin{pmatrix} 2 & -11 & 99 \\ 0 & 3 & 76 \\ 0 & 0 & 5 \end{pmatrix}$$

$$\text{Volume}(T(G)) = \text{Volume}(G) \cdot |\det(A)| = 24 \cdot 2 \cdot 3 \cdot 5$$

$$= 720.$$

Problem 6. Let

$$A=(A_{rs})_{3\times 3}=\begin{array}{c}\begin{array}{cccc} n_1 & \quad n_2 & & \quad n_3\end{array}\\ \left(\begin{array}{c|cc|c} a_{11} & a_{12} & a_{13} & a_{14}\\ \hline a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ \hline a_{41} & a_{42} & a_{43} & a_{44}\end{array}\right)\end{array} \quad \text{and} \quad B=(B_{rs})_{3\times 1}=\left(\begin{array}{cccc} b_{11} & b_{12} & b_{13} & b_{14}\\ \hline b_{21} & b_{22} & b_{23} & b_{24}\\ b_{31} & b_{32} & b_{33} & b_{34}\\ \hline b_{41} & b_{42} & b_{43} & b_{44}\end{array}\right)\begin{array}{c} m_1\\ m_2\\ \\ m_3\end{array}$$

1. Explain why the given matrix partitionings are compliable for block matrix multiplication of AB?

Column block-partitioning of A matches the row block-partitioning of B.

$n_1=1$, $n_2=2$, $n_3=1$, $n_1=m_1$, $n_2=m_2$, $n_3=m_3$.

$A_{11}=(a_{11})$, $A_{12}=(a_{12},a_{13})$, $A_{13}=(a_{14})$

$A_{21}=\begin{pmatrix}a_{21}\\a_{31}\end{pmatrix}$, $A_{22}=\begin{pmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{pmatrix}$, $A_{23}=\begin{pmatrix}a_{24}\\a_{34}\end{pmatrix}$

$A_{31}=(a_{41})$, $A_{32}=(a_{42},a_{43})$, $A_{33}=(a_{44})$

$B_{11}=(b_{11},b_{12},b_{13},b_{14})$

$B_{21}=\begin{pmatrix}b_{21} & b_{22} & b_{23} & b_{24}\\ b_{31} & b_{32} & b_{33} & b_{34}\end{pmatrix}$

$B_{31}=(b_{41}\ b_{42}\ b_{43}\ b_{44})$

2. Block multiply $A=(A_{rs})$ and $B=(B_{rs})$.

$$A=\begin{pmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}\begin{pmatrix}B_1\\B_2\\B_3\end{pmatrix}=\begin{pmatrix}A_{11}B_1+A_{12}B_2+A_{13}B_3\\ A_{21}B_1+A_{22}B_2+A_{23}B_3\\ A_{31}B_1+A_{32}B_2+A_{33}B_3\end{pmatrix}.$$