

A Quadrature Finite Element Galerkin Scheme for a Biharmonic Problem on a Rectangular Polygon

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A quadrature Galerkin scheme with the Bogner–Fox–Schmit element for a biharmonic problem on a rectangular polygon is analyzed for existence, uniqueness, and convergence of the discrete solution. It is known that a product Gaussian quadrature with at least three-points is required to guarantee optimal order convergence in Sobolev norms. In this article, optimal order error estimates are proved for a scheme based on the product two-point Gaussian quadrature by establishing a relation with an underdetermined orthogonal spline collocation scheme. © 2007 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 00: 000–000, 2007

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I. INTRODUCTION

Let $\Omega \subset R^2$ be an open rectangular polygon with the boundary $\partial\Omega$ aligned with the coordinate axes. In this article, we propose a two-point Gaussian quadrature finite-element Galerkin scheme with the Bogner–Fox–Schmit element (see Section 2.2 in [1]) for the numerical solution of the biharmonic Dirichlet boundary value problem (BVP)

$$\Delta^2 u = f \text{ in } \Omega, \quad \text{and } u = \partial_n u = 0 \quad \text{on } \partial\Omega, \quad (1.1)$$

where Δ is the Laplace operator, and ∂_n is the outer normal derivative. We study well-posedness of the discrete problem and obtain error estimates in Sobolev norms.

Efficient methods for the numerical solution of biharmonic problems are important in several applications. For example, the biharmonic equation describes vertical displacements of thin plates in plane elasticity. In fluid mechanics, the streamfunction of the steady-state Stokes flow satisfies the biharmonic equation, and biharmonic problems arise in the numerical solution of incompressible fluid flow models described by the Navier-Stokes equations.

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Finite element methods for solving biharmonic problems are usually of a direct or a mixed type depending, respectively, on whether the biharmonic equation is discretized directly or via the introduction of a pair of second order equations [1].

In actual implementations of finite element Galerkin methods, the stiffness matrix and the load vector are formed using numerical integration, and the resulting discrete scheme is called a quadrature finite element scheme. Although finite element methods for biharmonic problems have been extensively studied, the analysis of quadrature finite element schemes has received significantly less attention.

The main criterion in selecting an integration rule for a quadrature finite element Galerkin scheme is to preserve the accuracy of the finite element solution. It is known that a product Gaussian quadrature rule with at least three points should be used for a quadrature approximation of the finite element Galerkin method with the Bogner–Fox–Schmit rectangle (see [2, Theorem 8.9]). In this article, we propose the product two-point Gaussian quadrature scheme for the finite element Galerkin solution of BVP (1.1), prove well-posedness of the discrete problem, and obtain optimal order error estimates in Sobolev norms. The stiffness matrix and the load vector of the two-point scheme are formed much faster than those of the three-point scheme since fewer function evaluations are required. Our numerical results demonstrate superconvergence properties of the quadrature solution.

Our scheme is related to the OSC method (see [3]), which uses Gaussian quadrature nodes as collocation points and a finite element space of C^1 piecewise polynomial functions. Our scheme has advantages of simple formulation, efficient computation, and superconvergence, which are typical for the OSC method. The results in this article can be viewed as an extension of the error analysis of the OSC scheme for the biharmonic problem on a rectangle given in [4] to the case of a rectangular polygonal domain. We note that this extension is not straight forward since most of the previous studies of the OSC schemes have been carried out on a rectangle. Our analysis requires higher than optimal solution regularity assumptions which are typical for OSC schemes. The linear system of the quadrature Galerkin scheme can efficiently be solved by a multilevel method developed in [5]. In this article, we also demonstrate that the standard analysis of the quadrature finite element Galerkin scheme based on the First Strang Lemma yields a suboptimal order error estimate, whereas our analysis based on an equivalent OSC problem gives optimal order error estimates.

The outline of the rest of the article is as follows. An H^2 -error estimate of the finite element solution which is consistent with the regularity is presented in Section II. Well-posedness of the quadrature problem is proved in Section III. In Section IV, it is shown that the error analysis of the quadrature scheme based on the First Strang Lemma yields a suboptimal order error estimate. In Section V, the equivalence of the quadrature Galerkin scheme and a mixed OSC scheme is proved and optimal order error estimates are obtained. Numerical results are presented in Section VI, and our concluding remarks are given in Section VII.

II. FINITE ELEMENT GALERKIN PROBLEM

In this section, we present known results on regularity of the biharmonic problem and give the H^2 -error estimate of the finite element solution that is consistent with regularity. Let $D \subset R^2$ be an open, bounded set and $x = (x_1, x_2) \in R^2$. For an integer $m \geq 0$, let $C^m(\bar{D})$ be the Banach space of m times continuously differentiable functions with the standard norm $\|v\|_{C^m(\bar{D})}$. Let $L^2(D)$ be the Hilbert space of square integrable functions on D with the norm $\|v\|_{L^2(D)} = (\int_D v^2 dx)^{1/2}$. Let $H^m(D)$ be the Sobolev space with the standard norm $\|v\|_{H^m(D)}$ and

the seminorm $|v|_{H^m(D)} = (\sum_{|\alpha|=m} \|\partial^\alpha v\|_{L^2(D)}^2)^{1/2}$, where $|\alpha| = \alpha_1 + \alpha_2$ and $\partial^\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2}$ for a multi-index $\alpha = (\alpha_1, \alpha_2)$. Let $H_0^2(\Omega)$ be the closure of C^∞ -functions with compact support in Ω in the H^2 -norm. It is known that $|\cdot|_{H^2(\Omega)}$ and $\|\cdot\|_{H^2(\Omega)}$ are equivalent norms on $H_0^2(\Omega)$. For a real $s > 0$, let $H^s(\Omega)$ be the standard fractional order Sobolev space with the dual space $H^{-s}(\Omega)$. Throughout this article, $C > 0$ is a generic constant independent of the problem and discretization parameters.

