

Problem 4.

1. Find the change of coordinate matrix from basis
- $C = \{c_1, c_2\}$
- to basis
- $B = \{b_1, b_2\}$
- if

$$c_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 7 \\ -2 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Let $E = \{e_1, e_2\}$, where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$P_{E \leftarrow C} = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}, \quad P_{E \leftarrow B} = \begin{pmatrix} 7 & 2 \\ -2 & -1 \end{pmatrix}$$

$$P_{B \leftarrow C} = P_{B \leftarrow E} P_{E \leftarrow C} = P_{E \leftarrow B}^{-1} P_{E \leftarrow C}.$$

$$P_{E \leftarrow B}^{-1} = \begin{pmatrix} 7 & 2 \\ -2 & -1 \end{pmatrix}^{-1} = \frac{1}{-7+4} \begin{pmatrix} -1 & -2 \\ 2 & 7 \end{pmatrix} = -\frac{1}{3} \begin{pmatrix} -1 & -2 \\ 2 & 7 \end{pmatrix}$$

$$\begin{aligned} P_{B \leftarrow C} &= -\frac{1}{3} \begin{pmatrix} -1 & -2 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix} = -\frac{1}{3} \begin{pmatrix} -4-2 & -5-4 \\ 8+7 & 10+14 \end{pmatrix} \\ &= -\frac{1}{3} \begin{pmatrix} -6 & -9 \\ 15 & 24 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -5 & -8 \end{pmatrix}. \end{aligned}$$

2. Find the
- C
- coordinate vector of vector
- $x = (2, -1)^T$
- .

$$[x]_C = P_{C \leftarrow E} x \Rightarrow P_{E \leftarrow C} [x]_C = x$$

$$\begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix} [x]_C = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \leftarrow \text{Solve.}$$

$$\begin{pmatrix} 4 & 5 & 2 \\ 1 & 2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 4 & 5 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{pmatrix} \Rightarrow [x]_C = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

$$\text{Verify: } 3 \cdot \begin{pmatrix} 4 \\ 1 \end{pmatrix} - 2 \cdot \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 12-10 \\ 3-4 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \equiv x. \quad \checkmark$$