Problem 5. Let $\mathbf{a} = (2, -1, 3)^T$, and let $W = \text{span}(S)$, where

$$S = \left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

$v_1 \cdot v_2 = 1 - 2 + 1 = 0$

$\Rightarrow S$ is an orthogonal set

1. Find vectors $\mathbf{w} \in W$ and $\mathbf{z} \in W^{\perp}$ such that $\mathbf{a} = \mathbf{w} + \mathbf{z}$.

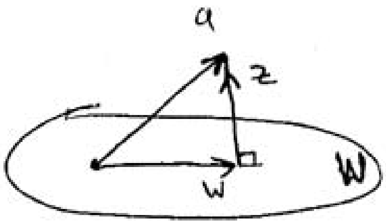
$$w = \frac{a \cdot v_1}{\|v_1\|^2} v_1 + \frac{a \cdot v_2}{\|v_2\|^2} v_2 = \frac{2-2+3}{1+4+1} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \frac{2+1+3}{1+1+1} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+4 \\ 2-4 \\ 1+4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 5/2 \\ -1 \\ 5/2 \end{pmatrix}.$$

$$z = a - w = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 5/2 \\ -1 \\ 5/2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 4-5 \\ -2+2 \\ 6-5 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix}.$$

$$a = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = w + z, \text{ where } w = \begin{pmatrix} 5/2 \\ -1 \\ 5/2 \end{pmatrix} \text{ and } z = \begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix}$$

<u>Check</u>. $w \cdot z = \begin{pmatrix} 5/2 \\ -1 \\ 5/2 \end{pmatrix} \cdot \begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix} = -\frac{5}{4} + \frac{5}{4} = 0.$



2. What is vector $\mathbf{w}$ called?

Vector w is called the orthogonal projection of vector $\bar{a}$ on $W = \text{span}(S)$.

3. Find the distance between vector $\mathbf{a}$ and W. (Hint: recall that the distance between vector $\mathbf{a}$ and W is the distance between $\mathbf{a}$ and its orthogonal projection on W.)

$$\text{dist}(a, W) = \|z\| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2}.$$