BVP (1.1) has the following variational form [1, Section 1.2]. Find $u \in H_0^2(\Omega)$ such that

$$a(u, v) = (f, v) \quad \text{for all } v \in H_0^2(\Omega), \tag{2.1}$$

where the bilinear and linear forms $a(\cdot, \cdot)$ and (f, \cdot) are defined by

$$a(w, v) = \int_\Omega \Delta w \Delta v \, dx \quad \text{and} \quad (f, v) = \int_\Omega f v \, dx. \tag{2.2}$$

Using the representation $a(v, v) = |v|_{H^2(\Omega)}^2$ for $v \in H_0^2(\Omega)$ (see (1.2.8) in [1]) and the norm equivalence $\|\cdot\|_{H^2(\Omega)} \sim |\cdot|_{H^2(\Omega)}$, it is easy to see that the bilinear form $a(\cdot, \cdot)$ is coercive and bounded on $H_0^2(\Omega)$; that is,

$$a(v, v) \geq C \|v\|_{H^2(\Omega)}^2, \quad v \in H_0^2(\Omega), \tag{2.3}$$

$$|a(v, w)| \leq C |v|_{H^2(\Omega)} |w|_{H^2(\Omega)}, \quad v, w \in H_0^2(\Omega). \tag{2.4}$$

It follows from (2.2)–(2.4) and the Riesz Representation Theorem that problem (2.1) has a unique solution $u \in H_0^2(\Omega)$ such that

$$\|u\|_{H^2(\Omega)} \leq C \|f\|_{H^{-2}(\Omega)}.$$

If Ω is a rectangle then

$$\|u\|_{H^{4-s}(\Omega)} \leq C \|f\|_{H^{-s}(\Omega)} \quad \text{for } f \in H^{-s}(\Omega), \quad s = 0, 1,$$

(see Theorem 2 in [6]). If Ω has the re-entrant corner with measure $3\pi/2$, then there exists $s_0 \in (0, 2)$ such that

$$\|u\|_{H^{2+2s}(\Omega)} \leq C \|f\|_{H^{-2+2s}(\Omega)} \quad \text{for } f \in H^{-2+2s}(\Omega), \quad 0 \leq 2s \leq s_0, \tag{2.5}$$

(see [7]).

Let \mathcal{T}_h be a regular rectangular partition of Ω , and let h be the largest diameter of elements in \mathcal{T}_h . Let \mathcal{Q}_3 be the vector space of bicubic polynomials, and let

$$X_h = \{v \in C^1(\overline{\Omega}) : v|_K \in \mathcal{Q}_3|_K, K \in \mathcal{T}_h\}$$

be the finite element space of Bogner–Fox–Schmit rectangles. Let

$$V_h = \{v \in X_h : v = \partial_n v = 0 \text{ on } \partial\Omega\}.$$

Note that ∇v and $v_{x_1 x_2}$ vanish on $\partial\Omega$ for any $v \in V_h$, and $V_h \subset H_0^2(\Omega)$ [1, Theorem 2.2.15].

The finite element Galerkin problem approximating the variational problem (2.1) is formulated as follows: find $U_h \in V_h$ such that

$$a(U_h, v) = (f, v) \quad \text{for all } v \in V_h, \tag{2.6}$$

where the forms $a(\cdot, \cdot)$ and (f, \cdot) are defined in (2.2). It follows from (2.3) that V_h is a Hilbert space with the inner product $a(\cdot, \cdot)$. Using the Riesz Representation Theorem, it is easy to see that problem (2.6) has a unique solution. If $u \in H^4(\Omega) \cap H_0^2(\Omega)$ then

$$\|u - U_h\|_{H^2(\Omega)} \leq Ch^2|u|_{H^4(\Omega)}, \quad (2.7)$$

where u is the solution of problem (2.1) [1, Theorem 6.1.6].

The regularity assumption for (2.7) is satisfied for a rectangle but not for a domain with a re-entrant corner (see [6]). Using (2.3)–(2.7) and the real method of interpolation of Sobolev spaces, we obtain the following error estimate consistent with the solution regularity.

Theorem 2.1. *Let u and U_h be the solutions of problems (2.1) and (2.6), respectively. There exists $s_0 \in (0, 2)$ such that, for $s \in (0, s_0/2)$ and $f \in H^{-2+2s}(\Omega)$,*

$$\|u - U_h\|_{H^2(\Omega)} \leq Ch^{2s}\|f\|_{H^{-2+2s}(\Omega)}. \quad (2.8)$$

III. EXISTENCE AND UNIQUENESS OF A QUADRATURE GALERKIN SOLUTION

In this section, we define our quadrature Galerkin scheme and prove that it has a unique solution. For an interval $I = (a, b)$, let $|I| = b - a$ and let

$$\mathcal{G}_I = \{a + |I|(3 - \sqrt{3})/6, a + |I|(3 + \sqrt{3})/6\}$$

be the Gauss points on I . The error of the two-point Gaussian quadrature

$$\sum_{\mathcal{G}_I} v = (|I|/2) \sum_{\xi \in \mathcal{G}_I} v(\xi)$$

is given by

$$\int_a^b v(t)dt = \sum_{\mathcal{G}_I} v + \frac{|I|^5}{4320} v^{(4)}(\eta), \quad \eta \in I, \quad v \in C^4(I) \quad (3.1)$$

(see (2.7.12) in [8]). For a rectangle $K = I_1 \times I_2$, let $\mathcal{G}_K = \mathcal{G}_{I_1} \times \mathcal{G}_{I_2}$ be the set of four Gauss points on K and let

$$\sum_{\mathcal{G}_K} v \equiv \sum_{\mathcal{G}_{I_1}} \sum_{\mathcal{G}_{I_2}} v = \frac{|K|}{4} \sum_{\xi \in \mathcal{G}_K} v(\xi), \quad v \in C(K), \quad (3.2)$$

be the product two-point Gaussian quadrature on K , where $|K|$ is the area of K . For any functions v and w defined on \mathcal{T}_h , let

$$(v, w)_h = \sum_{K \in \mathcal{T}_h} \sum_{\mathcal{G}_K} vw \quad \text{and} \quad \|v\|_h = \sqrt{(v, v)_h}. \quad (3.3)$$

Using the Cauchy–Schwarz inequality, it is easy to verify that

$$|(v, w)_h| \leq \|v\|_h \|w\|_h. \quad (3.4)$$

The quadrature finite element Galerkin problem is formulated as follows: find $u_h \in V_h$ such that

$$a_h(u_h, v) = (f, v)_h \text{ for all } v \in V_h, \tag{3.5}$$

where, by (3.3),

$$a_h(w, v) = (\Delta w, \Delta v)_h \text{ for } w, v \in V_h \tag{3.6}$$

To prove coercivity of the bilinear form a_h , we introduce the following two partitions of the domain

$$\bar{\Omega} = \bigcup_{l=1}^{L_V} \bar{R}_V^l = \bigcup_{l=1}^{L_H} \bar{R}_H^l, \tag{3.7}$$

where the sets $\{R_V^l\}_{l=1}^{L_V}$ and $\{R_H^l\}_{l=1}^{L_H}$ consist of open disjoint rectangles whose vertical and horizontal edges, respectively, belong to the boundary $\partial\Omega$ (see Fig. 1 for an example with $L_V = L_H = 3$).

