Problem 2. Is the set B a basis for P_2? Explain.

1. $B = \{1+t^2,\ 2-t+3t^2,\ 1+2t-4t^2\}$

dim$(P_2)=3$. The set B consists of 3 vectors.

The set B is a basis for P_2 if the set is linearly independent. Consider

$$c_1(1+t^2)+c_2(2-t+3t^2)+c_3(1+2t-4t^2)=0$$

$$(c_1+2c_2+c_3)+t(-c_2+2c_3)+t^2(c_1+3c_2-4c_3)=0,\ \forall t\in\mathbb{R}$$

$$\begin{cases} c_1+2c_2+c_3=0 \\ -c_2+2c_3=0 \\ c_1+3c_2-4c_3=0 \end{cases} \quad \text{Solve.} \quad \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -5 \end{pmatrix} R_3-R_1$$

$$\to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

All columns are pivot. The system has only the trivial solution $c_1=c_2=c_3=0$.

The set B is linearly independent.

The set B is a basis for P_2.

2. $B = \{-2+4t-2t^2,\ 2-t+3t^2,\ 1+t+2t^2\}$

Similarly, we obtain

$$\begin{cases} -2c_1+2c_2+c_3=0 \\ 4c_1-c_2+c_3=0 \\ -2c_1+3c_2+2c_3=0. \end{cases} \quad \begin{pmatrix} -2 & 2 & 1 \\ 4 & -1 & 1 \\ -2 & 3 & 2 \end{pmatrix} = A.$$

$$\det(A)=\begin{vmatrix} -2 & 2 & 1 \\ 6 & -3 & 0 \\ 2 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 6 & -3 \\ 2 & -1 \end{vmatrix} = -6+6=0.$$

Matrix A is singular. The linear system has a nontrivial solution.

Vectors are linearly dependent.

The set B is not a basis for P_2.