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# ORTHOGONAL SPLINE COLLOCATION FOR NONLINEAR DIRICHLET PROBLEMS*

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**Abstract.** We study the orthogonal spline collocation (OSC) solution of a homogeneous Dirichlet boundary value problem in a rectangle for a general nonlinear elliptic partial differential equation. The approximate solution is sought in the space of Hermite bicubic splines. We prove local existence and uniqueness of the OSC solution, obtain optimal order $H^1$ and $H^2$ error estimates, and prove the quadratic convergence of Newton's method for solving the OSC problem.

**Key words.** orthogonal spline collocation, Dirichlet problem, nonlinear, existence, uniqueness, error estimates, Newton's method

**AMS subject classifications.** 65N35, 65J15, 65N15

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## 1. Introduction.

The orthogonal spline collocation (OSC) method for the solution of nonlinear one-dimensional boundary value problems (BVPs) was introduced by de Boor and Swartz [6]. An extensive survey of spline collocation methods for solving partial differential equations is given in [5]. In comparison to finite element Galerkin methods, collocation methods do not involve integral approximations in the computation of the coefficients of the resulting algebraic equations. Moreover, the OSC solution has the superconvergence property at the partition nodes [3], [6], [9].

Analyses of the OSC solution of two-dimensional linear BVPs with optimal $H^2$ and optimal order $L^2$ and $H^1$ error estimates were given in [2], [4], [20], [21]. An OSC method with Hermite bicubic splines for the nonlinear equation $\Delta u + F(x,u) = 0$ was studied in [12], where existence and uniqueness of the OSC solution were proved and an optimal order $H^1$ error estimate obtained under the assumption that the exact solution is in $H^6(\Omega)$. Finite element Galerkin methods for BVPs with nonlinear elliptic equations in divergence form were studied in [10], [11], [19]. Douglas and Dupont [10] considered a mildly nonlinear equation and obtained optimal $L^2$ and $H^1$ error estimates. Frehse and Rannacher [11] considered linear finite element approximations for general nonlinear equations and derived "an almost optimal" convergence rate in $L^\infty$. Park [19] used mixed finite element methods and obtained $L^p$ error estimates for $2 \le p \le \infty$.

In this paper, we consider the OSC solution of

$$Lu\,(x) \;=\; f(x), \qquad x \in \Omega = (0,1)\times(0,1), \qquad u|_{\partial\Omega} \;=\; 0, \tag{1.1}$$

where $\partial\Omega$ is the boundary of $\Omega$, the nonlinear differential operator

$$Lu\,(x) \;=\; \sum_{i,\,j=1}^{2} a_{ij}(x,\,u,\,\nabla u)\,u_{x_i x_j} + a(x,\,u,\,\nabla u), \tag{1.2}$$

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$x = (x_1, x_2)$, and $\nabla u = (u_{x_1}, u_{x_2})$. The OSC scheme consists of finding a Hermite bicubic spline $u_h$ that vanishes on $\partial\Omega$ and satisfies the differential equation of (1.1) at the collocation points. In our analysis, we first obtain the following three basic results: 1) the consistency of the OSC scheme in a discrete norm using the approximation properties of the Hermite bicubic spline interpolant $\tilde{u}_h$ of $u$ [4]; 2) the Lipschitz continuity of the Fréchet derivative of the OSC operator; 3) the uniform boundedness of the inverse of the Fréchet derivative using Bernstein's transformation [16], a trick similar to that of Nitsche [17], and the Bramble–Hilbert lemma [7]. Then, using the contraction operator principle [15], in a way similar to that in [14], we prove existence and uniqueness of the OSC solution in the ball with center at $\tilde{u}_h$ and radius $\rho = \mathrm{O}(|\ln h|^{-(2+q)})$, where $q > 0$ is the exponent in the growth conditions and $h$ is the partition parameter. We obtain $H^1$ and $H^2$ error estimates using a generalization of Banach's theorem [13]. The quadratic convergence of Newton's method for the solution of the OSC problem is proved in a way similar to that in [22].

An outline of the paper is as follows. In section 2, we give assumptions on $L$, introduce the notation, and state basic results. In section 3, we formulate the OSC problem and prove a general result concerning this problem. The consistency of the OSC operator is established in section 4. In section 5, we prove the Lipschitz continuity of a Fréchet derivative of the OSC operator. In section 6, we obtain a bound on the inverse of the Fréchet derivative of the OSC operator. The main existence, uniqueness, and error estimate result for the OSC solution is presented in section 7. In section 8, we study the convergence of Newton's method for the iterative solution of the OSC problem.

**2. Preliminaries.** Concerning the BVP (1.1), we assume that the functions $a_{ij}(x, s)$ and $a(x, s)$, where $s = (s_0, s_1, s_2)$, are defined on $\overline{\Omega} \times R^3$,

$$a_{12}(x, s) = a_{21}(x, s), \qquad (x, s) \in \overline{\Omega} \times R^3, \tag{2.1}$$

and $f(x)$ is continuous on $\Omega$. In the following, the operator $L$ is uniformly elliptic, that is, there is $\nu > 0$ such that

$$\sum_{i,j=1}^{2} a_{ij}(x, s)\, \zeta_i\, \zeta_j \;\geq\; \nu\left(\zeta_1^2 + \zeta_2^2\right), \qquad (\zeta_1, \zeta_2) \in R^2,\ (x, s) \in \overline{\Omega} \times R^3. \tag{2.2}$$

Also, for $m$ and $\beta = (\beta_0, \beta_1, \beta_2)$ to be specified later, the functions $a_{ij}(x, s)$ and $a(x, s)$ will have continuous on $\overline{\Omega} \times R^3$ partial derivatives $\partial^{m+|\beta|} a_{ij}/\partial x_l^m\, \partial s^\beta$ and $\partial^{m+|\beta|} a/\partial x_l^m\, \partial s^\beta$, respectively, where $|\beta| = \beta_0 + \beta_1 + \beta_2$ and $\partial s^\beta = \partial s_0^{\beta_0}\, \partial s_1^{\beta_1}\, \partial s_2^{\beta_2}$. Moreover, we assume that there exists a function $\tilde{\mu}(t_0, t_1, t_2)$ defined for $t_0, t_1, t_2 \geq 0$, which is continuous and nondecreasing in each variable, and such that, for any $(x, s) \in \overline{\Omega} \times R^3$,

$$\left| \frac{\partial^{m+|\beta|} a_{ij}}{\partial x_l^m\, \partial s^\beta}\,(x, s) \right| \leq \tilde{\mu}(|s_0|, |s_1|, |s_2|), \qquad 1 \leq i, j, l \leq 2, \tag{2.3a}$$

$$\left| \frac{\partial^{m+|\beta|} a}{\partial x_l^m\, \partial s^\beta}\,(x, s) \right| \leq \tilde{\mu}(|s_0|, |s_1|, |s_2|), \qquad 1 \leq l \leq 2. \tag{2.3b}$$

We set

$$\bar{\mu}(t) = \tilde{\mu}(t, t, t), \qquad t \geq 0. \tag{2.4}$$

For convenience, we use the notation $\partial v/\partial x_0 = v_{x_0} = v$. For sufficiently smooth functions $y$, $w$, and $z$ defined on $\Omega$, and for $x \in \Omega$, we introduce

$$L_y\, w\,(x) = \sum_{i,\,j=1}^{2} a_{ij}(x,\, y,\, \nabla y)\, w_{x_i x_j} + \sum_{k=0}^{2} A_y^k\,(x)\, w_{x_k}, \tag{2.5}$$

$$A_y^k\,(x) = \sum_{i,\,j=1}^{2} \frac{\partial a_{ij}}{\partial s_k}(x,\, y,\, \nabla y)\, y_{x_i x_j} + \frac{\partial a}{\partial s_k}(x,\, y,\, \nabla y), \qquad k = 0, 1, 2. \tag{2.6}$$

The differential operator $L_y$ can be viewed as a formal first derivative of $L$ at $y$. If the functions $y$ and $w$ are twice differentiable at $x \in \Omega$, then

$$L(y + w)\,(x) - Ly\,(x) \;=\; \int_0^1 L_{y+t\,w}\, w\,(x)\, dt \tag{2.7}$$

(see [1, Lemma 4.1]).