Triangulation \mathcal{T}_h determines the partition $\pi_V^l = \pi_{V,1}^l \times \pi_{V,2}^l$ of the rectangle R_V^l for $l = 1, \dots, L_V$, where one-dimensional partitions $\pi_{V,i}^l, i = 1, 2$, consist of subintervals. Similarly, for $l = 1, \dots, L_H, \pi_H^l = \pi_{H,1}^l \times \pi_{H,2}^l$ is the partition of the rectangle R_H^l determined by \mathcal{T}_h . It follows from (3.7) that, for any set $\{s_K\}_{K \in \mathcal{T}_h} \subset R$,

$$\sum_{l=1}^{L_V} \sum_{I_1 \in \pi_{V,1}^l} \sum_{I_2 \in \pi_{V,2}^l} s_{I_1 \times I_2} = \sum_{K \in \mathcal{T}_h} s_K = \sum_{l=1}^{L_H} \sum_{I_1 \in \pi_{H,1}^l} \sum_{I_2 \in \pi_{H,2}^l} s_{I_1 \times I_2}. \tag{3.8}$$

The following two lemmas will be used to prove coercivity and boundedness of the approximate bilinear form $a_h(\cdot, \cdot)$ with respect to the H^2 -norm.

Lemma 3.1. For any $v \in X_h$ such that $v = 0$ on $\partial\Omega$,

$$\|v_{x_i x_i}\|_h \geq C \|v_{x_i x_i}\|_{L^2(\Omega)}, \quad i = 1, 2, \tag{3.9}$$

$$(v_{x_1 x_1}, v_{x_2 x_2})_h \geq \|v_{x_1 x_2}\|_{L^2(\Omega)}^2. \tag{3.10}$$

Proof. We prove (3.9) only for $i = 1$ since the case $i = 2$ is similar. Take any $v \in X_h$ vanishing on $\partial\Omega$. Take any $R_H^l = I_1^l \times I_2^l$ and $x_1 \in I_1^l$. Restricted to the vertical line segment

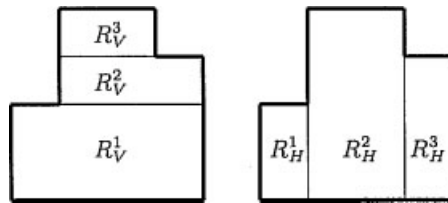


FIG. 1. Partitions $\{R_V^l\}_{l=1}^3$ and $\{R_H^l\}_{l=1}^3$.

$\{x_1\} \times I_2^l$, $v_{x_1x_1}$ is a piecewise Hermite cubic polynomial function vanishing at the end points of I_2^l . The inequality

$$\sum_{I_2 \in \pi_{H,2}^l} \sum_{\mathcal{G}_{I_2}} v_{x_1x_1}^2(x_1, \cdot) \geq C \sum_{I_2 \in \pi_{H,2}^l} \int_{I_2} v_{x_1x_1}^2(x_1, x_2) dx_2, \tag{3.11}$$

which follows directly from (2.6) in [9], plays the key role in proving (3.9).

Using (3.3), the second identity in (3.8), the fact that $v_{x_1x_1}^2$ is a polynomial of degree ≤ 2 in the x_1 -variable and the exactness of the two-point Gaussian quadrature, interchanging the order of the summation over \mathcal{G}_{I_2} and the integration over I_1 , we obtain

$$\begin{aligned} \|v_{x_1x_1}\|_h^2 &= \sum_{K \in \mathcal{T}_h} \sum_{\mathcal{G}_K} v_{x_1x_1}^2 = \sum_{l=1}^{L_H} \sum_{I_1 \in \pi_{H,1}^l} \sum_{I_2 \in \pi_{H,2}^l} \sum_{\mathcal{G}_{I_1}} \sum_{\mathcal{G}_{I_2}} v_{x_1x_1}^2 \\ &= \sum_{l=1}^{L_H} \sum_{I_1 \in \pi_{H,1}^l} \sum_{I_2 \in \pi_{H,2}^l} \sum_{\mathcal{G}_{I_2}} \int_{I_1} v_{x_1x_1}^2(x_1, \cdot) dx_1. \end{aligned}$$

Applying (3.11) to the right hand side of the last identity and using the second identity in (3.8), we get

$$\|v_{x_1x_1}\|_h^2 \geq C \sum_{l=1}^{L_H} \sum_{I_1 \in \pi_{H,1}^l} \sum_{I_2 \in \pi_{H,2}^l} \int_{I_1 \times I_2} v_{x_1x_1}^2 dx = C \sum_{K \in \mathcal{T}_h} \int_K v_{x_1x_1}^2 dx = C \|v_{x_1x_1}\|_{L^2(\Omega)}^2,$$

which is (3.9) for $i = 1$.

Let us prove (3.10). Using (3.3), the first identity in (3.8), and (3.1) in the x_1 -variable, we obtain

$$\begin{aligned} (v_{x_1x_1}, v_{x_2x_2})_h &= \sum_{K \in \mathcal{T}_h} \sum_{\mathcal{G}_K} v_{x_1x_1} v_{x_2x_2} = \sum_{l=1}^{L_V} \sum_{I_1 \in \pi_{V,1}^l} \sum_{I_2 \in \pi_{V,2}^l} \sum_{\mathcal{G}_{I_1}} \sum_{\mathcal{G}_{I_2}} v_{x_1x_1} v_{x_2x_2} \\ &= S_1 + 4320^{-1} S_2, \end{aligned} \tag{3.12}$$

where

$$S_1 = \sum_{l=1}^{L_V} \sum_{I_1 \in \pi_{V,1}^l} \sum_{I_2 \in \pi_{V,2}^l} \sum_{\mathcal{G}_{I_2}} \int_{I_1} (v_{x_1x_1} v_{x_2x_2})(x_1, \cdot) dx_1, \tag{3.13}$$

$$S_2 = - \sum_{l=1}^{L_V} \sum_{I_1 \in \pi_{V,1}^l} |I_1|^5 \sum_{I_2 \in \pi_{V,2}^l} \sum_{\mathcal{G}_{I_2}} (\partial^{(3,0)} v \partial^{(3,2)} v)(t_{I_1}(\cdot), \cdot), \tag{3.14}$$

with some $\{t_{I_1}(\xi_2)\}_{\xi_2 \in \mathcal{G}_{I_2}}$.

