Problem 4. The set $\mathcal{B} = \{p_1(t) = 1 + t + t^2,\ p_2(t) = 1 - 2t,\ p_3(t) = 1 - t + t^2\}$ is a basis for the vector space P_2 of polynomials of degree ≤ 2. Find the $\mathcal{B}$-coordinate vector of the polynomial $p(t) = 4 + 8t + 10t^2$.

Let $E = \{1, t, t^2\}$.

$[p_1]_E = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $[p_2]_E = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$, $[p_3]_E = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$P_{E \leftarrow B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & 0 & 1 \end{bmatrix}$, $[p]_E = \begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix}$

$P_{E \leftarrow B}[p]_B = [p]_E \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & 0 & 1 \end{bmatrix} [p]_B = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 1 & -2 & -1 & 8 \\ 1 & 0 & 1 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & -3 & -2 & 4 \\ 0 & -1 & 0 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & -6 \\ 0 & -3 & -2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & -2 & -14 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & -3 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 7 \end{pmatrix}$$

$$\Rightarrow \boxed{[p]_B = \begin{pmatrix} 3 \\ -6 \\ 7 \end{pmatrix}}$$