Problem 5. Diagonalize the following matrix if possible: $A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}$.

Find eigenvalues of A:

$$\det(A - tI) = \begin{vmatrix} 1-t & 2 & -1 \\ 1 & -t & 1 \\ 4 & -4 & 5-t \end{vmatrix} = \begin{vmatrix} 1-t & 2 & -1 \\ 2-t & 2-t & 0 \\ 4 & -4 & 5-t \end{vmatrix}$$

$$= (2-t)\begin{vmatrix} 1-t & 2 & -1 \\ 1 & 1 & 0 \\ 4 & -4 & 5-t \end{vmatrix} = (2-t)\begin{vmatrix} (1-t) & 0 & -1 \\ 1 & 1 & 0 \\ 8 & 0 & 5-t \end{vmatrix}$$

$$= (2-t)\left[-(1+t)(5-t)+8\right] = (2-t)\left(t^2-4t+3\right) = 0.$$

$\lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = 3$

Matrix is diagonalizable.

$\lambda_1 = 2$, $A - 2I = \begin{pmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \\ 4 & -4 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 4 & -4 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 4 & -4 & 3 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$\to \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x_3 = t,\ x_2 = 1/4\,t,\ x_1 = -1/2\,t \Rightarrow v_1 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$

$\lambda_2 = 1$ $\quad A - I = \begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 4 & -4 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1/2 \\ 1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$

$x_3 = t,\ x_2 = \frac{1}{2}t,\ x_1 = -\frac{1}{2}t \Rightarrow v_2 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

$\lambda_3 = 3$ $\quad A - 3I = \begin{pmatrix} -2 & 2 & -1 \\ 1 & -3 & 1 \\ 4 & -4 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 1 \\ -2 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\to \begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1/4 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x_3 = t,\ x_2 = t/4,\ x_1 = -t/4 \Rightarrow v_3 = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$

$A = SDS^{-1}$, $D = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}$, $S = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 2 & 4 & 4 \end{bmatrix}$.