Problem 3. Diagonalize the matrix if possible:

1. $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$

1) Find eigenvalues: $p_A(t) = \begin{vmatrix} 1-t & 0 & -2 \\ 0 & -t & 0 \\ -2 & 0 & 4-t \end{vmatrix} = -t \begin{vmatrix} 1-t & -2 \\ -2 & 4-t \end{vmatrix}$

$= -t\left[(1-t)(4-t)-4\right] = -t\left[t^2-5t\right] = t^2(5-t) = 0,\ \lambda_1 = 0,\ \lambda_2 = 5$

Algebraic multiplicity of λ_1 is $m_1 = 2$. $m_2 = 1 = n_2$

Find geometric multiplicities:

1) $A - 0t = A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \leftarrow$ 2 free variables $\Rightarrow n_2 = 2 = m_2$
$\Rightarrow A$ is diagonalizable.

$\begin{cases} x_3 = s \\ x_2 = t \\ x_1 = 2s \end{cases} \Rightarrow x = s\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

2) $A - \lambda_2 t = A - 5t = \begin{pmatrix} -4 & 0 & -2 \\ 0 & -5 & 0 \\ -2 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_3 = s \\ x_2 = 0 \\ x_1 = -\frac{s}{2} \end{cases} \Rightarrow v_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

$A = VDV^{-1}$, where $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ and $V = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.

2. $A = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$ $\quad p_A(t) = \begin{vmatrix} 1-t & 2 \\ -2 & -3-t \end{vmatrix} = (1-t)(-3-t)+4$
$= t^2 + 2t + 1 = (t+1)^2 = 0$
$\Rightarrow \lambda_1 = -1$, mult. $= 2 = m_1$

Find n_1 – geometr. multiplicity

$A - \lambda_1 I = \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow n_1 = 1$

\# of free variables.

Since $n_1 = 1 < m_1 = 2$, matrix is NOT diagonalizable.