Quiz 3

Name: Solution Key

Let $\beta = \{p_1(t) = 2 + 4t^2,\ p_2(t) = 4 + t - 4t^2,\ p_3(t) = -2 - t + 2t^2\}$.

1. Show that the set β is a basis for P_2.

Consider the equation $c_1 p_1(t) + c_2 p_2(t) + c_3 p_3(t) = 0$.
Combining the like terms, obtain

$$\begin{cases} 2c_1 + 4c_2 - 2c_3 = 0 \\ c_2 - c_3 = 0 \\ 4c_1 - 4c_2 + 2c_3 = 0 \end{cases} \quad \text{Let } A = \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & -1 \\ 4 & -4 & 2 \end{pmatrix}.$$

Row reduce A to determine if $c_1 = c_2 = c_3$ is a unique solution.

$$A \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 2 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & -6 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

All columns are pivot $\Rightarrow c_1 = c_2 = c_3 = 0$ is a unique solution $\Rightarrow$ β is linearly independent. Since β consists of 3 vectors and $\dim(P_2) = 3$, β is a basis for P_2.

2. Find the change of coordinate matrix from the β basis to the standard basis.

$$P_{\mathcal{E}\leftarrow\beta} = \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & -1 \\ 4 & -4 & 2 \end{pmatrix}, \text{ where } \mathcal{E} = \{1, t, t^2\}.$$

3. Find the β coordinates of $p(t) = 2 + t + 4t^2$. $\quad P_{\mathcal{E}\leftarrow\beta}[p]_\beta = [p]_{\mathcal{E}} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & -1 & 1 \\ 4 & -4 & 2 & 4 \end{array}\right) \to \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 2 & -2 & 1 & 2 \end{pmatrix}\begin{matrix} \frac{1}{2}R_1 \\ \\ R_3\frac{1}{2} \end{matrix} \to \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -6 & 3 & 0 \end{pmatrix}_{R_3 - 2R_1} \to \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -3 & 6 \end{pmatrix}_{R_3 + 6R_2}$$

$$\to \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}_{-\frac{1}{3}R_3} \to \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{pmatrix}\begin{matrix} R_1 + R_3 \\ R_2 + R_3 \\ \\ \end{matrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{pmatrix}^{R_1 - 2R_2}$$

$$\Rightarrow [p]_\beta = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}.$$

Check: $1 \cdot p_1(t) - 1 \cdot p_2(t) - 2 \cdot p_3(t) = 2 + 4t^2 - 4 - t + 4t^2 - 2(-2 - t + 2t^2)$
$= 2 + t + 4t^2 \equiv p(t)$. ✓