For positive integers $N_1$ and $N_2$, let $\pi_i = \{x_i^k\}_{k=0}^{N_i}$, $i = 1, 2$, be a nonuniform partition of the interval $[0, 1]$. We set $h_i^k = x_i^k - x_i^{k-1}$, $k = 1, \ldots, N_i$, $i = 1, 2$, and introduce $\underline{h}_i = \min_k h_i^k$, $\overline{h}_i = \max_k h_i^k$, and $h = \max\left\{\overline{h}_1, \overline{h}_2\right\}$. Let $\pi_h$ be the partition of $\Omega$ associated with the grid $\pi_1 \times \pi_2$. We consider a regular collection of partitions $\pi_h$, that is, we assume that there exist positive constants $\sigma_1$, $\sigma_2$, and $\sigma_3$, all independent of $h$, such that $\sigma_1 \le \overline{h}_1/\overline{h}_2 \le \sigma_2$ and $\underline{h}_i/\overline{h}_i \ge \sigma_3$ for $i = 1, 2$. Let $\mathcal{T}_h$ be the set of all open rectangles generated by the partition $\pi_h$, that is,

$$\mathcal{T}_h \;=\; \{\tau = (x_1^{k_1-1},\, x_1^{k_1}) \times (x_2^{k_2-1},\, x_2^{k_2}) :\ 1 \le k_i \le N_i,\ i = 1,\, 2\}.$$

The set of Gauss points in $\Omega$ corresponding to the partition $\pi_h$ is defined by

$$\mathcal{G}_h \;=\; \{(\xi_1^{m_1},\, \xi_2^{m_2}) :\ 1 \le m_i \le 2\, N_i,\ i = 1, 2\}\,,$$

where $\xi_i^{2k_i-1} = x_i^{k_i-1} + h_i^{k_i}\eta_1$, $\xi_i^{2k_i} = x_i^{k_i-1} + h_i^{k_i}\eta_2$, $1 \le k_i \le N_i$, $i = 1,\, 2$, and

$$\eta_1 \;=\; (3 - \sqrt{3})/6, \qquad \eta_2 \;=\; (3 + \sqrt{3})/6. \tag{2.8}$$

For any $v$ defined on $\mathcal{G}_h$, let

$$\sum_h v \;=\; \frac{1}{4} \sum_{\tau \in \mathcal{T}_h} \mathrm{mes}(\tau) \sum_{\xi \in \mathcal{G}_h \cap \tau} v(\xi), \qquad \|v\|_h^2 \;=\; \sum_h v^2.$$

For $E \subset R^2$, let $\|\cdot\|_{L^2(E)}$, $\|\cdot\|_{H^l(E)}$, and $\|\cdot\|_{C^l(E)}$ for integer $l \ge 0$ denote the standard norms in the indicated spaces.

We introduce the "broken" $C^2$-space

$$C^2(\mathcal{T}_h) \;=\; \{v \in L^2(\Omega) :\ v|_\tau \in C^2(\tau),\ \tau \in \mathcal{T}_h\}$$

and set $\|v\|_{C^2(\mathcal{T}_h)} = \max_{\tau \in \mathcal{T}_h} \|v\|_{C^2(\tau)}$ for any $v \in C^2(\mathcal{T}_h)$. Let $\mathcal{P}_h$ be the set of all piecewise bicubic polynomial functions defined on $\mathcal{T}_h$, that is, $\mathcal{P}_h = \{v : v|_\tau \in P_3 \otimes P_3,\ \tau \in \mathcal{T}_h\}$, where $P_3$ is the set of all polynomials of degree $\le 3$, and the symbol $\otimes$ denotes the tensor product. For the partition $\pi_h$, let $\mathcal{M}_h^0$ be the space of Hermite bicubic splines vanishing on $\partial\Omega$. We note that $\|\cdot\|_h$ is a norm in $\mathcal{M}_h^0$ since any element of $\mathcal{M}_h^0$ is uniquely determined by its values on $\mathcal{G}_h$ [20, Lemma 5.1].

Throughout this paper, $C$ denotes a generic positive constant independent of $h$; the value of $C$ may be different each time it is written. For $\alpha = (\alpha_1, \alpha_2)$, let $|\alpha| = \alpha_1 + \alpha_2$ and $\partial^\alpha = \partial^{\alpha_1+\alpha_2}/(\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2})$. For any $v \in C^2(\overline{\Omega})$ and partition $\pi_h$, $\tilde{v}_h \in \mathcal{M}_h$ denotes the Hermite bicubic spline interpolant of $v$.

LEMMA 2.1. *For $v \in H^4(\Omega)$ and $l = 0, 1, 2$,*

$$\|\tilde{v}_h\|_{C^2(\mathcal{T}_h)} \le C\, \|v\|_{H^4(\Omega)}, \tag{2.9}$$

$$\|v - \tilde{v}_h\|_{H^l(\Omega)} \le C\, h^{4-l}\, \|v\|_{H^4(\Omega)}, \tag{2.10}$$

$$\sum_{|\alpha|=l} \|\partial^\alpha (v - \tilde{v}_h)\|_h \le C\, h^{4-l}\, \|v\|_{H^4(\Omega)}, \tag{2.11}$$

*and, for $v \in H^5(\Omega)$,*

$$\sum_{|\alpha|=2} \|\partial^\alpha (v - \tilde{v}_h)\|_h \;\le\; C\, h^3\, \|v\|_{H^5(\Omega)}. \tag{2.12}$$

*Proof.* From [8, Theorem 3.1.6], we have $\|v - \tilde{v}_h\|_{C^2(\tau)} \le Ch\|v\|_{H^4(\tau)}$, $\tau \in \mathcal{T}_h$. Hence (2.9) follows from the triangle inequality for $\|\cdot\|_{C^2(\tau)}$ and the inequality $\|v\|_{C^2(\overline{\Omega})} \le C\|v\|_{H^4(\Omega)}$. Inequalities (2.10) and (2.11), (2.12) are proved in [8, Theorem 3.2.1] and [4, Lemma 4.2], respectively. ∎

If a function $G(x, t)$ defined on $\mathcal{G}_h \times [0, 1]$ is continuous in $t$ for all $x \in \mathcal{G}_h$, then

$$\left\| \int_0^1 G(\cdot, t)\, dt \right\|_h \;\le\; \int_0^1 \|G(\cdot, t)\|_h\, dt \tag{2.13}$$

(see [18, Lemma 3.2.11]).

## 3. The OSC problem and a general result.

We define the OSC operator $L_h$ from $\mathcal{M}_h^0$ into $\mathcal{M}_h^0$ and $f_h \in \mathcal{M}_h^0$ by

$$L_h v_h\,(x) = L v_h\,(x), \qquad x \in \mathcal{G}_h, \tag{3.1}$$

$$f_h(x) = f(x), \qquad x \in \mathcal{G}_h. \tag{3.2}$$

The OSC problem is formulated as follows: find $u_h \in \mathcal{M}_h^0$ such that

$$L_h u_h \;=\; f_h. \tag{3.3}$$

Hereafter, $L_h$ is viewed as an operator from $\mathcal{M}_h^0$ with the $H^2$-norm into $\mathcal{M}_h^0$ with the norm $\|\cdot\|_h$, and $L_{h,y}$ denotes the Fréchet derivative of $L_h$ at $y \in \mathcal{M}_h^0$. Also, $\|L_{h,y}\|$ and $\|L_{h,y}^{-1}\|$ are the corresponding operator norms of $L_{h,y}$ and $L_{h,y}^{-1}$, respectively.

For $z \in \mathcal{M}_h^0$ and $\rho \ge 0$, let $B_h(z, \rho) = \{w \in \mathcal{M}_h^0 : \|w - z\|_{H^2(\Omega)} \le \rho\}$. We prove the following general existence, uniqueness, and error estimate result for the OSC problem (3.3) (cf. [14, Theorems 2.6 and 3.6]).

THEOREM 3.1. *Let $u \in C^2(\overline{\Omega})$ be a solution of problem* (1.1)*, and suppose that, for some $h$ and some $\rho > 0$, $L_{h,y}$ exists for all $y \in B_h(\tilde{u}_h, \rho)$, $L_{h,\tilde{u}_h}^{-1}$ exists, and*

$$\|L_{h,\tilde{u}_h}^{-1}\| \le K_1, \tag{3.4}$$

$$\|L_{h,z} - L_{h,y}\| \le K_2\, \|z - y\|_{H^2(\Omega)}, \qquad y, z \in B_h(\tilde{u}_h, \rho), \tag{3.5}$$

$$\|Lu - L_h \tilde{u}_h\|_h \le K_3\, h^p, \tag{3.6}$$

$$\rho\, K_1\, K_2 \le 1/2, \tag{3.7}$$

$$K_1\, K_3\, h^p \le \rho/2, \tag{3.8}$$

*where positive* $K_1$ *and nonnegative* $K_2$, $K_3$, *and* $p$ *are constants. Then* (3.3) *has a unique solution* $u_h \in B_h(\tilde{u}_h, \rho)$, *and*

$$\|u_h - \tilde{u}_h\|_{H^2(\Omega)} \le 2\, K_1\, K_3\, h^p. \tag{3.9}$$

*Proof.* First we prove existence and uniqueness of the OSC solution. Since $B_h(\tilde{u}_h, \rho)$ is a convex set, it follows that

$$y + t\,(z - y) \in B_h(\tilde{u}_h, \rho), \qquad t \in [0, 1], \qquad y,\, z \in B_h(\tilde{u}_h, \rho). \tag{3.10}$$

Using (3.10) and the existence of $L_{h,y}$, $y \in B_h(\tilde{u}_h, \rho)$, we conclude that $L_{h,y+t\,(z-y)}$ is defined for all $t \in [0, 1]$ and for all $y,\, z \in B_h(\tilde{u}_h, \rho)$. By (3.5), for any $y,\, z \in B_h(\tilde{u}_h, \rho)$, we obtain

$$\|L_{h,y+t_1\,(z-y)} - L_{h,y+t_2\,(z-y)}\| \le K_2\, \|z - y\|_{H^2(\Omega)}\, |t_1 - t_2|, \qquad t_1,\, t_2 \in [0, 1],$$

which implies continuity of the mapping that assigns $L_{h,y+t\,(z-y)}$ to each $t \in [0, 1]$. Hence, for any $y,\, z \in B_h(\tilde{u}_h, \rho)$ and any $w \in \mathcal{M}_h^0$, $L_{h,y+t\,(z-y)}\, w$ is continuous as a mapping of $t \in [0, 1]$ into $\mathcal{M}_h^0$.

