Problem 3. Diagonalize the matrix if possible.

1. $A = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$ (Hint: Matrix A has two eigenvalues of $\lambda_1 = -2$ and $\lambda_2 = 4$.)

$A + 2I = \begin{pmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim(E_{\lambda_1}) = 1$ (1 free variable)

$A - 4I = \begin{pmatrix} -7 & 1 & -1 \\ -7 & 1 & -1 \\ -6 & 6 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ -7 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & -6 & 6 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$\dim(E_{\lambda_2}) = 1$ (1 free variable).

Matrix is not diagonalizable.

$\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) = 1 + 1 = 2 < 3.$

2. $A = \begin{pmatrix} 1 & 6 & -2 \\ 0 & 2 & 0 \\ 3 & 3 & -4 \end{pmatrix}$. $f_A(t) = \begin{vmatrix} 1-t & 6 & -2 \\ 0 & 2-t & 0 \\ 3 & 3 & -4-t \end{vmatrix} = (2-t)\begin{vmatrix} 1-t & -2 \\ 3 & -4-t \end{vmatrix}$

$= (2-t)\big((t-1)(t+4)+6\big) = (2-t)(t^2+3t+2) = (2-t)(t+1)(t+2)$

$\lambda_1 = -2,\ \lambda_2 = -1,\ \lambda_3 = 2$. A has 3 distinct eigenvalues $\Rightarrow$ A is diagonalizable.

$\lambda_1 = -2$: $A + 2I = \begin{pmatrix} 3 & 6 & -2 \\ 0 & 4 & 0 \\ 3 & 3 & -2 \end{pmatrix} \to \begin{pmatrix} 3 & 3 & -2 \\ 0 & 4 & 0 \\ 0 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & 3 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2/3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\boxed{v_1 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}}$.

$\lambda_2 = -1$: $A + I = \begin{pmatrix} 2 & 6 & -2 \\ 0 & 3 & 0 \\ 3 & 3 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\boxed{v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$

$\lambda_3 = 2$: $A - 2 = \begin{pmatrix} -1 & 6 & -2 \\ 0 & 0 & 0 \\ 3 & 3 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & -6 & 2 \\ 1 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -6 & 2 \\ 0 & 7 & -4 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -\frac{10}{7} \\ 0 & 1 & -4/7 \\ 0 & 0 & 0 \end{pmatrix}$

$\boxed{v_3 = \begin{pmatrix} 10 \\ 4 \\ 7 \end{pmatrix}} \Rightarrow A = SDS^{-1}$, $D = \begin{pmatrix} -2 & & \\ & -1 & \\ & & 2 \end{pmatrix}$, $S = \begin{pmatrix} 2 & 1 & 10 \\ 0 & 0 & 4 \\ 3 & 1 & 7 \end{pmatrix}$.