The estimate (3.10) follows from (3.12) and the inequalities

$$S_1 \geq \|v_{x_1x_2}\|_{L^2(\Omega)}^2, \tag{3.15}$$

$$S_2 \geq 0, \tag{3.16}$$

which we now prove. First, we obtain (3.15). Applying (3.1) in the x_2 -variable on S_1 in (3.13), we get

$$S_1 = S_{11} + 4320^{-1} S_{12}, \tag{3.17}$$

where

$$S_{11} = \sum_{l=1}^{L_V} \sum_{I_1 \in \pi_{V,1}^l} \sum_{I_2 \in \pi_{V,2}^l} \int_{I_1 \times I_2} v_{x_1 x_1} v_{x_2 x_2} dx = \sum_{l=1}^{L_V} \int_{R_V^l} v_{x_1 x_1} v_{x_2 x_2} dx, \tag{3.18}$$

$$S_{12} = - \sum_{l=1}^{L_V} \sum_{I_1 \in \pi_{V,1}^l} \sum_{I_2 \in \pi_{V,2}^l} |I_2|^5 \int_{I_1} (\partial^{(2,3)} v \partial^{(0,3)} v)(x_1, t_{I_2}(x_1)) dx_1, \tag{3.19}$$

with some $t_{I_2}(x_1) \in I_2$ for any $x_1 \in I_1$. Since the function $\partial^{(2,3)} v \partial^{(0,3)} v$ is constant in the x_2 -variable on $I_1 \times I_2$, we set

$$t_{I_2}(x_1) = t_{I_1 \times I_2} = \text{const}, \quad \text{for all } x_1 \in I_1. \tag{3.20}$$

From (3.19), using integration by parts in the x_1 -variable, (3.20), continuity in the x_1 -variable of $(\partial^{(1,3)} v \partial^{(0,3)} v)$, and the fact that $\partial^{(0,3)} v$ vanishes on the vertical edges of R_V^l , we obtain

$$S_{12} = \sum_{l=1}^{L_V} \sum_{I_1 \in \pi_{V,1}^l} \sum_{I_2 \in \pi_{V,2}^l} |I_2|^5 \int_{I_1} (\partial^{(1,3)} v)^2(x_1, t_{I_1 \times I_2}) dx_1 \geq 0. \tag{3.21}$$

We now prove

$$S_{11} = \|v_{x_1 x_2}\|_{L^2(\Omega)}^2, \tag{3.22}$$

where S_{11} is defined by (3.18). Using integration by parts in the x_1 -variable, continuity of $v_{x_1} v_{x_2 x_2}$ in the x_1 -variable, the fact that $v_{x_2 x_2}$ vanishes on the vertical edges of R_V^l , and the first representation of Ω in (3.7), we obtain

$$S_{11} = - \sum_{l=1}^{L_V} \sum_{R_V^l} \int_{R_V^l} v_{x_1} v_{x_1 x_2 x_2} dx = - \int_{\Omega} v_{x_1} v_{x_1 x_2 x_2} dx.$$

Similarly, using the second representation of Ω in (3.7), integration by parts in the x_2 -variable, continuity of $v_{x_1} v_{x_1 x_2}$, and the fact that v_{x_1} vanishes on the horizontal edges of R_H^l , we get (3.22). Relations (3.17), (3.22), and (3.21), imply (3.15).

It remains to prove (3.16), where S_2 is defined by (3.14) and used in (3.12). Since function $\partial^{(3,0)} v \partial^{(3,2)} v$ is constant in the x_1 -variable on $I_1 \times I_2$, in (3.14), we set

$$t_{I_1}(x_2) = t_{I_1 \times I_2} = \text{const}, \quad \text{for all } x_2 \in I_2. \tag{3.23}$$

From (3.14), using (3.23), (3.8), and (3.1) in x_2 -direction, we obtain

$$S_2 = - \sum_{l=1}^{L_H} \sum_{I_1 \in \pi_{H,1}^l} |I_1|^5 \sum_{I_2 \in \pi_{H,2}^l} \sum_{\mathcal{G}I_2} (\partial^{(3,0)} v \partial^{(3,2)} v)(t_{I_1 \times I_2}, \cdot) = S_{21} + 4320^{-1} S_{22}, \tag{3.24}$$

where

$$S_{21} = - \sum_{l=1}^{L_H} \sum_{I_1 \in \pi_{H,1}^l} |I_1|^5 \sum_{I_2 \in \pi_{H,2}^l} \int_{I_2} (\partial^{(3,0)} v \partial^{(3,2)} v)(t_{I_1 \times I_2}, x_2) dx_2, \tag{3.25}$$

$$S_{22} = \sum_{l=1}^{L_H} \sum_{I_1 \in \pi_{H,1}^l} |I_1|^5 \sum_{I_2 \in \pi_{H,2}^l} |I_2|^5 (\partial^{(3,3)} v)^2|_{I_1 \times I_2} \geq 0. \tag{3.26}$$

We note that, in (3.26), $\partial^{(3,3)} v$ is constant on $I_1 \times I_2$.

From (3.25), using integration by parts in x_2 -direction, continuity in the x_2 -variable of $(\partial^{(3,0)} v \partial^{(3,1)} v)$, and the fact that $\partial^{(3,0)}$ vanishes on the horizontal boundaries of R_H^l , we obtain

$$S_{21} = \sum_{l=1}^{L_H} \sum_{I_1 \in \pi_{H,1}^l} |I_1|^5 \sum_{I_2 \in \pi_{H,2}^l} \int_{I_2} (\partial^{(3,1)} v)^2(t_{I_1 \times I_2}, x_2) dx_2 \geq 0. \tag{3.27}$$

Identity (3.24) and the inequalities (3.27) and (3.26) imply (3.16). ■

Lemma 3.2.

$$\|\partial^\alpha v\|_h \leq C \|\partial^\alpha v\|_{L^2(\Omega)} \quad \text{for } v \in X_h \text{ and } |\alpha| \leq 2. \tag{3.28}$$

Proof. Take any $v \in X_h$. Applying the inverse inequality

$$\|w\|_{C(K)} \leq Ch^{-1} \|w\|_{L^2(K)}, \quad K \in \mathcal{T}_h, \quad w \in X_h,$$

(see (3.2.33) in [1]) and the inclusion $X_h \subset H^2(\Omega)$, we obtain

$$\|\partial^\alpha v\|_h^2 = \sum_{K \in \mathcal{T}_h} \frac{|K|}{4} \sum_{\xi \in \mathcal{G}_K} |(\partial^\alpha v)(\xi)|^2 \leq C \sum_{K \in \mathcal{T}_h} \|\partial^\alpha v\|_{L^2(K)}^2 = C \|\partial^\alpha v\|_{L^2(\Omega)}^2. \tag{3.29}$$

The following lemma states that the approximate bilinear form $a_h(\cdot, \cdot)$ is coercive and bounded relative to the H^2 -norm.

