Problem 2. Let $A = \begin{pmatrix} 3 & -2 & -1 & -3 & -6 \\ 2 & 4 & 10 & -2 & -4 \\ 1 & 2 & 5 & -2 & -5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -10 \\ -5 \end{pmatrix}$.

1. Solve the linear system $A\mathbf{x} = \mathbf{b}$ and write the solution in the vector-parametric form.

Row-reduce $[A|b]$ to REF.

$$\begin{bmatrix} 3 & -2 & -1 & -3 & -6 & 1 \\ 2 & 4 & 10 & -2 & -4 & -10 \\ 1 & 2 & 5 & -2 & -5 & -5 \end{bmatrix} \to \begin{array}{c} \begin{bmatrix} 1 & 2 & 5 & -2 & -5 & -5 \\ 1 & 2 & 5 & -1 & -2 & -5 \\ 3 & -2 & -1 & -3 & -6 & 1 \end{bmatrix} \end{array} \begin{array}{l} R_3 \\ \frac{1}{2}R_2 \\ R_1 \end{array}$$

$$\to \begin{bmatrix} 1 & 2 & 5 & -2 & -5 & -5 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & -8 & -16 & 0 & 0 & 16 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 - R_1 \\ R_3 - 3R_1 \end{array} \to \begin{bmatrix} 1 & 2 & 5 & -2 & -5 & -5 \\ 0 & 1 & 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ -\frac{1}{8}R_3 \\ R_2 \end{array}$$

System is consistent. Obtain RREF

$$\to \begin{bmatrix} 1 & 2 & 5 & 0 & 1 & -5 \\ 0 & 1 & 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{bmatrix} \begin{array}{l} R_1 + 2R_3 \\ R_2 \\ R_3 \end{array} \to \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right]$$

x_1, x_2, x_4 - basic

$$\begin{cases} x_5 = t \\ x_4 = -3t \\ x_3 = s \\ x_2 = -2 - 2s \\ x_1 = -1 - s - t \end{cases} \Rightarrow \bar{x} = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \quad s, t \in R.$$

2. Specify a particular solution of the linear system and the general solution of the corresponding homogeneous linear system.

Take $s = t = 0 \Rightarrow \bar{x}_p = \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

$$\bar{x}_h = s \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \quad s, t \in R$$

3. Does the linear system $A\mathbf{x} = \mathbf{b}$ have a solution for any $\mathbf{b} \in R^3$? Explain.

Yes, it does. Every row of A has a pivot.