Problem 5. Let $A = \begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 3 & 6 & 1 & -1 & 2 \\ 4 & 8 & 3 & -2 & 4 \end{pmatrix}$

1. Find bases for the nullspace Null(A), the column space Col(A), and the row space Row(A) of matrix A.

Row reduce A:

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 3 & 6 & 1 & -1 & 2 \\ 4 & 8 & 3 & -2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 3 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix} \text{ (RREF)}$$

Columns 1, 3, 4 are pivot $\Rightarrow \left\{ \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} \right\}$ - Basis for Col(A)

$\{(1,2,0,0,1), (0,0,1,0,2), (0,0,0,1,3)\}$ - Basis for Row (A

To find Basis for Null(A), solve $Ax=0$.

x_1, x_3, x_4 - basic variables, x_2, x_5 - free variables.

$$\begin{cases} x_5 = t \\ x_4 = -3t \\ x_3 = -2t \\ x_2 = s \\ x_1 = -2s - t \end{cases} \Rightarrow x = t\begin{pmatrix} -1 \\ 0 \\ -2 \\ -3 \\ 1 \end{pmatrix} + s\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad \left\{ \begin{pmatrix} -1 \\ 0 \\ -2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Basis for Null (A)

2. Specify the nullity and the rank of matrix A.

Nullity(A) = dim(Null(A)) = 2

rank(A) = dim(Col(A)) = 3

3. Does a linear system $Ax = b$ have a solution for any $b \in R^3$? Explain.

Yes, it does. Linear system is consistent for any $b \in R^3$ since there is a pivot position in every row of A.