Quiz 4

Name: Solution Key

1. Prove

$$f(x) = \frac{6x^5 - 2x^3 + 5}{2x^3 - 2x - 4} = \Theta(x^2).$$

Since $6x^5 - 2x^3 + 5 = \Theta(x^5)$, $\exists\ C_1, C_2 > 0$ and $k > 0$ such that

$C_1|x^5| \le |6x^5 - 2x^3 + 5| \le C_2|x^5|$, $\forall\ |x| > k$.

Since $2x^3 - 2x - 4 = \Theta(x^3)$, $\exists\ \bar{C}_1, \bar{C}_2 > 0$ and $\bar{k} > 0$ such that

$\bar{C}_1|x^3| \le |2x^3 - 2x - 4| \le \bar{C}_2|x^3|$, for all $|x| > \bar{k}$.

Note

$$|f(x)| = \left|\frac{6x^5 - 2x^3 + 5}{2x^3 - 2x - 4}\right| \le \frac{C_2|x^5|}{\bar{C}_1|x^3|} = \frac{C_2}{\bar{C}_1}x^2, \text{ for } |x| > \max\{k, \bar{k}\}$$

$$|f(x)| \ge \frac{C_1|x^5|}{\bar{C}_2|x^3|} = \frac{C_1}{\bar{C}_2}x^2, \text{ for } |x| > \max\{k, \bar{k}\}.$$

Therefore,

$$f(x) = \Theta(x^2).$$

2. Using Θ-notation, estimate the number of print statements in the following algorithm: (Hint: use $1 + 2 + \ldots + n = n(n+1)/2$)

```
for i = 1,... m
    print "hello"
    for j = i,... n
        print "bye"
    end
end
```

Consider two cases: $m \le n$ and $m > n$.

1. Let $m \le n$.
   Let C = # of print statements.
   $C \le mn$. $\Rightarrow C(m,n) = O(mn)$.

$$C = \sum_{i=1}^{m}\left(1 + \sum_{j=i}^{n} 1\right) = \sum_{i=1}^{m}(1 + n - i + 1) = (n+2)\sum_{i=1}^{m} 1 - \sum_{i=1}^{m} i = m(n+2) - \frac{m(m+1)}{2}$$

$$\overset{(m \le n)}{\ge} mn - \frac{m(n+1)}{2} = \frac{mn}{2} - \frac{m}{2} = \frac{mn}{4} + \left(\frac{mn}{4} - \frac{m}{2}\right) \ge \frac{mn}{4} \text{ for } n \ge 2.$$

Therefore, $\boxed{C = \Theta(mn)}$

2. Let $m > n$. Consider $i = 1, \ldots, n$ and $i = n+1, \ldots, m$.

$$C = \sum_{i=1}^{n}\left(\sum_{j=i}^{n} 1 + 1\right) + \sum_{i=n+1}^{m} 1 = \sum_{i=1}^{n}(n+2-i) + m - n = (n+2)n - \frac{n(n+1)}{2} - n + m$$

$$= \frac{n^2}{2} + \frac{n}{2} + m.$$

Note, $\exists\ M_1, M_2 > 0$ such that $M_1 n^2 \le \frac{n^2}{2} + \frac{n}{2} \le M_2 n^2$ for sufficiently large n.

Therefore $M_1 n^2 + m \le C \le M_2 n^2 + m$ and

$$\boxed{C = \Theta(n^2 + m)}$$