Using (2.13), (3.5), and (3.10), for any $y,\, z \in B_h(\tilde{u}_h, \rho)$ and any $w \in \mathcal{M}_h^0$, we obtain

$$\begin{aligned} \left\| \int_0^1 \left[L_{h,\tilde{u}_h} - L_{h,y+t\,(z-y)}\right] w\, dt \right\|_h &\le \int_0^1 \left\| \left[L_{h,\tilde{u}_h} - L_{h,y+t\,(z-y)}\right] w \right\|_h dt \\ &\le \int_0^1 \left\| L_{h,\tilde{u}_h} - L_{h,y+t\,(z-y)} \right\| \, \|w\|_{H^2(\Omega)}\, dt \le \rho\, K_2\, \|w\|_{H^2(\Omega)}. \end{aligned} \tag{3.11}$$

It follows from a result in [13, Chapter XVII, section 1.7] that

$$L_h z - L_h y = \int_0^1 L_{h,y+t\,(z-y)}\, (z - y)\, dt, \qquad y,\, z \in B_h(\tilde{u}_h, \rho). \tag{3.12}$$

Let $G_h$ be an operator from $\mathcal{M}_h^0$ into $\mathcal{M}_h^0$ defined by

$$G_h\, v = v - L_{h,\tilde{u}_h}^{-1}(L_h v - f_h), \qquad v \in \mathcal{M}_h^0. \tag{3.13}$$

Using (3.13) and (3.12), for any $y,\, z \in B_h(\tilde{u}_h, \rho)$, we get

$$\begin{aligned} \|G_h\, z - G_h\, y\|_{H^2(\Omega)} &= \left\| z - y - L_{h,\tilde{u}_h}^{-1}(L_h z - L_h y) \right\|_{H^2(\Omega)} \\ &\le \left\| L_{h,\tilde{u}_h}^{-1} \right\| \left\| \int_0^1 \left[L_{h,\tilde{u}_h} - L_{h,y+t\,(z-y)}\right] (z - y)\, dt \right\|_h. \end{aligned}$$

Thus (3.4), (3.11) with $w = z - y$, and (3.7) give

$$\|G_h\, z - G_h\, y\|_{H^2(\Omega)} \le (1/2)\, \|z - y\|_{H^2(\Omega)}, \qquad y,\, z \in B_h(\tilde{u}_h, \rho). \tag{3.14}$$

Using (3.13), (3.4), (3.2), (1.1), (3.6), and (3.8), we obtain

$$\begin{aligned} \|\tilde{u}_h - G_h\, \tilde{u}_h\|_{H^2(\Omega)} &= \left\| L_{h,\tilde{u}_h}^{-1}(L_h \tilde{u}_h - f_h) \right\|_{H^2(\Omega)} \le \left\| L_{h,\tilde{u}_h}^{-1} \right\| \, \|L_h \tilde{u}_h - f_h\|_h \\ &\le K_1\, \|L_h \tilde{u}_h - L u\|_h \le K_1\, K_3\, h^p \le \rho/2. \end{aligned} \tag{3.15}$$

From (3.14), (3.15), and the contraction operator principle [15, Theorem 1.2], we conclude that there is a unique $u_h \in B_h(\tilde{u}_h, \rho)$ such that $G_h u_h = u_h$, which, by (3.13), shows that (3.3) has a unique solution $u_h \in B_h(\tilde{u}_h, \rho)$.

We now prove the estimate (3.9). We fix $y,\, z \in B_h(\tilde{u}_h, \rho)$ and view $L_h(y, z)$, defined by

$$L_h(y,z)\, w \;\equiv\; \int_0^1 L_{h,y+t\,(z-y)}\, w\, dt, \qquad w \in \mathcal{M}_h^0, \tag{3.16}$$

as a linear operator from $\mathcal{M}_h^0$ with the $H^2$-norm into $\mathcal{M}_h^0$ with the norm $\|\cdot\|_h$.

Using (3.16) and (3.11), we obtain

$$\|[L_{h,\tilde{u}_h} - L_h(y,z)]\, w\|_h \;=\; \left\| \int_0^1 \left[L_{h,\tilde{u}_h} - L_{h,y+t\,(z-y)}\right]\, w\, dt \right\|_h \;\le\; \rho\, K_2\, \|w\|_{H^2(\Omega)}$$

for any $w \in \mathcal{M}_h^0$, which implies

$$\|L_h(y,z) - L_{h,\tilde{u}_h}\| \;\le\; \rho\, K_2. \tag{3.17}$$

From (3.17), (3.7), and (3.4), we have

$$\|L_h(y,z) - L_{h,\tilde{u}_h}\| \;\le\; \rho\, K_1\, K_2\, K_1^{-1} \;\le\; (2\, K_1)^{-1} \;<\; \|L_{h,\tilde{u}_h}^{-1}\|^{-1}.$$

Hence, using the existence of $L_{h,\tilde{u}_h}^{-1}$, (3.4), and the generalization of Banach's theorem on the existence of an inverse operator [13, Chapter V, section 4, Theorem 4], we conclude that $L_h(y,z)$ has an inverse and, by (3.4), (3.17), and (3.7),

$$\|[L_h(y,z)]^{-1}\| \;\le\; \|L_{h,\tilde{u}_h}^{-1}\| \left(1 - \|L_{h,\tilde{u}_h}^{-1}\|\, \|L_h(y,z) - L_{h,\tilde{u}_h}\|\right)^{-1} \;\le\; 2\, K_1. \tag{3.18}$$

Inequality (3.18) is equivalent to

$$\|w\|_{H^2(\Omega)} \;\le\; 2\, K_1\, \|L_h(y,z)\, w\|_h, \qquad w \in \mathcal{M}_h^0. \tag{3.19}$$

Setting $w = z - y$ in (3.19) and using (3.16) and (3.12), we obtain

$$\|z - y\|_{H^2(\Omega)} \;\le\; 2\, K_1\, \|L_h z - L_h y\|_h, \qquad y, z \in B_h(\tilde{u}_h, \rho). \tag{3.20}$$

Finally, setting $y = \tilde{u}_h$ and $z = u_h$ in (3.20), and using (3.3), (3.2), (1.1), and (3.6), we obtain (3.9). ∎

Inequality (3.5) is called the Lipschitz continuity of a Fréchet derivative of $L_h$ in $B_h(\tilde{u}_h, \rho)$ with the Lipschitz constant $K_2$. If $p$ and $K_3$ are independent of $h$, then the inequality (3.6) is called the consistency of the OSC operator $L_h$ at $u$.

**4. Consistency.** In this section, we prove the consistency of the operator $L_h$.

LEMMA 4.1. *Suppose that* (2.3a) *and* (2.3b) *hold for all* $\beta$ *with* $|\beta| \le 1$ *and* $|\beta| = 1$*, respectively, and for* $m = 0$*. Then*

$$\|Lv - L_h\tilde{v}_h\|_h \;\le\; K_3\, h^{k-2}, \qquad v \in H^k(\Omega), \;\; k = 4, 5, \tag{4.1}$$

*where* $K_3$ *is independent of* $h$ *but depends on* $\|v\|_{H^k(\Omega)}$.

*Proof.* If $v \in H^4(\Omega)$, then $v \in C^2(\overline{\Omega})$. Using (3.1) and (2.7) with $y = \tilde{v}_h$ and $w = v - \tilde{v}_h$, we have

$$Lv\,(x) - L_h\tilde{v}_h\,(x) \;=\; Lv\,(x) - L\tilde{v}_h\,(x) \;=\; \int_0^1 L_{y_h(t)}\,(v - \tilde{v}_h)\,(x)\, dt, \qquad x \in \mathcal{G}_h, \tag{4.2}$$

where $y_h(t) = \tilde{v}_h + t\,(v - \tilde{v}_h)$. Using (2.5), (2.6) and the smoothness of $a_{ij}$, $1 \le i, j \le 2$, and $a$, it is easy to verify that, for any $x \in \mathcal{G}_h$, $L_{y_h(t)}\,(v - \tilde{v}_h)\,(x)$ is continuous for $t \in [0, 1]$. Hence, (4.2) and (2.13) imply that

$$\|Lv - L_h\tilde{v}_h\|_h \;\le\; \int_0^1 \left\|L_{y_h(t)}\,(v - \tilde{v}_h)\right\|_h dt. \tag{4.3}$$

With $\bar{\mu}$ defined by (2.4), the inequality

$$|L_{y_h(t)}(v - \tilde{v}_h)(x)| \le \bar{\mu}\left(\|y_h(t)\|_{C^1(\overline{\Omega})}\right)\left(1 + \|y_h(t)\|_{C^2(\mathcal{T}_h)}\right) \times \sum_{|\alpha|\le 2} |\partial^\alpha (v - \tilde{v}_h)(x)|, \qquad x \in \mathcal{T}_h$$

(see [1, Lemma 4.2, (ii)]), and the triangle inequality for $\|\cdot\|_h$ give

$$\|L_{y_h(t)}\,(v - \tilde{v}_h)\|_h \;\le\; \bar{\mu}\left(\|y_h(t)\|_{C^1(\overline{\Omega})}\right)\,\left(1 + \|y_h(t)\|_{C^2(\mathcal{T}_h)}\right)\,W, \qquad t \in [0,\,1], \tag{4.4}$$

where $W = \sum_{|\alpha|\le 2} \|\partial^\alpha (v - \tilde{v}_h)\|_h$. Using (2.11), we have

$$W \le Ch^2\|v\|_{H^4(\Omega)}. \tag{4.5}$$

Since $v,\ \tilde{v}_h \in C^2(\mathcal{T}_h)$, on applying the triangle inequality for $\|\cdot\|_{C^2(\mathcal{T}_h)}$, the inequality $\|v\|_{C^2(\overline{\Omega})} \le C\|v\|_{H^4(\Omega)}$, and (2.9), we have

$$\|y_h(t)\|_{C^2(\mathcal{T}_h)} \le C\|v\|_{H^4(\Omega)}, \qquad t \in [0, 1]. \tag{4.6}$$

Hence, from (4.4), (4.5), and (4.6), we obtain

$$\left\|L_{y_h(t)}\,(v - \tilde{v}_h)\right\|_h \;\le\; C\,\|v\|_{H^4(\Omega)}\,\bar{\mu}\,(C\,\|v\|_{H^4(\Omega)})\,(1 + \|v\|_{H^4(\Omega)})\;h^2, \qquad t \in [0,\,1],$$

which, along with (4.3), gives (4.1) for $k = 4$.

