

Problem 5. Let $S = \{v_1, v_2, v_3, v_4\}$, where

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -3 \end{bmatrix}, v_4 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 6 \end{bmatrix}.$$

1. Determine whether the set S is linearly independent or linearly dependent.

S is linearly independent $\Leftrightarrow [v_1, v_2, v_3, v_4]x = 0$
has only the trivial solution. Row reduce $[v_1, v_2, v_3, v_4]$.

$$\begin{bmatrix} 0 & 1 & -1 & 3 \\ 1 & 2 & 1 & 2 \\ 2 & -1 & 1 & 1 \\ 0 & 3 & -3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & 3 \\ 2 & -1 & 1 & 1 \\ 0 & 1 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & -5 & -1 & -3 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -6 & 12 \\ 0 & 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The set S is linearly independent
since there is a pivot in every column of $[v_1, v_2, v_3, v_4]$.

2. Does the set S span \mathbb{R}^4 ?

Yes. There is a pivot in every row of $[v_1, v_2, v_3, v_4]$.
Linear system $Ax = b$, where $A = [v_1, v_2, v_3, v_4]$, $x \in \mathbb{R}^4$, $b \in \mathbb{R}^4$
has a solution for any $b \in \mathbb{R}^4$.