

Problem 3. Let $A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 3 & 2 & 4 & 0 \\ 5 & 1 & -2 & 5 \\ 1 & 2 & 3 & 0 \end{pmatrix}$

1. Compute the determinant of matrix A .

$$\begin{aligned}
 |A| &= \begin{vmatrix} 1 & 0 & -1 & 1 \\ 3 & 2 & 4 & 0 \\ 5 & 1 & -2 & 5 \\ 1 & 2 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & 1 \\ 3 & 2 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 2 & 3 & 0 \end{vmatrix} \xrightarrow{R_3 - 5R_1} (-1) \cdot 1 \begin{vmatrix} 3 & 2 & 4 \\ 0 & 1 & 3 \\ 1 & 2 & 3 \end{vmatrix} \\
 &= - \begin{vmatrix} 0 & -4 & -5 \\ 0 & 1 & 3 \\ 1 & 2 & 3 \end{vmatrix} \xrightarrow{R_1 - 3R_2} (-1)^{3+1} \cdot 1 \begin{vmatrix} -4 & -5 \\ 1 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 5 \\ 1 & 3 \end{vmatrix} \\
 &= 12 - 5 = 7
 \end{aligned}$$

2. Find the determinants of the following matrices:

(a) $A^T A$

$$\det(A^T A) = \det(A^T) \det(A) = [\det(A)]^2 = 7^2 = 49$$

(b) $A^{-T} = (A^T)^{-1}$

$$\det(A^{-T}) = \det((A^T)^{-1}) = \frac{1}{\det(A^T)} = \frac{1}{\det(A)} = \frac{1}{7}$$