If $v \in H^5(\Omega)$, then (2.11) for $l = 0,\,1$, and (2.12) imply $W \le Ch^3\|v\|_{H^5(\Omega)}$ which gives (4.1) for $k = 5$. ∎

**5. Lipschitz continuity of a Fréchet derivative.** Let $L_y\,w\,(x)$ be given by (2.5) and (2.6), and let

$$\lambda_{y,z} \;=\; \|y\|_{H^2(\Omega)} + \|z\|_{H^2(\Omega)}, \qquad y,\,z \in H^2(\Omega). \tag{5.1}$$

LEMMA 5.1. *Suppose that $h \in (0, e^{-2}]$ and that* (2.3a) *and* (2.3b) *hold for all $\beta$ with $|\beta| = 1, 2$ and $|\beta| = 2$, respectively, and $m = 0$. Then $L_h$ has a Fréchet derivative $L_{h,y}$ for each $y \in \mathcal{M}_h^0$, and*

$$L_{h,y}\,w\,(x) \;=\; L_y\,w\,(x), \qquad x \in \mathcal{G}_h, \qquad w \in \mathcal{M}_h^0. \tag{5.2}$$

*Moreover, for any $v \in H^4(\Omega)$ and any $\rho_0 > 0$,*

$$\|L_{h,z} - L_{h,y}\| \;\le\; K_2\,(h, \rho_0, v)\,\|z - y\|_{H^2(\Omega)}, \qquad y, z \in B_h(\tilde{v}_h, \rho_0), \tag{5.3}$$

*where*

$$K_2\,(h, \rho_0, v) \;\equiv\; (\ln^2 h)\,(C + \gamma)\,\tilde{\mu}\,(\gamma,\,|\ln h|\,\gamma,\,|\ln h|\,\gamma)\,, \tag{5.4}$$

*and* $\gamma = C\,(\|v\|_{H^4(\Omega)} + \rho_0)$.

*Proof.* We take any $y$ and $w \neq 0$ in $\mathcal{M}_h^0$. Using (2.7), we have

$$\|L(y+w) - Ly - L_y\, w\|_h \;=\; \left\| \int_0^1 (L_{y+t\,w} - L_y)\, w\, dt \right\|_h . \tag{5.5}$$

Since the functions $a_{ij}$ and $a$ are sufficiently smooth, $L_{y+t\,w} w\,(x)$ is continuous in $t \in [0,1]$ for all $x \in \mathcal{G}_h$. Using (2.13) with $G(x,t) = (L_{y+t\,w}\, w - L_y\, w)\,(x)$, we obtain

$$\left\| \int_0^1 (L_{y+t\,w} - L_y)\, w\, dt \right\|_h \;\le\; \int_0^1 \|(L_{y+t\,w} - L_y)\, w\|_h \; dt. \tag{5.6}$$

In [1, Lemma 6.1], we proved that, for $h \in (0, e^{-2}]$,

$$\|(L_{y+z} - L_y)\, w\|_h \;\le\; \Psi_h\,[y,\, z]\, \|z\|_{H^2(\Omega)}\, \|w\|_{H^2(\Omega)}, \qquad z \in \mathcal{M}_h^0, \tag{5.7}$$

where

$$\Psi_h\,[y,\, z] \;=\; C\,(\ln^2 h)\, \tilde{\mu}\,(C\,\lambda_{y,z},\, C\,|\ln h|\,\lambda_{y,z},\, C\,|\ln h|\,\lambda_{y,z})\;(1+\lambda_{y,z}), \tag{5.8}$$

and $\lambda_{y,z}$ is given by (5.1). Applying (5.7) with $z = t\,w$ and the inequality $\Psi_h\,[y,\, t\,w] \le \Psi_h\,[y,\, w]$, $t \in [0,1]$, we get

$$\begin{aligned} \int_0^1 \|(L_{y+t\,w} - L_y)\, w\|_h \; dt \;&\le\; \int_0^1 t\,\Psi_h\,[y,\, t\,w]\, \|w\|^2_{H^2(\Omega)}\, dt \\ &\le\; (1/2)\,\Psi_h\,[y,\, w]\, \|w\|^2_{H^2(\Omega)}. \end{aligned} \tag{5.9}$$

From (5.5), (5.6), and (5.9), it follows that

$$\|L(y+w) - Ly - L_y\, w\|_h \;\le\; (1/2)\,\Psi_h\,[y,\, w]\, \|w\|^2_{H^2(\Omega)}. \tag{5.10}$$

We define the linear operator $L_{y,h}$ from $\mathcal{M}_h^0$ into $\mathcal{M}_h^0$ by

$$L_{y,h}\, z\,(x) \;=\; L_y\, z\,(x), \qquad x \in \mathcal{G}_h. \tag{5.11}$$

Using (5.10), (3.1), and (5.11), we obtain

$$\|L_h(y+w) - L_h y - L_{y,h}\, w\|_h / \|w\|_{H^2(\Omega)} \;\le\; (1/2)\,\Psi_h\,[y,\, w]\, \|w\|_{H^2(\Omega)}. \tag{5.12}$$

Since $\tilde{\mu}$ is nondecreasing, it follows from (5.8) and (5.1) that $\Psi_h\,[y,\, w]\, \|w\|_{H^2(\Omega)} \to 0$ as $\|w\|_{H^2(\Omega)} \to 0$. The operator $L_{y,h}$ is bounded since it is linear and $\mathcal{M}_h^0$ is finite dimensional. Therefore, (5.12) implies that $L_{y,h}$ is a Fréchet derivative of $L_h$ at $y$. Hence (5.2) follows from (5.11).

To prove (5.3), we take any $v \in H^4(\Omega)$, any $\rho_0 > 0$, and any $y,\ z \in B_h(\tilde{v}_h, \rho_0)$. Using (5.7) with $z$ replaced by $z - y$, we have

$$\|(L_z - L_y)\, w\|_h \;\le\; \Psi_h\,[y,\, z-y]\, \|z-y\|_{H^2(\Omega)}\, \|w\|_{H^2(\Omega)}, \qquad w \in \mathcal{M}_h^0. \tag{5.13}$$

Since $y,\ z \in B_h(\tilde{v}_h, \rho_0)$, using the triangle inequality and (2.9), we have

$$\|y\|_{H^2(\Omega)} \le \|\tilde{v}_h\|_{H^2(\Omega)} + \|y - \tilde{v}_h\|_{H^2(\Omega)} \le C\,\|\tilde{v}_h\|_{C^2(\mathcal{T}_h)} + \rho_0 \le C\,\|v\|_{H^4(\Omega)} + \rho_0 \tag{5.14}$$

and

$$\|z-y\|_{H^2(\Omega)} \le \|z - \tilde{v}_h\|_{H^2(\Omega)} + \|y - \tilde{v}_h\|_{H^2(\Omega)} \;\le 2\,\rho_0. \tag{5.15}$$

Using (5.1), (5.14), (5.15), and the definition of $\gamma$, we obtain $\lambda_{y,z-y} \le \gamma$. Therefore, from (5.8), we have

$$\Psi_h\,[y,\, z-y] \;\le\; (\ln^2 h)(C+\gamma)\;\tilde{\mu}\,(\gamma,\,|\ln h|\,\gamma,\,|\ln h|\,\gamma)\,. \tag{5.16}$$

From (5.13) and (5.16), we get (5.3) and (5.4). ∎

The Lipschitz constant $K_2\,(h,\rho_0,v)$ in (5.4) is independent of $h$ if the coefficients $a_{ij}$ of the differential operator $L$ do not depend on $\nabla u$ (see [1, Lemma 6.4]).

**6. Bound on the inverse of the Fréchet derivative.** In this section, we prove that, if $v \in H^4(\Omega)$, then $L^{-1}_{h,\tilde{v}_h}$ exists and its norm is uniformly bounded for $h$ sufficiently small. This result follows from the two auxiliary inequalities obtained in the following two subsections. Throughout this section, $\lambda_v = \|v\|_{H^4(\Omega)}$ for $v \in H^4(\Omega)$.

