Problem 4. Let $A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 3 & 2 \\ 3 & 3 & 3 & 3 \end{pmatrix}$

1. Find bases for the nullspace, the column space, and the row space of matrix A.

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 3 & 2 \\ 3 & 3 & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & 0 & -3 \\ 0 & -4 & 0 & -4 \\ 0 & -3 & 0 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\begin{cases} x_4 = t \\ x_3 = s \\ x_2 = -t \\ x_1 = -s \end{cases} \qquad x = t\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Basis for null(A): $\left\{ \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

Basis for Col(A): $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix} \right\}$

Basis for Row(A): $\{(1,0,1,0), (0,1,0,1)\}$.

2. Determine nullity(A), rank(A), rank(A^T), and nullity(A^T).

$$\text{nullity}(A) = \dim(\text{null}(A)) = 2$$
$$\text{rank}(A) = \dim(\text{Col}(A)) = 2$$
$$\text{rank}(A^T) = \text{rank}(A) = 2$$
$$\text{nullity}(A^T) = 4 - \text{rank}(A^T) = 4 - 2 = 2$$

3. Is matrix A invertible? Explain.

A is not invertible since nullity$(A) \neq 0$.