Lemma 3.3.

$$C \|v\|_{H^2(\Omega)}^2 \leq a_h(v, v) \quad \text{for } v \in V_h, \tag{3.29}$$

$$|a_h(v, w)| \leq C |v|_{H^2(\Omega)} |w|_{H^2(\Omega)} \quad \text{for } v, w \in V_h. \tag{3.30}$$

Proof. Inequality (3.29) follows from the representation

$$a_h(v, v) = \|v_{x_1 x_1}\|_h^2 + 2(v_{x_1 x_1}, v_{x_2 x_2})_h + \|v_{x_2 x_2}\|_h^2, \quad v \in V_h,$$

Lemma 3.1, and the fact that $|\cdot|_{H^2(\Omega)}$ is equivalent to $\|\cdot\|_{H^2(\Omega)}$ on $H_0^2(\Omega)$. Using (3.6), (3.4), the Cauchy-Schwarz inequality in R^4 , and (3.28), we obtain, for any $v, w \in V_h$,

$$\begin{aligned} |a_h(v, w)| &\leq \sum_{i,j=1}^2 \|v_{x_i x_i}\|_h \|w_{x_j x_j}\|_h \\ &\leq 2(\|v_{x_1 x_1}\|_h^2 + \|v_{x_2 x_2}\|_h^2)^{1/2} (\|w_{x_1 x_1}\|_h^2 + \|w_{x_2 x_2}\|_h^2)^{1/2} \\ &\leq C|v|_{H^2(\Omega)}|w|_{H^2(\Omega)}, \end{aligned}$$

which is (3.30). ■

Theorem 3.4. *For any $f \in C(\Omega)$, the quadrature finite element Galerkin scheme (3.5) has a unique solution u_h and*

$$\|u_h\|_{H^2(\Omega)} \leq C\|f\|_h.$$

Proof. It follows from (3.6) and Lemma 3.3 that V_h is a Hilbert space with the inner product $a_h(\cdot, \cdot)$. A linear functional $(f, \cdot)_h$ is bounded on V_h . Therefore, the statement of the theorem follows from the Riesz Representation Theorem. ■

IV. AN H^2 -ERROR ESTIMATE BY THE FIRST STRANG LEMMA

In this section, we present the H^2 error estimate of the quadrature solution obtained using the standard approach based on the First Strang Lemma [1, Theorem 4.1.1]. To begin with, we introduce a piecewise Hermite bicubic interpolant and state some of its properties. For any $v \in C^2(K)$, let $\Pi_K v \in Q_3|_K$ be the bicubic Hermite interpolant of v defined by

$$\partial^\alpha(\Pi_K v)(a_i) = \partial^\alpha v(a_i), \quad 1 \leq i \leq 4, \quad |\alpha_j| \leq 1, \quad j = 1, 2, \tag{4.1}$$

where $\{a_i\}_{i=1}^4$ are the vertices of rectangle K . For $v \in C^2(\Omega)$, let $\Pi_h v \in X_h$ be the piecewise Hermite bicubic interpolant of v defined by

$$(\Pi_h v)|_K = \Pi_K(v|_K), \quad \text{for all } K \in \mathcal{T}_h. \tag{4.2}$$

The following two consistency results are proved using the Bramble-Hilbert Lemma [1, Theorem 4.1.3]), and the proofs are similar to those of Theorems 4.1.4 and 4.1.5 in [1].

Lemma 4.1. *If $u \in H^4(\Omega)$ then*

$$|a(\Pi_h u, w) - a_h(\Pi_h u, w)| \leq Ch\|u\|_{H^4(\Omega)}|w|_{H^2(\Omega)}, \quad w \in V_h. \tag{4.3}$$

Note that the estimate (4.3) has a suboptimal order $O(h)$.

Lemma 4.2. *If $f \in C^2(\Omega)$ then*

$$|(f, v) - (f, v)_h| \leq Ch^2\|f\|_{C^2(\Omega)}\|v\|_{H^1(\Omega)}, \quad v \in V_h.$$

Theorem 4.3. *Let u and u_h be the solutions of problems (2.1) and (3.5), respectively. If $u \in H^4(\Omega)$ and $f \in C^2(\Omega)$ then*

$$\|u - u_h\|_{H^2(\Omega)} \leq Ch^2\|f\|_{C^2(\Omega)} + Ch\|u\|_{H^4(\Omega)}. \tag{4.4}$$

Proof. The statement follows from the First Strang Lemma (Theorem 4.1.1 in [1]), coercivity and boundedness of the approximate bilinear form $a_h(\cdot, \cdot)$ proved in Lemma 3.3, and the consistency results presented in Lemma 4.1 and Lemma 4.2. ■

The error estimate (4.4) has a suboptimal order $O(h)$. We note that the H^2 error estimate of the quadrature scheme based on the three-point Gaussian quadrature is optimal [2, Theorem 8.9]. In the next section, we obtain optimal order estimates of H^1 - and H^2 -errors by introducing an auxiliary orthogonal spline collocation problem and assuming higher than optimal regularity.

V. AN EQUIVALENT MINIMUM NORM LEAST SQUARES OSC PROBLEM

In this section, we introduce a mixed orthogonal spline collocation scheme, show its relation with our quadrature Galerkin scheme, and obtain optimal order error estimates of the quadrature solution. As in [4], we consider the following coupled form of problem (1.1):

$$\Delta u = v, \Delta v = f \quad \text{in } \Omega, \quad \text{and} \quad u = \partial_n u = 0 \quad \text{on } \partial\Omega. \tag{5.1}$$

Let $\mathcal{G}_h = \cup_{K \in \mathcal{T}_h} \mathcal{G}_K$ be the set of Gauss points in Ω . We consider the following orthogonal spline collocation scheme for problem (5.1): find $u_h \in V_h$ and $v_h \in X_h$, such that

$$\Delta u_h(\xi) = v_h(\xi), \quad \text{for all } \xi \in \mathcal{G}_h, \tag{5.2}$$

$$\Delta v_h(\xi) = f(\xi), \quad \text{for all } \xi \in \mathcal{G}_h. \tag{5.3}$$

In [4], the mixed problem (5.1) with $\Omega = (0, 1)^2$ is approximated by the OSC scheme (5.2)–(5.3) with the additional eight constraints

$$v_h(a, b) = \frac{\partial v_h}{\partial x_2}(a, b) = 0 \quad \text{for } a, b = 0, 1. \tag{5.4}$$

The authors proved existence and uniqueness of the solution and obtained optimal order H^k -error estimates for $k = 0, 1, 2$. We obtain a similar result for the quadrature scheme (3.5).

To prove existence of a solution of problem (5.2)–(5.3), we require the following two lemmas. The first lemma presents a discrete version of Green’s formula.

Lemma 5.1.

$$(\Delta v, w)_h = (v, \Delta w)_h, \quad v \in V_h, w \in X_h. \tag{5.5}$$

Proof. Take any $v \in V_h$ and $w \in X_h$. It suffices to prove

$$(v_{x_1 x_1}, w)_h = (v, w_{x_1 x_1})_h, \tag{5.6}$$

since the proof of $(v_{x_2x_2}, w)_h = (v, w_{x_2x_2})_h$ is similar. Using (3.3), (3.2), (3.1), and integration by parts in the x_1 -variable, continuity of $v_{x_1}w$, and the fact $v_{x_1}|_{\partial\Omega} = 0$, we get

$$\begin{aligned} (v_{x_1x_1}, w)_h &= \sum_{I_1 \times I_2 \in \mathcal{T}_h} \sum_{\mathcal{G}_{I_2}} \sum_{\mathcal{G}_{I_1}} v_{x_1x_1} w \\ &= \sum_{I_1 \times I_2 \in \mathcal{T}_h} \sum_{\mathcal{G}_{I_2}} \int_{I_1} v_{x_1x_1} w(x_1, \cdot) dx_1 + E = S + E, \end{aligned} \tag{5.7}$$