**6.1. First auxiliary inequality.** The first auxiliary inequality is given in the following lemma.

Lemma 6.1. *Suppose that* (2.3a) *holds for $m = 0$ and all $\beta$ such that $|\beta| \le 1$, and that* (2.3b) *holds for $m = 0$ and all $\beta$ such that $|\beta| = 1$. If $v \in H^4(\Omega)$, then, for any $w \in \mathcal{M}^0_h$,*

$$\|w\|_{H^2(\Omega)} \;\le\; C\,\nu^{-2}\,\bar{\mu}\,(C\lambda_v)\;\|L_{h,\tilde{v}_h}\,w\|_h + C\,\nu^{-4}\,\bar{\mu}^4\,(C\lambda_v)\,(1+\lambda_v)^2\;\|w\|_{L^2(\Omega)}. \tag{6.1}$$

*Proof.* The proof is similar to that of Lemma 3.1 in [2], and it involves the application of Bernstein's transformation [16, p. 452]. For $x \in \mathcal{G}_h$, we introduce

$$b_{ij}(x) = a_{ij}(x,\,\tilde{v}_h(x),\,\nabla\tilde{v}_h(x)), \qquad i,j = 1,2, \tag{6.2}$$

$$b_k(x) = A^k_{\tilde{v}_h}(x), \qquad k = 0,1,2, \tag{6.3}$$

where $A^k_{\tilde{v}_h}$ is given by (2.6). For any $w \in \mathcal{M}^0_h$ and $x \in \mathcal{G}_h$, (2.5), (6.2), and (6.3) imply that

$$\sum_{i,\,j\,=\,1}^{2} b_{ij}\,w_{x_i x_j} \;=\; L_{\tilde{v}_h} w - \sum_{k=0}^{2} b_k\,w_{x_k} \;\equiv\; \Phi, \tag{6.4}$$

where $b_{ij}$, $b_k$, $\Phi$, $w$ and its derivatives are evaluated at $x$.

First, using (6.4), we bound

$$|w|_{2,h} \;\equiv\; \left(\|w_{x_1x_1}\|_h^2 + 2\,\|w_{x_1x_2}\|_h^2 + \|w_{x_2x_2}\|_h^2\right)^{1/2} \tag{6.5}$$

in terms of $\|\Phi\|_h$. Inequality (2.2) with $(\zeta_1,\zeta_2) = (1,0)$ and $(\zeta_1,\zeta_2) = (0,1)$ implies that $b_{ii} \ge \nu > 0$, $i = 1,2$. Therefore, multiplying (6.4) by $b^{-1}_{22} w_{x_1x_1} + b^{-1}_{11} w_{x_2x_2}$, we obtain

$$I \equiv \sum_{i,\,j\,=\,1}^{2} b_{ij} w_{x_ix_j}\left(b^{-1}_{22} w_{x_1x_1} + b^{-1}_{11} w_{x_2x_2}\right) = \Phi(x)\left(b^{-1}_{22} w_{x_1x_1} + b^{-1}_{11} w_{x_2x_2}\right). \tag{6.6}$$

Since (6.2) and (2.1) imply $b_{12} = b_{21}$, (6.6) gives

$$\begin{aligned} I = {} & 2\,\left(w_{x_1x_1}\,w_{x_2x_2} - w^2_{x_1x_2}\right) + b^{-1}_{22}\left(b_{11}\,w^2_{x_1x_1} + 2\,b_{12}\,w_{x_1x_1}\,w_{x_1x_2} + b_{22}\,w^2_{x_1x_2}\right) \\ & + b^{-1}_{11}\left(b_{11}\,w^2_{x_1x_2} + 2\,b_{12}\,w_{x_1x_2}\,w_{x_2x_2} + b_{22}\,w^2_{x_2x_2}\right). \end{aligned} \tag{6.7}$$

It follows easily from (6.2) and (2.2) that

$$\sum_{i,j=1}^{2} b_{ij}\,\zeta_i\zeta_j \;\geq\; \nu\,(\zeta_1^2+\zeta_2^2), \qquad (\zeta_1,\zeta_2)\in R^2. \tag{6.8}$$

Equations (6.7) and (6.8) applied twice give

$$I \;\geq\; \nu b_{22}^{-1}\left(w_{x_1x_1}^2+w_{x_1x_2}^2\right)+\nu b_{11}^{-1}\left(w_{x_1x_2}^2+w_{x_2x_2}^2\right)+2\left(w_{x_1x_1}w_{x_2x_2}-w_{x_1x_2}^2\right). \tag{6.9}$$

Using (6.2), (2.3a) with $m=0$ and $|\beta|=0$, (2.4), and (2.9), we obtain

$$b_{ii} \;\leq\; \tilde{\mu}\left(|\tilde{v}_h(x)|\,,\,|(\tilde{v}_h)_{x_1}(x)|\,,\,|(\tilde{v}_h)_{x_2}(x)|\right) \;\leq\; \bar{\mu}\left(\|\tilde{v}_h\|_{C^1(\overline{\Omega})}\right) \;\leq\; \mu_v, \quad i=1,2, \tag{6.10}$$

where $\mu_v \equiv \bar{\mu}(C\lambda_v)$. Therefore, from (6.9) and (6.10), we have

$$I \;\geq\; \nu\,\mu_v^{-1}\left(w_{x_1x_1}^2+2\,w_{x_1x_2}^2+w_{x_2x_2}^2\right)+2\left(w_{x_1x_1}\,w_{x_2x_2}-w_{x_1x_2}^2\right). \tag{6.11}$$

Using (6.11), (6.6), and $b_{ii}\geq\nu$, we have

$$\begin{aligned}\nu^{-1}\,|\Phi|\left(|w_{x_1x_1}| \;+\; |w_{x_2x_2}|\right) \geq \nu\,\mu_v^{-1}&\left(w_{x_1x_1}^2+2\,w_{x_1x_2}^2+w_{x_2x_2}^2\right)\\ &+2\left(w_{x_1x_1}\,w_{x_2x_2}-w_{x_1x_2}^2)\right).\end{aligned} \tag{6.12}$$

Multiplying (6.12) by the Gaussian quadrature weights and summing over all $x\in\mathcal{G}_h$, we obtain

$$\nu\mu_v^{-1}\,|w|_{2,h}^2+2\sum_h\left(w_{x_1x_1}\,w_{x_2x_2}-w_{x_1x_2}^2\right) \;\leq\; \nu^{-1}\sum_h|\Phi|\left(|w_{x_1x_1}| \;+\; |w_{x_2x_2}|\right), \tag{6.13}$$

where $|w|_{2,h}$ is defined in (6.5). It easily follows from (6.5) that

$$\|w_{x_1x_1}\|_h^2+\|w_{x_2x_2}\|_h^2 \;\leq\; |w|_{2,h}^2. \tag{6.14}$$

The Cauchy–Schwarz inequality in $R^2$ and (6.14) give

$$\|w_{x_1x_1}\|_h+\|w_{x_2x_2}\|_h \;\leq\; \sqrt{2}\left(\|w_{x_1x_1}\|_h^2+\|w_{x_2x_2}\|_h^2\right)^{1/2} \;\leq\; \sqrt{2}\,|w|_{2,h}. \tag{6.15}$$

The Cauchy–Schwarz inequality in $\mathcal{M}_h^0$ and (6.15) imply

$$\sum_h|\Phi|\left(|w_{x_1x_1}| \;+\; |w_{x_2x_2}|\right) \;\leq\; \sqrt{2}\,\|\Phi\|_h\,|w|_{2,h}. \tag{6.16}$$

Using (6.13), the inequality

$$\sum_h z_{x_1x_1}\,z_{x_2x_2}-\sum_h z_{x_1x_2}^2 \;\geq\; 0, \qquad z\in\mathcal{M}_h^0,$$

which is (2.5) in [2], and (6.16), we obtain

$$|w|_{2,h} \;\leq\; \sqrt{2}\,\nu^{-2}\,\mu_v\,\|\Phi\|_h. \tag{6.17}$$

Next we bound $\|\Phi\|_h$. Using (6.4) and the triangle inequality, we obtain

$$\|\Phi\|_h \;\leq\; \|L_{\tilde{v}_h}\,w\|_h+\left(\max_{k=0,1,2}\,\max_{x\in\mathcal{G}_h}|b_k(x)|\right)\sum_{k=0}^{2}\|w_{x_k}\|_h. \tag{6.18}$$

Since $\mathcal{M}_h^0 \subset C^2(\mathcal{T}_h)$, (6.3), (2.4), and the inequality

$$|A^k_{\tilde{v}_h}(x)| \;\le\; \bar{\mu}\left(\|\tilde{v}_h\|_{C^1(\overline{\Omega})}\right)\left(1+\|\tilde{v}_h\|_{C^2(\mathcal{T}_h)}\right), \qquad x \in \mathcal{G}_h,\ k=0,1,2$$

(see [1, Lemma 4.2, (i)]) give

$$\max_{x\in\mathcal{G}_h} |b_k(x)| \le \bar{\mu}\left(\|\tilde{v}_h\|_{C^1(\overline{\Omega})}\right)\left(1+\|\tilde{v}_h\|_{C^2(\mathcal{T}_h)}\right), \qquad k=0,1,2.$$

Hence, from (2.9), we have

$$\max_{k=0,1,2}\max_{x\in\mathcal{G}_h} |b_k(x)| \;\le\; C\,\mu_v\,(1+\lambda_v)\,. \tag{6.19}$$

Using Cauchy–Schwarz's inequality in $R^2$, the inequality

$$\|z_{x_1}\|_h^2 + \|z_{x_2}\|_h^2 \;\le\; -\sum_h (\Delta z)\, z, \qquad z \in \mathcal{M}_h^0,$$

which is (2.6) in [2], and the Cauchy–Schwarz inequality in $\mathcal{M}_h^0$, we obtain

$$\|w_{x_1}\|_h + \|w_{x_2}\|_h \;\le\; \sqrt{2}\,\left(\|w_{x_1}\|_h^2 + \|w_{x_2}\|_h^2\right)^{1/2} \;\le\; \sqrt{2}\,\|\Delta w\|_h^{1/2}\,\|w\|_h^{1/2}.$$

Hence, the Cauchy inequality $a\,b \;\le\; \epsilon\,a^2 + (4\,\epsilon)^{-1}\,b^2$, $\epsilon > 0$, gives

$$\|w_{x_1}\|_h + \|w_{x_2}\|_h \;\le\; \epsilon\,\|\Delta w\|_h \;+\; (2\,\epsilon)^{-1}\,\|w\|_h.$$

