Problem 5. Let

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} \right\}.$$

1. Is the set S linearly dependent or independent? Explain.

S consists of 4 3-tuples. The set is linearly dependent.
# of vectors is greater that # of entries in a vector.

2. Find a linearly independent subset Q of the set S that generates the span of S.

$$A = \begin{pmatrix} 1 & 3 & 2 & 2 \\ 2 & -3 & 4 & -5 \\ 2 & 1 & -2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 2 & 2 \\ 0 & -9 & 0 & -9 \\ 0 & -5 & -6 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 5 & 6 & 5 \end{pmatrix}.$$

$$\to \begin{pmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 6 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$\uparrow \; \uparrow \; \uparrow$

Columns 1,2,3 are pivot.

$$Q = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} \right\}$$

– linearly independent subset of S that generates span(S).

3. Express the last vector as a linear combination of vectors in Q (Hint: use the calculation results in part 2).

$$\begin{pmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\Rightarrow c_1 = -1, \; c_2 = 1, \; c_3 = 0$

$$\begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

4. Does the set S span R^3? Explain. Yes, it does. Every row of matrix A has a pivot. The equation $Ax = b$ has a solution for every $b \in R^3$. That is, every vector $b \in R^3$ is a linear combination of the columns of matrix A.