Problem 4. 1. Show that the set $\beta = \{p_1(t) = 3 - 4t + 5t^2,\ p_2(t) = 1 - t,\ p_3(t) = 1\}$ is a basis for the vector space P_2 of polynomials of degree ≤ 2.

Let $\mathcal{E} = \{1, t, t^2\}$ be the standard basis for P_2.

$[p_1]_{\mathcal{E}} = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$, $[p_2]_{\mathcal{E}} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $[p_3]_{\mathcal{E}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Note, vectors $[p_1]_{\mathcal{E}}, [p_2]_{\mathcal{E}}, [p_3]_{\mathcal{E}}$ are linearly independent
$\Rightarrow \{p_1, p_2, p_3\}$ is linearly independent, as well.
Since $\dim(P_2) = 3$ and β consists of 3 linearly independent vectors, the set β is a basis for P_2.

2. Find the polynomial with the β-coordinate vector $(5, 2, -3)^T$.

$p(t) = 5(3 - 4t + 5t^2) + 2(1 - t) - 3 \cdot 1 = t^2(25) + t(-20 - 2) + (15 + 2 - 3)$

$p(t) = 25t^2 - 22t + 14$.

3. Find the β-coordinate vector of the polynomial $p(t) = 10t^2 - 11t + 2$. $\Rightarrow [p]_{\mathcal{E}} = \begin{pmatrix} 2 \\ -11 \\ 10 \end{pmatrix}$

$[p]_\beta = P_{\beta \leftarrow \mathcal{E}} [p]_{\mathcal{E}}$, $P_{\mathcal{E} \leftarrow \beta} [p]_\beta = [p]_{\mathcal{E}} \Rightarrow \begin{pmatrix} 3 & 1 & 1 \\ -4 & -1 & 0 \\ 5 & 0 & 0 \end{pmatrix} \bar{x} = \begin{pmatrix} 2 \\ -11 \\ 10 \end{pmatrix}$

Let $y_1 = x_3$, $y_2 = x_2$, $y_3 = x_1$. $\Rightarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & 5 \end{pmatrix} \bar{y} = \begin{pmatrix} 2 \\ -11 \\ 10 \end{pmatrix}$.

$y_3 = 2$

$-y_2 = -11 + 4y_3 = -11 + 8 = -3 \Rightarrow y_2 = 3$

$y_1 = 2 - y_2 - 3y_3 = 2 - 3 - 6 = -7$

$\Rightarrow [p]_\beta = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_3 \\ y_2 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -7 \end{bmatrix}$.