Problem 6. Use the normal equation to find the least squares best fit straight line on the plane to the data points

$(-1,1),\ (1,1),\ (2,3).$

x	-1	1	2
y	1	1	3

$\Rightarrow \quad A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \ \bar{y} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \ \bar{x} = \begin{pmatrix} c \\ d \end{pmatrix}$

Solve $\min_{x \in \mathbb{R}^2} \|A\bar{x} - \bar{y}\|$. Then $y(x) = c + dx$.

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}.$$

$$A^T \bar{y} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

Solve $A^T A \bar{x} = A^T y$.

$$\left[\begin{array}{cc|c} 3 & 2 & 5 \\ 2 & 6 & 6 \end{array}\right] \to \begin{bmatrix} 1 & 3 & 3 \\ 3 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 \\ 0 & -7 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & \frac{4}{7} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 3 + (-3)\frac{4}{7} \\ 0 & 1 & \frac{4}{7} \end{bmatrix} \to \left[\begin{array}{cc|c} 1 & 0 & 3 \cdot \frac{3}{7} \\ 0 & 1 & 4/7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 9/7 \\ 0 & 1 & 4/7 \end{array}\right].$$

$$\Rightarrow \bar{x} = \begin{pmatrix} 9/7 \\ 4/7 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \Rightarrow \boxed{y(x) = \frac{9}{7} + \frac{4}{7}x}$$