Applying the triangle inequality to bound $\|\Delta w\|_h$ and using (6.15), we obtain

$$\|w_{x_1}\|_h + \|w_{x_2}\|_h \;\le\; \sqrt{2}\,\epsilon\,|w|_{2,h} + (2\,\epsilon)^{-1}\,\|w\|_h. \tag{6.20}$$

Thus, from (6.18), (6.19), and (6.20), we have

$$\|\Phi\|_h \;\le\; \|L_{\tilde{v}_h}\,w\|_h + C\,\mu_v\,(1+\lambda_v)\left[\epsilon\,|w|_{2,h} + \left(\epsilon^{-1}+1\right)\,\|w\|_h\right]. \tag{6.21}$$

Multiplying (6.21) by $\sqrt{2}\,\nu^{-2}\,\mu_v$ and setting $\epsilon = \nu^2\left(2\,\sqrt{2}\,C\,\mu_v^2\,(1+\lambda_v)\right)^{-1}$, we obtain

$$\sqrt{2}\,\nu^{-2}\,\mu_v\,\|\Phi\|_h \;\le\; \sqrt{2}\,\nu^{-2}\,\mu_v\,\|L_{\tilde{v}_h}\,w\|_h + (1/2)\,|w|_{2,h} + M\,\|w\|_h, \tag{6.22}$$

where

$$M \;\equiv\; \sqrt{2}\,C\,\nu^{-2}\,\mu_v^2\,(1+\lambda_v)\,\left[2\,\sqrt{2}\,C\,\nu^{-2}\,\mu_v^2\,(1+\lambda_v)+1\right]. \tag{6.23}$$

Combining (6.17) and (6.22), we get

$$(1/2)\,|w|_{2,h} \;\le\; \sqrt{2}\,\nu^{-2}\,\mu_v\,\|L_{\tilde{v}_h}\,w\|_h + M\,\|w\|_h. \tag{6.24}$$

It follows from (2.2) with $\zeta_1 = 1$, $\zeta_2 = 0$, $s = (t,\, t,\, t)$, and (2.3a) for $i = j = 1$ and $m + |\beta| = 0$ that $\nu \le a_{11}(x,t,t,t) \le \tilde{\mu}(t,t,t)$ for any $x \in \overline{\Omega}$ and $t \ge 0$ which, along with (2.4), implies $\bar{\mu}(t) \ge \nu$ for all $t \ge 0$. Therefore, by (6.10) and $b_{ii} \ge \nu$, we have $\nu \le \mu_v$, and hence (6.23) gives

$$M \;\le\; C\,\nu^{-4}\,\mu_v^2\,(\mu_v^2+\nu^2)\,(1+\lambda_v)^2 \;\le\; C\,\nu^{-4}\,\mu_v^4\,(1+\lambda_v)^2. \tag{6.25}$$

Using the inequality

$$\|z\|^2_{H^2(\Omega)} \;\le\; C \sum_{i=1}^{2} \|z_{x_i x_i}\|_h^2, \qquad z \in \mathcal{M}_h^0,$$

which is (2.7) in [2], (6.14), (6.24), and (6.25), we obtain

$$\|w\|_{H^2(\Omega)} \;\le\; C\,\nu^{-2}\,\mu_v\,\|L_{\tilde{v}_h}\,w\|_h + C\,\nu^{-4}\,\mu_v^4\,(1+\lambda_v)^2\,\|w\|_h. \tag{6.26}$$

Finally, (6.1) follows from (6.26), (5.2), and the inequality $\|w\|_h \le C\|w\|_{L^2(\Omega)}$ (see [1, Lemma B.2]). ∎

**6.2. Second auxiliary inequality.** To obtain the second auxiliary inequality, we introduce a formal adjoint of the linear differential operator $L_v$ given by (2.5) and (2.6). For sufficiently smooth $v$ defined on $\Omega$, let

$$L_v^*\,\varphi\,(x) \;=\; \sum_{i,\,j\,=\,1}^{2} \left[\tilde{a}_{ij}(x)\,\varphi\right]_{x_j x_i} - \sum_{k=0}^{2} \left[A_v^k\,(x)\,\varphi\right]_{x_k}, \qquad x \in \Omega, \tag{6.27}$$

where

$$\tilde{a}_{ij}(x) \;\equiv\; a_{ij}(x,\, v(x),\, \nabla v(x)), \qquad i,j = 1,2, \tag{6.28}$$

and $A_v^k$ is given by (2.6). Assume that $v \in H^4(\Omega)$ and that $a_{ij}$ and $a$ are twice continuously differentiable on $\overline{\Omega} \times R^3$. Then $\tilde{a}_{ij} \in C^1(\overline{\Omega}) \cap H^2(\Omega)$ and $A_v^k \in C(\overline{\Omega}) \cap H^1(\Omega)$. Hence, using (2.1), $L_v^*$ can be written in the form

$$L_v^*\,\varphi\,(x) \;=\; \sum_{i,\,j\,=\,1}^{2} \tilde{a}_{ij}(x)\,\varphi_{x_i x_j} + \sum_{k=0}^{2} \tilde{a}_k(x)\,\varphi_{x_k}, \qquad x \in \Omega,$$

for some $\tilde{a}_k \in C(\overline{\Omega})$, $k = 1,2$, and $\tilde{a}_0 \in L^2(\overline{\Omega})$. It follows from (6.28) and (2.2) that $L_v^*$ is uniformly elliptic. Introducing the normed space

$$H^{2,0}(\Omega) \;=\; \{v \in H^2(\Omega) :\; v|_{\partial\Omega} = 0\}$$

with the $\|\cdot\|_{H^2(\Omega)}$-norm, we see that $L_v^*$ is a linear operator from $H^{2,0}(\Omega)$ into $L^2(\Omega)$. Moreover, $L_v^*$ is a formal adjoint of $L_v$, that is,

$$\int_\Omega (L_v^*\,\varphi)\,w\,dx \;=\; \int_\Omega \varphi\,L_v\,w\,dx, \qquad \varphi,\; w \in H^{2,0}(\Omega). \tag{6.29}$$

Indeed, for any $\varphi$ and $w \in H^{2,0}(\Omega)$, the traces of $\varphi_{x_i}$, $w_{x_i}$, $i = 1,2$, on $\partial\Omega$ are in $L^2(\partial\Omega)$. Hence, using Green's formula (see [8, equation (1.2.4)]) and $w = \varphi = 0$ on $\partial\Omega$, we obtain

$$\int_\Omega \left[\tilde{a}_{ij}(x)\,\varphi\right]_{x_j x_i}\,w\,dx \;=\; \int_\Omega \varphi\,\tilde{a}_{ij}(x)\,w_{x_i x_j}\,dx, \qquad 1 \le i,\, j \le 2. \tag{6.30}$$

Similarly,

$$-\int_\Omega \left[A_v^k\,(x)\,\varphi\right]_{x_k}\,w\,dx \;=\; \int_\Omega \varphi\,A_v^k\,(x)\,w_{x_k}\,dx, \qquad k = 1,\, 2. \tag{6.31}$$

Therefore, using (6.27), (6.30), (6.31), (6.28), and (2.5), we have (6.29).

The proof of the second auxiliary inequality is based on the following result.

LEMMA 6.2 (see [1, Lemma C.3]). *For $l$ a positive integer, suppose that $g(x,s)$ is $l$-times continuously differentiable on $\Omega \times R^3$ and*

$$\left| \frac{\partial^{m+|\beta|} g}{\partial x_i^m \, \partial s^\beta}\,(x,s) \right| \;\le\; \tilde\sigma(|s_0|,|s_1|,|s_2|), \qquad m+|\beta| \le l,\; i=1,2,\; (x,s)\in\Omega\times R^3, \tag{6.32}$$

*where $\tilde\sigma(t_0,t_1,t_2)$, $t_0,t_1,t_2 \ge 0$, is continuous and nondecreasing in each variable. Then, for $g_w\,(x) = g(x,w(x),\nabla w(x))$, $x\in\Omega$, we have*

$$\left\| \frac{\partial^l g_w}{\partial x_i^l} \right\|_{C(\tau)} \le C h^{1-l} \bar\sigma\left(\|w\|_{C^1(\tau)}\right)\left(1 + \|w\|^l_{C^2(\tau)}\right),\; i=1,2,\; w\in\mathcal{P}_h,\; \tau\in\mathcal{T}_h, \tag{6.33}$$

*where $\bar\sigma(t) = \tilde\sigma(t,t,t)$, $t\ge 0$, and $C$ is independent of $g$ and $w$.*

In the proof of the second auxiliary inequality, we also use the following inverse inequalities: for $\tau\in\mathcal{T}_h$, $z\in\mathcal{P}_h$, and $i=1,2$,

$$\|z_{x_i}\|_{L^2(\tau)} \le C\,h^{-1}\|z\|_{L^2(\tau)}, \tag{6.34}$$

$$\|z_{x_i}\|_{C(\tau)} \le Ch^{-1}\|z\|_{C(\tau)} \tag{6.35}$$

(see [8, Theorem 3.2.6, equation (3.2.33)]).

The second auxiliary inequality is formulated in the following lemma.

