Test 3

Problem	1	2	3	4	5	Score
Points						

NAME: Solution Key

Show all your work for the full credit. Crib sheets and notes are not allowed.

Problem 1. Find an **orthonormal** basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Let $a_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, a_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}, a_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix}.$

Use the Gram-Schmidt Process:

$$v_1 = a_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \|v_1\|^2 = 4, \ \|v_1\| = 2$$

$$\hat{v}_2 = a_2 - \frac{a_2 \cdot v_1}{\|v_1\|^2} v_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}}{4} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2-1+1}{4} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{pmatrix}.$$

Let $v_2 = 2\hat{v}_2 = \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix}$. $\|v_2\|^2 = 9 + 9 + 1 + 1 = 20$, $\|v_2\| = \sqrt{20} = 2\sqrt{5}$

$$\hat{v}_3 = a_3 - \frac{a_3 \cdot v_1}{\|v_1\|^2} v_1 - \frac{a_3 \cdot v_2}{\|v_2\|^2} v_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}}{4} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix}}{20} \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix} - \frac{2-2-1+2}{4} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} - \frac{6+6+1+2}{20} \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} - \frac{3}{4} \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 8-1-9 \\ 8+1-9 \\ 4+1-3 \\ 8-1-3 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -2 \\ 0 \\ 2 \\ 4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

Let $v_3 = 2\hat{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$.

$\|v_3\|^2 = 1 + 1 + 4 = 6$.

$\|v_3\| = \sqrt{6}$

Orthogonal set $\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\}$.

Orthonormal set $\left\{ \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} \frac{3}{2\sqrt{5}} \\ \frac{3}{2\sqrt{5}} \\ \frac{1}{2\sqrt{5}} \\ \frac{1}{2\sqrt{5}} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \right\}$

Problem 2. Find a basis for $W^\perp$ if $W = \{\overset{v_1}{(1,3,1,1,1)^T},\ \overset{v_2}{(1,2,0,1,-1)^T},\ \overset{v_3}{(-1,1,0,2,10)^T}\}$.

$$W^\perp = \{ x \in \mathbb{R}^5 \mid x \cdot v_i = 0,\ i = 1,2,3 \}.$$

$$\begin{cases} v_1^T x = 0 \\ v_2^T x = 0 \\ v_3^T x = 0 \end{cases} \Rightarrow A^T x = 0, \text{ where } A = \begin{pmatrix} 1 & 1 & -1 \\ 3 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 10 \end{pmatrix}$$

Row reduce A^T:

$$\begin{pmatrix} 1 & 3 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 & -1 \\ -1 & 1 & 0 & 2 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 & -2 \\ 0 & 4 & 1 & 3 & 11 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & -3 & 3 & 3 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 3 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 0 & 2 & 2 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 & -7 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}$$

x_1, x_2, x_3 – basic variables, x_4, x_5 – free.

$$\begin{cases} x_5 = t \\ x_4 = s \\ x_3 = s + t \\ x_2 = -s - 3t \\ x_1 = s + 7t \end{cases} \qquad x = s \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 7 \\ -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Basis for $W^\perp$: $\beta = \left\{ \begin{pmatrix} +1 \\ -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$

Problem 3. Let W be the subspace of R^4 spanned by the orthogonal vectors $\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{v_3} \right\}$.

Write the vector $y = \begin{bmatrix} 3 \\ -2 \\ 4 \\ -3 \end{bmatrix}$ as the sum of the orthogonal projection $\hat{y}$ of y on W and the vector $z \in W^{\perp}$.

$y = \hat{y} + z$, where $\hat{y} = \frac{y \cdot v_1}{\|v_1\|^2} v_1 + \frac{y \cdot v_2}{\|v_2\|^2} v_2 + \frac{y \cdot v_3}{\|v_3\|^2} v_3$

and $z = y - \hat{y}$.

$$\hat{y} = \frac{\begin{pmatrix} 3 \\ -2 \\ 4 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}{1+1} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{\begin{pmatrix} 3 \\ -2 \\ 4 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}}{4} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + \frac{\begin{pmatrix} 3 \\ -2 \\ 4 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{3-2}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{3+2-4-3}{4} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + \frac{4-3}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1-1+0 \\ 1+1+0 \\ 0+1+1 \\ 0-1+1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{y} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad z = y - \hat{y} = \begin{pmatrix} 3 \\ -2 \\ 4 \\ -3 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \\ -3 \end{pmatrix}.$$

Problem 4. Diagonalize the following matrix if possible: $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

$$p_A(t) = \begin{vmatrix} 1-t & 2 & 1 \\ -1 & -t & 1 \\ 1 & 1 & -t \end{vmatrix} = -\begin{vmatrix} 1 & 1 & -t \\ -1 & -t & 1 \\ 1-t & 2 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & -t \\ 0 & 1-t & 1-t \\ 0 & t-1+2 & -t(t-1)+1 \end{vmatrix}$$

$$= -\begin{vmatrix} 1-t & 1-t \\ t+1 & -t(t-1)+1 \end{vmatrix} = (t-1)\begin{vmatrix} 1 & 1 \\ t+1 & -t(t-1)+1 \end{vmatrix}$$

$$= (t-1)\left[-t(t-1)+1-(t+1)\right] = (t-1)\left(-t^2+t+1-t-1\right) = t^2(1-t)$$

$p_A(t) = 0 \Rightarrow \lambda_1 = 0,\ m_1 = 2$

$\lambda_2 = 1,\ m_2 = 1$

Find $n_1 = \dim(E_{\lambda_1}) = \text{nullity}\,(A - \lambda_1 I)$

$$A - \lambda_1 I = A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow n_1 = 1 \neq m_1 = 2 \Rightarrow$ matrix is not diagonalizable.

Problem 5. The matrix $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ has the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$. Explain why the matrix A is orthogonally diagonalizable, and then orthogonally diagonalize it.

Matrix A is symmetric $\Rightarrow$ A is orthogonally diagonalizable.

Let's find bases for the eigenspaces.

$\underline{\lambda_1 = 2.}$

$$A - \lambda_1 I = A - 2I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad v_1^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \; v_2^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

$$v_1^{(1)} \cdot v_2^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1 \neq 0 \Rightarrow \{v_1^{(1)}, v_2^{(1)}\} \text{ is not an orthogonal basis}$$

Orthogonalize using the Gram-Schmidt process:

$$v_1^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}. \quad \tilde{v}_2^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix}$$

Use $v_2^{(1)} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

Normalize:
$$\hat{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$\hat{v}_2 = \frac{1}{\sqrt{1+1+4}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix}.$$

$\underline{\lambda_2 = 5}$

$$A - 5I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = t \\ x_2 = t \\ x_3 = t \end{cases} \Rightarrow v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Normalize v_3: $\|v_3\| = \sqrt{3}$.

$$\hat{v}_3 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}.$$

Thus, $A = QDQ^T$, where

$$Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \text{ and } D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$