*Math 254-2– **Intro. Linear Algebra**, Instructor: Rakhim Aitbayev, 30 November 2006*

Problem 6. Let $T : R^2 \to R^2$ be a linear transformation defined by $T(x) = Ax$ for $x \in R^2$, whe
$A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$, and let $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$.

1. Show that $\mathcal{B}$ is a basis for R^2.

$P = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \Rightarrow \det(P) = 6-5 = 1 \neq 0 \Rightarrow P$ is invertible
$\Rightarrow \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ are linearly independent.
2 lin. ind. vectors in R^2 is a basis.

2. Find the coordinate change matrix from the standard basis for R^2 to the $\mathcal{B}$-basis.

Let $\mathcal{E} = \{e_1, e_2\}$ be the standard basis for R^2.
Find $P_{\mathcal{B} \leftarrow \mathcal{E}}$. Since $P_{\mathcal{E} \leftarrow \mathcal{B}} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$. then

$$P_{\mathcal{B} \leftarrow \mathcal{E}} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}^{-1} = \frac{1}{6-5} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}}.$$

3. Find $[x]_{\mathcal{B}}$ for $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. $\quad x = P_{\mathcal{E} \leftarrow \mathcal{B}} [x]_{\mathcal{B}} \Rightarrow [x]_{\mathcal{B}} = P_{\mathcal{B} \leftarrow \mathcal{E}}\, x$

$$= \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3-2 \\ -5+4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \boxed{[x]_{\mathcal{B}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

4. Find the matrix of the linear transformation T in the $\mathcal{B}$-basis.

$$[T]_{\mathcal{B}} = P_{\mathcal{B} \leftarrow \mathcal{E}}\, A\, P_{\mathcal{E} \leftarrow \mathcal{B}} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 12-2 & -3-1 \\ -20+4 & 5+2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 10 & -4 \\ -16 & 7 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 20-20 & 10-12 \\ -32+35 & -16+21 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 3 & 5 \end{pmatrix}$$

5. Find $[T(x)]_{\mathcal{B}}$ for $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. $\quad [T(x)]_{\mathcal{B}} = [T]_{\mathcal{B}} [x]_{\mathcal{B}}$

$$= \begin{pmatrix} 0 & -2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3-5 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}.$$