LEMMA 6.3. *Suppose that* (2.3a) *holds for $0 \le m \le 4$ and all $\beta$ such that $m+|\beta| \le 5$, and that* (2.3b) *holds for $0\le m\le 4$ and all $\beta$ such that $m+|\beta|\le 5$ and $|\beta|\ge 1$. Suppose that $v\in H^4(\Omega)$ and that the operator $L_v^*$ of* (6.27) *is from $H^{2,0}(\Omega)$ onto $L^2(\Omega)$, has a bounded inverse, and $C_v = \|(L_v^*)^{-1}\|$. Then, for any $h\in(0,\,e^{-2}]$,*

$$\|w\|_{L^2(\Omega)} \;\le\; C\,C_v\,\|L_{h,\tilde v_h}\,w\|_h \;+\; h\,C\,C_v\,\bar\mu(C\,\lambda_v)\,\left(1+\lambda_v^5\right)\,\|w\|_{H^2(\Omega)},\; w\in\mathcal{M}_h^0. \tag{6.36}$$

*Proof.* We adapt the approach used in the proof of Lemma 3.2 of [2]. First, we observe that $L_v^*$ is well defined since $v\in H^4(\Omega)$. Next we take any $w\in\mathcal{M}_h^0$. Since $\mathcal{M}_h^0 \subset L^2(\Omega)$ and $L_v^*$ has a bounded inverse, there is unique $\varphi\in H^{2,0}(\Omega)$ such that

$$L_v^*\,\varphi = w, \tag{6.37}$$

$$\|\varphi\|_{H^2(\Omega)} \le C_v\,\|w\|_{L^2(\Omega)}. \tag{6.38}$$

Let $\overline\varphi_h$ be the piecewise constant interpolant of $\varphi$ such that

$$\|\varphi - \overline\varphi_h\|_{L^2(\Omega)} \;\le\; C\,h\,\|\varphi\|_{H^1(\Omega)}. \tag{6.39}$$

Using (6.39) and (6.38), we obtain

$$\|\varphi - \overline\varphi_h\|_{L^2(\Omega)} \;\le\; C\,C_v\,h\,\|w\|_{L^2(\Omega)}. \tag{6.40}$$

Also, using the exactness property of Gauss quadrature for a piecewise constant function, the triangle inequality, (6.40), and (6.38), we have

$$\|\overline\varphi_h\|_h \;=\; \|\overline\varphi_h\|_{L^2(\Omega)} \;\le\; \|\varphi-\overline\varphi_h\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)} \;\le\; C\,C_v\,\|w\|_{L^2(\Omega)}. \tag{6.41}$$

As in Nitsche's trick [17], using (6.37) and (6.29), we obtain

$$\|w\|^2_{L^2(\Omega)} \;=\; \int_\Omega w^2\,dx \;=\; \int_\Omega (L_v^*\,\varphi)\,w\,dx \;=\; \int_\Omega \varphi\,L_v\,w\,dx \;=\; \sum_{i=1}^4 J_i, \tag{6.42}$$

where

$$J_1 \equiv \int_\Omega (\varphi - \overline{\varphi}_h)\, L_v\, w\, dx, \qquad J_2 \;\equiv\; \int_\Omega \overline{\varphi}_h\, (L_v\, w - L_{\tilde{v}_h}\, w)\; dx,$$
$$J_3 \equiv \int_\Omega \overline{\varphi}_h\, L_{\tilde{v}_h}\, w\, dx - \sum_h \overline{\varphi}_h\, L_{\tilde{v}_h}\, w, \qquad J_4 \;\equiv\; \sum_h \overline{\varphi}_h\, L_{\tilde{v}_h}\, w.$$

The Cauchy–Schwarz inequality, (6.40), and the estimate

$$\|L_v\, w\|_{L^2(\Omega)} \;\le\; C\, \bar{\mu}(C\, \lambda_v)\, (1+\lambda_v)\, \|w\|_{H^2(\Omega)}$$

(see [1, Lemma 7.2]) give

$$J_1 \le \left| \int_\Omega (\varphi - \overline{\varphi}_h)\, L_v w\, dx \right| \;\le\; \|\varphi - \overline{\varphi}_h\|_{L^2(\Omega)}\, \|L_v w\|_{L^2(\Omega)}, \tag{6.43}$$
$$\le C\, C_v\, h\, \bar{\mu}(C\, \lambda_v)\, (1+\lambda_v)\, \|w\|_{H^2(\Omega)}\, \|w\|_{L^2(\Omega)}.$$

The Cauchy–Schwarz inequality, (6.41), and the inequality

$$\|(L_v - L_{\tilde{v}_h})\, w\|_{L^2(\Omega)} \;\le\; C\, h\, \bar{\mu}\, (C\, \lambda_v)\; \lambda_v\, (1+\lambda_v)\, \|w\|_{H^2(\Omega)}$$

(see [1, Lemma 7.2]) give

$$J_2 \le \left| \int_\Omega \overline{\varphi}_h\; (L_v w - L_{\tilde{v}_h}\, w\,)\; dx \right| \;\le\; \|\overline{\varphi}_h\|_{L^2(\Omega)}\, \|(L_v - L_{\tilde{v}_h})\, w\|_{L^2(\Omega)} \tag{6.44}$$
$$\le C\, C_v\, h\, \bar{\mu}\, (C\, \lambda_v)\; \lambda_v\, (1+\lambda_v)\, \|w\|_{H^2(\Omega)}\, \|w\|_{L^2(\Omega)}.$$

The Cauchy–Schwarz inequality in $\mathcal{M}_h^0$, (6.41), and (5.2) give

$$J_4 \;\le\; \|\overline{\varphi}_h\|_h\, \|L_{\tilde{v}_h}\, w\|_h \le CC_v\, \|w\|_{L^2(\Omega)} \|L_{\tilde{v}_h}\, w\|_h \le CC_v\, \|L_{h,\tilde{v}_h} w\|_h \|w\|_{L^2(\Omega)}. \tag{6.45}$$

Suppose that

$$J_3 \;\le\; C\, C_v\, h\, \bar{\mu}(C\, \lambda_v)\, \left(1+\lambda_v^5\right)\, \|w\|_{H^2(\Omega)}\, \|w\|_{L^2(\Omega)}. \tag{6.46}$$

Then, using (6.42)–(6.46) and

$$t^p \;\le\; t^q + 1, \qquad t \ge 0, \qquad 0 < p \le q, \tag{6.47}$$

we obtain (6.36).

It remains to prove (6.46). Using the triangle inequality and setting $g(x) = \overline{\varphi}_h(x)\, L_{\tilde{v}_h} w\, (x)$, we get

$$J_3 \;\le\; \left| \int_\Omega \overline{\varphi}_h\, L_{\tilde{v}_h} w\, dx - \sum_h \overline{\varphi}_h\, L_{\tilde{v}_h} w \right| \;\le \sum_{\tau \in \mathcal{T}_h} \left| \int_\tau g(x)\, dx - \frac{1}{4}\; \text{mes}\; (\tau) \sum_{x \in \tau \cap \mathcal{G}_h} g(x) \right|. \tag{6.48}$$

We fix $\tau = (x_1^{k_1-1},\, x_1^{k_1}) \times (x_2^{k_2-1},\, x_2^{k_2}) \in \mathcal{T}_h$. For $\eta_1$ and $\eta_2$ given in (2.8) and $\zeta = (\zeta_1,\, \zeta_2) \in \Omega$, we introduce the linear functional

$$F(z) \;=\; \int_\Omega z(\zeta)\, d\zeta - \frac{1}{4} \sum_{l_1,\, l_2=1}^{2} z(\eta_{l_1},\, \eta_{l_2}), \qquad z \in H_1^4(\Omega).$$

Making a change of variables, we obtain

$$\int_\tau g(x)\,dx - \frac{1}{4}\,h_1^{k_1}\,h_2^{k_2} \sum_{x\in\tau\cap\mathcal{G}_h} g(x) \;=\; h_1^{k_1}\,h_2^{k_2}\,F(\tilde g), \tag{6.49}$$

where $\tilde g(\zeta) = g(x_1^{k_1-1} + h_1^{k_1}\,\zeta_1,\, x_2^{k_2-1} + h_2^{k_2}\,\zeta_2)$. Since $\tilde g \in H_1^4(\Omega)$, $|F(z)| \le C\|z\|_{H_1^4(\Omega)}$, $z \in H_1^4(\Omega)$, and $F(p) = 0$, $p \in P_3 \otimes P_3$, it follows from the Bramble–Hilbert lemma [7, Theorem 2] that

$$|F(\tilde g)| \;\le\; C \int_\Omega \left( \left|\frac{\partial^4 \tilde g}{\partial \zeta_1^4}\right| + \left|\frac{\partial^4 \tilde g}{\partial \zeta_2^4}\right| \right) d\zeta. \tag{6.50}$$

Using (6.48)–(6.50), making a change of variables, and taking into account the fact that $\overline{\varphi}_h\,|_\tau$ is a constant for any $\tau \in \mathcal{T}_h$, we have

$$J_3 \;\le\; C\,h^4 \sum_{k=1}^{2} \sum_{\tau\in\mathcal{T}_h} \int_\tau \left|\overline{\varphi}_h\,\partial_k^4\,L_{\tilde v_h} w\right|\,dx,$$

where $\partial_k^l = \partial^l/\partial x_k^l$. Applying the Cauchy–Schwarz inequality in $L^2(\tau)$ and in $R^{N_1N_2}$ and using (6.41), we get

$$\begin{aligned} J_3 &\le C\,h^4 \sum_{k=1}^{2} \sum_{\tau\in\mathcal{T}_h} \|\overline{\varphi}_h\|_{L^2(\tau)}\, \left\|\partial_k^4\,L_{\tilde v_h}\,w\right\|_{L^2(\tau)} \\ &\le C\,C_v\,h^4\,\|w\|_{L^2(\Omega)} \sum_{k=1}^{2} \sqrt{\sum_{\tau\in\mathcal{T}_h} \|\partial_k^4\,L_{\tilde v_h}\,w\|^2_{L^2(\tau)}}. \end{aligned} \tag{6.51}$$

