Test 2

Problem	1	2	3	4	5	6		Grade
Points	/10	/10	/10	/10	/10	/10		/80

NAME: Solution key

Show all your work for full credit.

Problem 1. Prove

1. $$\frac{x^3 + 3x^2 + 2x}{3x - 1} = \Theta(x^2)$$

We need to prove that there are positive constants C_1, C_2, K such that $C_1 |3x-1| x^2 \le |x^3 + 3x^2 + 2x| \le C_2 |3x-1| x^2$.

We will use the fact: $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = \Theta(x^n)$ if $a_n \ne 0$.

Note $3x - 1 = \Theta(x) \Rightarrow \exists C_1, C_2, K_1 : C_1|x| \le |3x-1| \le C_2|x|$ for $|x| \ge K_1$

$x^3 + 3x^2 + 2x = \Theta(x^3) \Rightarrow \exists M_1, M_2, K_2 : M_1|x|^3 \le |x^3 + 3x^2 + 2x| \le M_2 |x|^3$, for $|x| \ge K_2$

Note

$$\left.\begin{aligned} \frac{|x^3+3x^2+2x|}{|3x-1|} &\le \frac{M_2}{C_1}\frac{|x|^3}{|x|} = \frac{M_2}{C_1} x^2 \\ \frac{|x^3+3x^2+2x|}{|3x-1|} &\ge \frac{M_1 |x|^3}{C_2 |x|} = \frac{M_1}{C_2} x^2 \end{aligned}\right\} \Rightarrow \frac{x^3+3x^2+2x}{3x-1} = \Theta(x^2).$$

2. $$x^3 - 3x + 5 = \Omega(x^3)$$

We need to prove that $\exists C, K > 0$ such that $|x^3 - 3x + 5| \ge C|x^3|$ for $|x| \ge K$.

$$|x^3 - 3x + 5| = \left|\tfrac{1}{2}x^3 + \left(\tfrac{1}{4}x^3 - 3x\right) + \left(\tfrac{1}{4}x^3 + 5\right)\right| \ge \tfrac{1}{2}|x|^3 + \left(\tfrac{1}{4}|x^3| - 3|x|\right) + \left(\tfrac{1}{4}|x|^3 - 5\right)$$

$$\ge \tfrac{1}{2}|x|^3 \quad \text{for } |x| \ge \sqrt{12} \text{ since}$$

$\frac{1}{4}|x|^3 \ge 3|x| \Rightarrow x^2 \ge 12 \Rightarrow |x| \ge \sqrt{12} \approx 3.5$

$\frac{1}{4}|x|^3 \ge 5 \Rightarrow |x|^3 \ge 20 \Rightarrow |x| \ge \sqrt[3]{20} \approx 2.7$.

and $\sqrt{12} > \sqrt[3]{20}$.