where

$$\begin{aligned} E &= - \sum_{I_1 \times I_2 \in \mathcal{T}_h} \sum_{\mathcal{G}_{I_2}} C_1 |I_1|^5 (\partial^{(3,0)} v \partial^{(3,0)} w)(t_{I_1 \times I_2}, \cdot), \\ S &= - \sum_{I_1 \times I_2 \in \mathcal{T}_h} \sum_{\mathcal{G}_{I_2}} \int_{I_1} v_{x_1} w_{x_1}(x_1, \cdot) dx_1, \end{aligned}$$

and $C_1 > 0$ is a constant. Similarly, using continuity of $w_{x_1}v$, $v|_{\partial\Omega} = 0$, and the fact that $(\partial^{(3,0)} v \partial^{(3,0)} w)|_{I_1 \times I_2}$ is constant in the x_1 -variable, we obtain $(w_{x_1x_1}, v)_h = S + E$, which, along with (5.7), gives (5.6). ■

The following lemma states that a bicubic polynomial p is uniquely determined by the values of p and Δp at the Gauss points in a square and the values of p at the vertices of any edge of the square. We note that the polynomial is not unique if its values are given at vertices that do not belong to the same edge.

Lemma 5.2. *Let $\hat{K} = (-1, 1)^2$, and let η_1 and η_2 be the vertices of any edge of \hat{K} . Let $p(x_1, x_2)$ be a bicubic polynomial such that*

$$\begin{aligned} p(\xi) = \Delta p(\xi) = 0 \quad \text{for all } \xi \in \mathcal{G}_{\hat{K}}, \\ p(\eta_i) = p_{x_1}(\eta_i) = p_{x_2}(\eta_i) = p_{x_1x_2}(\eta_i) = 0, \quad i = 1, 2. \end{aligned} \tag{5.8}$$

Then, $p = 0$.

Proof. A bicubic polynomial has 16 coefficients, and there are 16 equations in (5.8). Let $M \in R^{16 \times 16}$ be the coefficient matrix corresponding to the equations in (5.8). Using a computer algebra system, we obtain

$$\det(M) = \pm 268435456/59049,$$

which implies that $p = 0$. ■

Lemma 5.3. *Let $f \in C(\Omega)$. The OSC problem (5.2)–(5.3) is under-determined and has a solution. For any solution $\{u_h, v_h\}$ of (5.2)–(5.3), function u_h is the unique solution of the quadrature Galerkin problem (3.5).*

Proof. Let us prove that the OSC system (5.2)–(5.3) is under-determined by eight constraints. Let N_e , N_i , and N_b denote, respectively, the numbers of elements, internal nodes, and boundary nodes in the triangulation \mathcal{T}_h . The set \mathcal{G}_h has $4N_e$ points; hence, there are $8N_e$ equations

in (5.2)–(5.3). By mathematical induction, it follows that $N_b = 2(N_e - N_i + 1)$. Thus, the system (5.2)–(5.3) has $8N_e = 8N_i + 4N_b - 8$ constraints. Since the dimensions of V_h and X_h are $4N_i$ and $4(N_i + N_b)$, respectively, the system (5.2)–(5.3) has $8N_i + 4N_b$ degrees of freedom, and the OSC system (5.2)–(5.3) is underdetermined by eight constraints.

Let η_1 and η_2 be the vertices of any element $K_\eta \in \mathcal{T}_h$. To prove that problem (5.2)–(5.3) has a solution, we show that the problem given by (5.2)–(5.3), and the additional eight constraints

$$v_h(\eta_i) = (v_h)_{x_1}(\eta_i) = (v_h)_{x_2}(\eta_i) = (v_h)_{x_1 x_2}(\eta_i) = 0, \quad i = 1, 2, \tag{5.9}$$

has a unique solution $\{u_h, v_h\}$. Since the numbers of degrees of freedom and constraints are equal, it suffices to show that the homogeneous system consisting of the equations (5.2), (5.9), and

$$\Delta v_h(\xi) = 0 \quad \text{for all } \xi \in \mathcal{G}_h, \tag{5.10}$$

has only the zero solution. Let u_h and v_h be solutions of equations (5.2), (5.9), and (5.10). Using (3.6), (5.2), Lemma 5.1, and (5.10), we obtain

$$a_h(u_h, u_h) = (\Delta u_h, \Delta u_h)_h = (v_h, \Delta u_h)_h = (\Delta v_h, u_h)_h = 0,$$

which implies $u_h = 0$ by (3.29). Using (5.2) with $u_h = 0$, (5.9), (5.10), and Lemma 5.2, we obtain $v_h|_{K_\eta} = 0$. By a similar argument, $v_h|_K = 0$ for any element $K \in \mathcal{T}_h$ adjacent to K_η . By recursion, $v_h|_K = 0$ for any $K \in \mathcal{T}_h$, that is, $v_h = 0$.

Let $\{u_h, v_h\}$ be a solution of problem (5.2)–(5.3). Using (3.6), (5.2), Lemma 5.1, and (5.3), we obtain

$$a_h(u_h, w) = (\Delta u_h, \Delta w)_h = (v_h, \Delta w)_h = (\Delta v_h, w)_h = (f, w)_h, \quad \text{for any } w \in V_h,$$

that is, u_h is a solution of problem (3.5), which is unique by Theorem 3.4. ■

Thus, the solution of the quadrature problem (3.5) is the solution of the minimal norm least squares problem for the linear system arising from (5.2)–(5.3).

Theorem 5.4. *Let $f \in C(\Omega)$ and let u and u_h be the solutions of the problems (2.1) and (3.5), respectively. If $u \in H^{8-k}(\Omega)$, then*

$$\|u - u_h\|_{H^k(\Omega)} \leq Ch^{4-k} |u|_{H^{8-k}(\Omega)}, \quad k = 1, 2. \tag{5.11}$$

Before presenting the proof of Theorem 5.4, we obtain the following generalization of Lemma 4.2 in [10].

