Problem 3. Diagonalize the following matrices if possible.

1. $A = \begin{pmatrix} 4 & 2 & -4 \\ 1 & 5 & 2 \\ 0 & 0 & 6 \end{pmatrix}$, (Hint: use the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 6$).

$\underline{\lambda_1 = 3}$

$$A - \lambda_1 I = A - 3I = \begin{pmatrix} 1 & 2 & -4 \\ 1 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim(E_{\lambda_1}) = 1$$

$$\begin{cases} x_3 = 0 \\ x_2 = s \\ x_1 = -2s + 4x_3 = -2s \end{cases} \Rightarrow \bar{x} = s\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$\underline{\lambda_2 = 6}$

$$A - 6I = \begin{pmatrix} -2 & 2 & -4 \\ 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim(E_{\lambda_2}) = 2, \ \bar{x} = s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow \dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) = 3 =$ matrix size $\Rightarrow A$ is diagonalizable.

Let $P = \begin{pmatrix} -2 & 1 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$. Then $A = PDP^{-1}$.

2. $B = \begin{pmatrix} 5 & 1 & -2 \\ 0 & 5 & 1 \\ 0 & 0 & 4 \end{pmatrix}$. B is upper triangular $\Rightarrow \lambda_1 = 5, \lambda_2 = 4$.
Note, $m_1 = 2$, $m_2 = 1$.

$\underline{\lambda_1 = 5}$

$$B - 5I = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim(E_{\lambda_1}) = 1 \neq 2 = m_1.$$

Matrix is not diagonalizable.