Problem 2.

Let $\beta = \left\{ \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ and $\gamma = \left\{ \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \end{pmatrix} \right\}$ be bases for R^2.

1. Find the change of coordinate matrices from β-basis to γ-basis and from γ-basis to β-basis (note which one is which). Let $\mathcal{E} = \{e_1, e_2\}$ be the standard basis for R^2.

$$P_{\mathcal{E}\leftarrow\beta} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$$

$$P_{\mathcal{E}\leftarrow\gamma} = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}, \quad P^{-1}_{\mathcal{E}\leftarrow\beta} = \frac{1}{6-5}\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$$

$$P_{\beta\leftarrow\gamma} = P^{-1}_{\mathcal{E}\leftarrow\beta} P_{\mathcal{E}\leftarrow\gamma} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 8 & 11 \\ -21 & -29 \end{pmatrix}$$

$$P_{\gamma\leftarrow\beta} = \frac{1}{8(-29)-11(-21)}\begin{pmatrix} -29 & -11 \\ 21 & 8 \end{pmatrix} = -\begin{pmatrix} -29 & -11 \\ 21 & 8 \end{pmatrix} = \begin{pmatrix} 29 & 11 \\ -21 & -8 \end{pmatrix}.$$

$P_{\beta\leftarrow\gamma}$ - change of coordinate matrix from γ-basis to β-basis.

$P_{\gamma\leftarrow\beta}$ - change of coordinate matrix from β-basis to γ basis.

2. Find the β-coordinate vector $[\mathbf{a}]_\beta$ of the vector $\mathbf{a} = (2, -1)^T$. $\bar{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$[\bar{a}]_\beta = P_{\beta\leftarrow\mathcal{E}}\bar{a} = P^{-1}_{\mathcal{E}\leftarrow\beta}\bar{a} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -13 \end{pmatrix}.$$