Lemma 5.5. *If $v \in H^4(\Omega)$ then*

$$\|\partial^\alpha (v - \Pi_h v)\|_h \leq Ch^{4-|\alpha|} \|v\|_{H^4(\Omega)}, \quad |\alpha| \leq 2. \tag{5.12}$$

Moreover, if $v \in H^5(\Omega)$ then

$$\|\partial^\alpha (v - \Pi_h v)\|_h \leq Ch^3 \|v\|_{H^5(\Omega)}, \quad |\alpha| = 2. \tag{5.13}$$

Proof. We prove (5.12) first. Take any $v \in H^4(\Omega)$ and let α be a multi-index such that $|\alpha| \leq 2$. By (3.3),

$$\|\partial^\alpha(v - \Pi_h v)\|_h^2 = \sum_{K \in \mathcal{T}_h} \sum_{\mathcal{G}_K} [\partial^\alpha(v - \Pi_K v)]^2. \tag{5.14}$$

Take any $K \in \mathcal{T}_h$, let $F : \hat{K} = (0, 1)^2 \rightarrow K$ be an invertible affine mapping, and let $\hat{v} = v|_K \circ F$. Changing variables by the transformation F , we obtain

$$\sum_{\mathcal{G}_K} |\partial^\alpha(v - \Pi_K v)|^2 \leq Ch^{2-2|\alpha|} \sum_{\xi \in \mathcal{G}_{\hat{K}}} |l_\xi(\hat{v})|^2, \tag{5.15}$$

where, for any $\xi \in \mathcal{G}_{\hat{K}}$,

$$l_\xi(\hat{w}) = \partial^\alpha(\hat{w} - \Pi_{\hat{K}} \hat{w})(\xi), \quad \text{for all } \hat{w} \in C^2(\hat{K}). \tag{5.16}$$

Let us show that

$$|l_\xi(\hat{v})| \leq C|\hat{v}|_{H^4(\hat{K})}, \quad \xi \in \mathcal{G}_{\hat{K}}. \tag{5.17}$$

Take $\xi \in \mathcal{G}_{\hat{K}}$. The linear functional $l_\xi(\cdot)$ is bounded on $H^4(\hat{K})$ since $H^4(\hat{K}) \subset C^2(\hat{K})$ and $|\alpha| \leq 2$, and it vanishes on polynomials of degree ≤ 3 . Applying the Bramble-Hilbert Lemma, we obtain

$$|l_\xi(\hat{w})| \leq C|\hat{w}|_{H^4(\hat{K})}, \quad \hat{w} \in H^4(\hat{K}),$$

which implies (5.17).

Substituting (5.17) in (5.15) and changing the variables by the transformation F^{-1} , we get

$$\sum_{\mathcal{G}_K} |\partial^\alpha(v - \Pi_K v)|^2 \leq Ch^{2(4-|\alpha|)} |v|_{H^4(K)}^2.$$

The last estimate along with (5.14) gives

$$\|\partial^\alpha(v - \Pi_h v)\|_h^2 \leq Ch^{2(4-|\alpha|)} \sum_{K \in \mathcal{T}_h} |v|_{H^4(K)}^2 = Ch^{2(4-|\alpha|)} |v|_{H^4(\Omega)}^2,$$

which implies (5.12). Similarly, we obtain estimate (5.13) using the fact that $l_\xi(\cdot)$ in (5.16) vanishes on polynomials of degree ≤ 4 when $|\alpha| = 2$. ■

Proof of Theorem 5.4. It follows from Theorem 3.4 and Lemma 5.3, that the quadrature scheme (3.5) has a unique solution u_h , which is also the solution of the OSC scheme $\{(5.2)–(5.3), (5.9)\}$. The proof of (5.11) is similar to that of Theorem 3.1 in [4]. Since $u \in H^6(\Omega)$, u and $v = \Delta u \in C^2(\Omega)$ are solutions of the coupled problem (5.1). Let

$$U = u_h - \Pi_h u \quad \text{and} \quad V = v_h - \Pi_h v, \tag{5.18}$$

where v_h is the solution of the OSC scheme $\{(5.2)–(5.3)\}, (5.9)$. We note that

$$U \in V_h \quad \text{and} \quad V \in X_h. \tag{5.19}$$

It follows from (5.2) and $\Delta u = v$ that

$$\begin{aligned} \Delta U(\xi) - V(\xi) &= -\Delta(\Pi_h u)(\xi) + (\Pi_h)v(\xi) \\ &= \Delta(u - (\Pi_h)u)(\xi) - (v - (\Pi_h)v)(\xi), \text{ for any } \xi \in \mathcal{G}_h. \end{aligned} \tag{5.20}$$

Similarly, using (5.3) and $\Delta v = f$, we obtain

$$\Delta V(\xi) = \Delta(v - \Pi_h v)(\xi), \text{ for any } \xi \in \mathcal{G}_h. \tag{5.21}$$

Taking the product $(\cdot, \cdot)_h$ of (5.20) and (5.21) with ΔU and U , respectively, we obtain

$$\begin{aligned} \|\Delta U\|_h^2 - (V, \Delta U)_h &= (\Delta(u - \Pi_h u), \Delta U)_h - (v - \Pi_h v, \Delta U)_h, \\ (\Delta V, U)_h &= (\Delta(v - \Pi_h v), U)_h. \end{aligned}$$

Summing the last two identities and using (5.19) and Lemma 5.1, we obtain

$$\|\Delta U\|_h^2 = (\Delta(u - \Pi_h u), \Delta U)_h - (v - \Pi_h v, \Delta U)_h + (\Delta(v - \Pi_h v), U)_h.$$

Using the Cauchy-Schwarz inequality for $(\cdot, \cdot)_h$, estimate (3.28) with $\alpha = (0, 0)$ and (3.29), we get

$$\|\Delta U\|_h \leq \|\Delta(u - \Pi_h u)\|_h + \|v - \Pi_h v\|_h + C\|\Delta(v - \Pi_h v)\|_h. \tag{5.22}$$

Using (3.29), (5.22), Lemma 5.5, and $v = \Delta u$, we obtain

$$\begin{aligned} \|U\|_{H^2(\Omega)} &\leq C\|\Delta U\|_h \leq Ch^{4-k}\|u\|_{H^{6-k}(\Omega)} + Ch^4\|v\|_{H^4(\Omega)} \\ &\quad + Ch^{4-k}\|v\|_{H^{6-k}(\Omega)} \leq Ch^{4-k}\|u\|_{H^{8-k}(\Omega)}, \quad k = 1, 2. \end{aligned} \tag{5.23}$$

Theorem 3.1.6 in [1] (also, see estimate (6.1.7)) implies

$$\|v - \Pi_K v\|_{H^m(K)} \leq Ch^{4-m}|v|_{H^4(K)}, \quad 0 \leq m \leq 4, \quad K \in \mathcal{T}_h, \quad v \in H^4(\Omega). \tag{5.24}$$

Using the triangle inequality for the H^k -norm, (5.24) with $m = 1, 2$, (5.18), and (5.23), we get

$$\|u - u_h\|_{H^k(\Omega)} \leq \|u - \Pi_h u\|_{H^k(\Omega)} + \|\Pi_h u - u_h\|_{H^2(\Omega)} \leq Ch^{4-k}\|u\|_{H^{8-k}(\Omega)}$$

for $k = 1, 2$, which is (5.11). ■

VI. NUMERICAL RESULTS

In this section, we present numerical results for two test problems, one with a square domain Ω and the other with an L-shaped domain. Let N_h be the interior node set of the triangulation \mathcal{T}_h , and let

$$e_h \equiv \max_{x \in N_h} |u(x) - u_h(x)| \quad \text{and} \quad p_h = \log_2 \frac{e_{2h}}{e_h}$$

be the maximal nodal error and the approximate convergence order of the numerical solution u_h , respectively. We carried out a series of computations for decreasing values of h obtained by halving to determine the errors of the numerical solution and its derivatives and the approximate convergence orders. Linear systems resulting from the discretization were solved using the LU decomposition method.

