Problem 2.

Let $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \right\}$. Let $v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$.

1. Explain why β is an orthogonal basis for R^3.

β consists of nonzero vectors.

$v_1 \cdot v_2 = 3 - 3 = 0$

$v_1 \cdot v_3 = 2 - 2 = 0$ $\Rightarrow \beta$ is an orthogonal set.

$v_2 \cdot v_3 = 6 - 12 + 6 = 0$

$\Rightarrow \beta$ is a linearly independent set of 3 vectors.

Since $\dim(R^3) = 3$, β is a spanning set for R^3

$\Rightarrow \beta$ is a basis for R^3 (orthogonal).

2. Normalize basis β.

$$\|v_1\|^2 = 1 + 1 = 2 \Rightarrow \tilde{v}_1 = \frac{1}{\sqrt{2}} v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}.$$

$$\|v_2\|^2 = 9 + 16 + 9 = 34 \Rightarrow \tilde{v}_2 = \frac{1}{\sqrt{34}} v_2 = \begin{pmatrix} 3/\sqrt{34} \\ -4/\sqrt{34} \\ 3/\sqrt{34} \end{pmatrix}.$$

$$\|v_3\|^2 = 4 + 9 + 4 = 17 \Rightarrow \tilde{v}_3 = \frac{1}{\sqrt{17}} v_3 = \begin{pmatrix} 2/\sqrt{17} \\ 3/\sqrt{17} \\ 2/\sqrt{17} \end{pmatrix}.$$

$\tilde{\beta} = \{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ is an orthonormal basis for R^3.

3. Using orthogonality of basis β, find the β-coordinate vector $[\mathbf{a}]_\beta$ of the vector

$\mathbf{a} = (2, -1, 1)^T$.

$$\bar{a} = \frac{a \cdot v_1}{\|v_1\|^2} v_1 + \frac{a \cdot v_2}{\|v_2\|^2} v_2 + \frac{a \cdot v_3}{\|v_3\|^2} v_3$$

$a \cdot v_1 = 2 - 1 = 1$, $a \cdot v_2 = 6 + 4 + 3 = 13$, $a \cdot v_3 = 4 - 3 + 2 = 3$.

$$\Rightarrow [a]_\beta = \left(\frac{1}{2}, \frac{13}{34}, \frac{3}{17} \right)^T.$$