Problem 5. Let

$$S = \left\{ \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 7 \\ -4 \\ -1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} -8 \\ 1 \\ -6 \end{bmatrix} \right\} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -5 \\ 0 \\ -5 \end{bmatrix}.$$

1. Determine if vector $\mathbf{b}$ belongs to the span of vectors in set S? If yes, then express $\mathbf{b}$ as a linear combination of vectors in set S.

Let $A = [v_1, v_2, v_3, v_4]$. Determine if $Ax = b$ is consistent.

$$[A|b] = \begin{bmatrix} 2 & -1 & 7 & -8 & -5 \\ 1 & 2 & -4 & 1 & 0 \\ 4 & 3 & -1 & -6 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -4 & 1 & 0 \\ 0 & -5 & 15 & -10 & -5 \\ 0 & -5 & 15 & -10 & -5 \end{bmatrix} R_2$$

$$\to \begin{bmatrix} 1 & 2 & -4 & 1 & 0 \\ 0 & 1 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{consistent} \Rightarrow b \in \text{span}(S).$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 2 & -3 & -2 \\ 0 & 1 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1, x_2 - \text{basic.} \\ x_3, x_4 - \text{free} \end{array}$$

Set $x_4 = x_3 = 0$.

$x_2 = 1$, $x_1 = -2$

$\Rightarrow 1 \cdot v_2 - 2v_1 = b. \Rightarrow \bar{b} = v_2 - 2v_1.$

Check: $\begin{pmatrix} -5 \\ 0 \\ -5 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ -5 \end{pmatrix}.$

2. Give a linearly independent subset of S that generates span(S).

$S' = \{v_1, v_2\}.$

3. Does the set S generate R^3? Explain.

No, it doesn't. Third row doesn't have a pivot. For some $b \in R^3$, system is inconsistent.