TABLE I. Example 1: Maximum nodal errors and approximate convergence orders.

h	$u - u_h$	Order	$(u - u_h)_{x_1}$	Order	$(u - u_h)_{x_2}$	Order	$(u - u_h)_{x_1 x_2}$	Order
1/8	1.3E-3	4.3	7.8E-4	4.6	7.8E-4	4.6	1.7E-2	4.4
1/16	7.9E-5	4.1	4.2E-5	4.2	4.2E-5	4.2	9.4E-4	4.1
1/32	4.9E-6	4.0	2.5E-6	4.1	2.5E-6	4.1	5.8E-5	4.0
1/64	3.0E-7	4.0	1.5E-7	4.0	1.6E-7	4.0	3.6E-6	4.0
1/128	1.1E-8	4.8	3.3E-8	2.2	3.5E-8	2.1	3.1E-7	3.6
1/256	1.7E-7	-4.0	5.7E-7	-4.1	5.6E-7	-4.0	1.8E-6	-2.6

TABLE II. Example 1: Sobolev norm errors and approximate convergence orders.

h	L^2 -error	L^2 -order	H^1 -error	H^1 -order	H^2 -error	H^2 -order
1/8	6.9E-4	4.1	2.2E-2	3.1	1.1E+0	2.0
1/16	4.2E-5	4.0	2.7E-3	3.0	2.8E-1	2.0
1/32	2.6E-6	4.0	3.4E-4	3.0	6.9E-2	2.0
1/64	1.6E-7	4.0	4.2E-5	3.0	1.7E-2	2.0
1/128	1.1E-8	3.9	5.2E-6	3.0	4.3E-3	2.0
1/256	7.0E-8	-2.7	7.5E-7	2.8	1.1E-3	2.0

Example 1. The first test problem is the same as that in Example 1 in [5] and [11], and Problem 2 in [12]. The BVP (1.1) is formulated on the unit square $\Omega = (0, 1)^2$, and $u(x) = 4 \sin^2(\pi x_1) \sin^2(\pi x_2)$ is the exact solution. Confirming the statement of Lemma 5.3, the computed solution was identical, up to round-off errors, to that of Example 1 in [5] and [11].

In Table I, we present the maximum nodal errors and the corresponding approximate convergence orders of the quadrature Galerkin solution and its derivatives. We observe that all approximate convergence orders are close to 4. Higher than optimal convergence orders for the derivatives point to a superconvergence property of the quadrature solution. The approximate convergence orders for $h \leq 1/128$ decrease because of the effect of round-off errors. The condition number κ_h of the stiffness matrix is $O(h^{-4})$, and $\kappa_h \approx 10^7$ for $h = 1/128$.

In Table II, we present Sobolev norm errors and their approximate convergence orders, and note that the convergence orders of the H^1 - and H^2 -errors are close to the optimal values 3 and 2, respectively. Thus, the numerical results confirm the error estimate (5.11). An L^2 -error analysis of the quadrature scheme for Dirichlet biharmonic problem has not yet been developed. Our numerical results indicate on the optimal convergence order 4 of the L^2 -error.

Example 2. In this example, we solve numerically the BVP (1.1) on the L-shaped domain Ω presented in Figure 2 in Section III and with the exact solution $u(x) = 4 \sin^2(2\pi x_1) \sin^2(2\pi x_2)$.

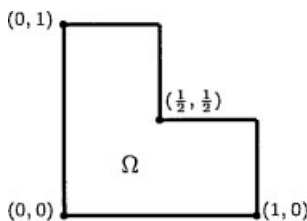


FIG. 2. L-shaped domain Ω .

TABLE III. Example 2: Maximum nodal errors and approximate convergence orders.

h	e_h	Order	$(e_h)_{x_1}$	Order	$(e_h)_{x_2}$	Order	$(e_h)_{x_1 x_2}$	Order
1/8	2.4E-2	5.4	7.8E-2	5.0	7.8E-2	5.0	1.5E+0	-2.0
1/16	1.2E-3	4.3	4.3E-3	4.1	4.3E-3	4.1	7.5E-2	4.3
1/32	7.0E-5	4.1	2.7E-4	4.0	2.7E-4	4.0	4.3E-3	4.1
1/64	4.3E-6	4.0	1.7E-5	4.0	1.7E-5	4.0	2.6E-4	4.0
1/128	2.7E-7	4.0	1.1E-6	4.0	1.1E-6	4.0	1.6E-5	4.0
1/256	1.6E-8	4.1	1.1E-7	3.3	1.0E-7	3.4	1.2E-6	3.7

TABLE IV. Example 2: Sobolev norm errors and approximate convergence orders.

h	L^2 -error	L^2 -order	H^1 -error	H^1 -order	H^2 -error	H^2 -order
1/8	1.0E-2	4.9	3.2E-1	3.2	1.5E+1	1.6
1/16	5.8E-4	4.1	3.8E-2	3.1	3.8E+0	2.0
1/32	3.6E-5	4.0	4.7E-3	3.0	9.6E-1	2.0
1/64	2.2E-6	4.0	5.8E-4	3.0	2.4E-1	2.0
1/128	1.4E-7	4.0	7.3E-5	3.0	6.0E-2	2.0
1/256	9.7E-9	3.8	9.1E-6	3.0	1.5E-2	2.0

The maximum nodal errors and the corresponding approximate convergence orders of the quadrature solution are presented in Table III. Note that the convergence orders are close to 4 as in Example 1, and the results point to a superconvergence property of the quadrature solution. At $h = 1/128$, the machine precision has been reached.

In Table IV, we present errors in Sobolev norms and their approximate convergence orders. The results are similar to those in Table II for the test problem in Example 1. We note that the errors for the problem on the L -shaped domain are larger by approximately the factor of 10 than the corresponding errors for the problem on the square domain. This difference might have been caused by the presence of the re-entrant corner in the domain of Example 2.

VII. CONCLUSION

It is known that at least a three-point Gaussian quadrature should be used for approximating the integrals in the finite element Galerkin solution of the biharmonic problem with the Bogner–Fox–Schmit element. In this article, we proved that the two-point Gaussian quadrature Galerkin scheme is well-posed and has optimal order error estimates in Sobolev norms. The solution of the proposed quadrature scheme is the same as that of the minimum norm least squares problem for the corresponding underdetermined OSC scheme. Forming the stiffness matrix and the load vector of the quadrature scheme is faster than with the three-point scheme since fewer function evaluations are required.

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