Quiz 3

Name: Solution Key

1. Evaluate the determinant

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 5 & 5 \\ 0 & 1 & -3 & -4 \end{vmatrix} \begin{matrix} \\ \\ R_3+2R_2 \\ R_4-R_2 \end{matrix} = -1 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 5 \\ 1 & -3 & -4 \end{vmatrix}$$

$$= -5 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -7 \end{vmatrix}_{R_3-R_1} = 5 \begin{vmatrix} 1 & 1 \\ 5 & 7 \end{vmatrix} = 5(7-5) = 10.$$

2. Use Cramer's rule to solve the linear system:

$$\begin{aligned} 2x + 3y &= 2 \\ 3x + 2y &= 5 \end{aligned}$$

$A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, $A_1 = \begin{pmatrix} 2 & 3 \\ 5 & 2 \end{pmatrix}$, $A_2 = \begin{pmatrix} 2 & 2 \\ 3 & 5 \end{pmatrix}$.

$|A| = 4 - 9 = -5$, $|A_1| = 4 - 15 = -11$, $|A_2| = 10 - 6 = 4$.

$x = \dfrac{|A_1|}{|A|} = \dfrac{-11}{-5} = \dfrac{11}{5}$, $y = \dfrac{|A_2|}{|A|} = \dfrac{4}{-5} = -\dfrac{4}{5}$.

Check: $2x + 3y = 2 \cdot \frac{11}{5} + 3\left(-\frac{4}{5}\right) = \frac{22-12}{5} = 2.$

$3x + 2y = \frac{33-8}{5} = \frac{25}{5} = 5.$ ✓

3. A linear transformation $T : R^3 \to R^3$ maps the unit vectors e_1, e_2, and e_3 to the vectors

$$v_1 = (1, 2, 3), \quad v_2 = (0, -1, 1), \quad \text{and} \quad v_3 = (0, 1, 2),$$

respectively. Find the volume of the tetrahedron determined by the vectors v_1, v_2, v_3, and the origin. (Hint: the volume of the tetrahedron determined by the vectors e_1, e_2, e_3, and the origin equals 1/6.)

Let D be the tetrahedron determined by v_1, v_2, v_3 and 0, and let E be the tetrahedron determined by e_1, e_2, e_3 and 0.

Volume (D) = $|\det(v_1, v_2, v_3)| \cdot$ Volume (E). $\begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{vmatrix} = |-2-1| = 3$

Volume (E) = $\frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{6}$.

Volume (D) = $3 \cdot \frac{1}{6} = \boxed{\frac{1}{2}}$