

**Problem 4.** 1. Show that the set  $\mathcal{B} = \{p_1(t) = 1 + 2t + t^2, p_2(t) = 1 - 2t + t^2, p_3(t) = 1 - t - t^2\}$  is a basis for the vector space  $P_2$  of polynomials of degree  $\leq 2$ .

$\dim(P_2) = 3$ ,  $\mathcal{B}$  consists of 3 polynomials.

If  $\mathcal{B}$  is linearly independent then  $\mathcal{B}$  is a basis for  $P_2$ .

$[p_1]_E = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ ,  $[p_2]_E = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ ,  $[p_3]_E = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ , where  $E = \{1, t, t^2\}$  standard basis for  $P_2$ .

$P_{E \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -2 & -1 \\ 1 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -4 & -3 \\ 0 & 0 & -2 \end{bmatrix} \Rightarrow P_{E \leftarrow \mathcal{B}}$  has 3 pivot columns

$\Rightarrow \{[p_1]_E, [p_2]_E, [p_3]_E\}$  are linearly independent.

$\Rightarrow \{p_1, p_2, p_3\}$  is linearly independent.

2. Find the  $\mathcal{B}$ -coordinate vector of the polynomial  $p(t) = 7 - 3t + 9t^2$ .

$[p]_E = \begin{pmatrix} 7 \\ -3 \\ 9 \end{pmatrix}$ .  $P_{E \leftarrow \mathcal{B}} [p]_{\mathcal{B}} = [p]_E \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & -2 & -1 \\ 1 & 1 & -1 \end{bmatrix} [p]_{\mathcal{B}} = \begin{pmatrix} 7 \\ -3 \\ 9 \end{pmatrix}$

$\begin{bmatrix} 1 & 1 & 1 & 7 \\ 2 & -2 & -1 & -3 \\ 1 & 1 & -1 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & -4 & -3 & -17 \\ 0 & 0 & -2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & 4 & 3 & 17 \\ 0 & 0 & 1 & -1 \end{bmatrix}$  Solve  $\rightarrow$

$\rightarrow \begin{bmatrix} 1 & 1 & 0 & 8 \\ 0 & 4 & 0 & 20 \\ 0 & 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 8 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow [p]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$

3. Verify your result in Part 2 by showing that  $p(t)$  is a linear combination of the basis polynomials with the appropriate coefficients.

$$p = 3p_1 + 5p_2 - p_3 ?$$

$$\begin{aligned} 7 - 3t + 9t^2 &= 3(1 + 2t + t^2) + 5(1 - 2t + t^2) - (1 - t - t^2) \\ &= (3 + 5 - 1) + t(6 - 10 + 1) + t^2(3 + 5 + 1) \\ &= 7 - 3t + 9t^2 \end{aligned}$$