Using (2.5), (2.6), and the triangle inequality, we obtain

$$\left\|\partial_k^4\,L_{\tilde v_h} w\right\|_{L^2(\tau)} \;\le\; \sum_{i,\,j\,=1}^{2} P_{k,\tau}^{(i,j)} \;+\; \sum_{\nu=0}^{2} \sum_{i,\,j\,=1}^{2} Q_{\nu,k,\tau}^{(i,j)} + \sum_{\nu=0}^{2} R_{\nu,k,\tau}, \tag{6.52}$$

where, for $\tau \in \mathcal{T}_h$, $i, j, k = 1,\, 2$, and $0 \le \nu \le 2$,

$$P_{k,\tau}^{(i,j)} = \left\|\partial_k^4\left[a_{ij}(\cdot,\,\tilde v_h,\,\nabla\tilde v_h)\,w_{x_ix_j}\right]\right\|_{L^2(\tau)}, \tag{6.53}$$

$$Q_{\nu,k,\tau}^{(i,j)} = \left\|\partial_k^4\left[\frac{\partial a_{ij}}{\partial s_\nu}(\cdot,\,\tilde v_h,\,\nabla\tilde v_h)\,(\tilde v_h)_{x_ix_j}\,w_{x_\nu}\right]\right\|_{L^2(\tau)}, \tag{6.54}$$

$$R_{\nu,k,\tau} = \left\|\partial_k^4\left[\frac{\partial a}{\partial s_\nu}(\cdot,\,\tilde v_h,\,\nabla\tilde v_h)\,w_{x_\nu}\right]\right\|_{L^2(\tau)}. \tag{6.55}$$

Next we estimate the terms in (6.53)–(6.55). We fix $i, j, k = 1,\, 2$, $\tau \in \mathcal{T}_h$ and bound $P_{k,\tau}^{(ij)}$. Using Leibniz's formula, for any $x \in \tau$, we have

$$\partial_k^4\left[a_{ij}(x,\,\tilde v_h,\,\nabla\tilde v_h)\,w_{x_ix_j}\right] \;=\; \sum_{l=0}^{4} C_4^l\,[\partial_k^l\,a_{ij}(x,\,\tilde v_h,\,\nabla\tilde v_h)]\,\partial_k^{4-l}\,w_{x_ix_j}, \tag{6.56}$$

where $C_n^m = n!/(m!\,(n-m)!)$. Since $w$ is a bicubic polynomial on $\tau$, the term in (6.56) corresponding to $l = 0$ is 0. Using (6.53), (6.56), and the triangle inequality, we get

$$P_{k,\tau}^{(ij)} \;\le\; C \sum_{l=1}^{4} \left\|\partial_k^l\,a_{ij}(\cdot,\,\tilde v_h,\,\nabla\tilde v_h)\right\|_{C(\tau)} \left\|\partial_k^{4-l}\,w_{x_ix_j}\right\|_{L^2(\tau)}. \tag{6.57}$$

We apply Lemma 6.2 with $g = a_{ij}$, $1 \le l \le 4$, $\tilde\sigma = \tilde\mu$, and $w = \tilde v_h$. We note that (6.32) follows from condition (2.3a) for all $m$ and $\beta$ such that $m+|\beta| \le 4$. Therefore, using (6.33) and (2.4), we have

$$\left\|\partial_k^l\, a_{ij}(\cdot,\, \tilde v_h,\, \nabla \tilde v_h)\right\|_{C(\tau)} \;\le\; C\, h^{1-l}\, \bar\mu\,(\|\tilde v_h\|_{C^1(\tau)})\left[1 + \|\tilde v_h\|^l_{C^2(\tau)}\right], \qquad 1 \le l \le 4.$$

Applying (2.9) and (6.47), we obtain

$$\left\|\partial_k^l\, a_{ij}(\cdot,\, \tilde v_h,\, \nabla \tilde v_h)\right\|_{C(\tau)} \;\le\; C\, h^{1-l}\, \bar\mu\,(C\,\lambda_v)\left(1 + \lambda_v^4\right), \qquad 1 \le l \le 4. \tag{6.58}$$

Using (6.34) $(4-l)$-times, we get

$$\left\|\partial_k^{4-l}\, w_{x_i x_j}\right\|_{L^2(\tau)} \;\le\; C\, h^{l-4}\, \|w_{x_i x_j}\|_{L^2(\tau)}, \qquad 1 \le l \le 4. \tag{6.59}$$

Thus, from (6.57)–(6.59), we have

$$P^{(ij)}_{k,\tau} \;\le\; C\, h^{-3}\, \bar\mu(C\,\lambda_v)\left(1 + \lambda_v^4\right)\, \|w\|_{H^2(\tau)}, \qquad i, j, k = 1,\, 2, \qquad \tau \in \mathcal{T}_h. \tag{6.60}$$

We fix $i, j, k = 1,\, 2$, $0 \le \nu \le 2$, $\tau \in \mathcal{T}_h$ and bound $Q^{(ij)}_{\nu,k,\tau}$ given by (6.54). Applying Leibniz's formula twice and taking into account the fact that $w$ and $\tilde v_h$ are bicubic polynomials on $\tau$, we have

$$\begin{aligned}
&\partial_k^4 \left( \frac{\partial a_{ij}}{\partial s_\nu}(x,\, \tilde v_h,\, \nabla \tilde v_h)\, (\tilde v_h)_{x_i x_j}\, w_{x_\nu} \right) \\
&\quad = \sum_{n=1}^{4} \sum_{l=1}^{n} C_4^n\, C_n^l \left( \partial_k^l\, \frac{\partial a_{ij}}{\partial s_\nu}(x,\, \tilde v_h,\, \nabla \tilde v_h) \right) [\partial_k^{n-l}\, (\tilde v_h)_{x_i x_j}]\, \partial_k^{4-n}\, w_{x_\nu} \\
&\qquad + \sum_{n=1}^{3} C_4^n\, C_n^0\, \frac{\partial a_{ij}}{\partial s_\nu}(x,\, \tilde v_h,\, \nabla \tilde v_h)\, [\partial_k^n\, (\tilde v_h)_{x_i x_j}]\, \partial_k^{4-n}\, w_{x_\nu}.
\end{aligned} \tag{6.61}$$

Using (6.54), (6.61), and the triangle inequality, we obtain

$$Q^{(ij)}_{\nu,k,\tau} \;\le\; C\,(T_1 + T_2), \tag{6.62}$$

where

$$\begin{aligned}
T_1 &\equiv \sum_{n=1}^{4} \sum_{l=1}^{n} \left\| \partial_k^l\, \frac{\partial a_{ij}}{\partial s_\nu}(\cdot,\, \tilde v_h,\, \nabla \tilde v_h) \right\|_{C(\tau)} \left\|\partial_k^{n-l}\, (\tilde v_h)_{x_i x_j}\right\|_{C(\tau)} \left\|\partial_k^{4-n}\, w_{x_\nu}\right\|_{L^2(\tau)}, \\
T_2 &\equiv \sum_{n=1}^{3} \left\| \frac{\partial a_{ij}}{\partial s_\nu}(\cdot,\, \tilde v_h,\, \nabla \tilde v_h) \right\|_{C(\tau)} \left\|\partial_k^{n}\, (\tilde v_h)_{x_i x_j}\right\|_{C(\tau)} \left\|\partial_k^{4-n} w_{x_\nu}\right\|_{L^2(\tau)}.
\end{aligned}$$

First we bound $T_1$. We apply Lemma 6.2 with $1 \le l \le 4$, $g = \partial a_{ij}/\partial s_\nu$, $\tilde\sigma = \tilde\mu$, and $w = \tilde v_h$. We note that (6.32) follows from condition (2.3a) for all $m$ and $\beta$ such that $m + |\beta| \le 5$, $0 \le m \le 4$, and $|\beta| \ge 1$. Therefore, using (6.33), (2.4), (2.9), and (6.47), we have

$$\left\| \partial_k^l\, \frac{\partial a_{ij}}{\partial s_\nu}(\cdot,\, \tilde v_h,\, \nabla \tilde v_h) \right\|_{C(\tau)} \;\le\; C\, h^{1-l}\, \bar\mu(C\,\lambda_v)\left(1 + \lambda_v^4\right), \qquad 1 \le l \le 4. \tag{6.63}$$

For $1 \le n \le 4$ and $1 \le l \le n$, using (6.35) $(n-l)$-times and (2.9), we get

$$\left\|\partial_k^{n-l}\, (\tilde v_h)_{x_i x_j}\right\|_{C(\tau)} \;\le\; C\, h^{l-n}\, \|(\tilde v_h)_{x_i x_j}\|_{C(\tau)} \;\le\; C\, h^{l-n}\, \lambda_v. \tag{6.64}$$

We note that (6.64) also holds for $l = 0$. For $1 \le n \le 4$, using (6.34) $(4-n)$-times, we obtain

$$\left\|\partial_k^{4-n} w_{x_\nu}\right\|_{L^2(\tau)} \le C\,h^{n-4}\,\|w_{x_\nu}\|_{L^2(\tau)} \le C\,h^{n-4}\,\|w\|_{H^1(\tau)}. \tag{6.65}$$

Inequalities (6.63)–(6.65) imply that

$$T_1 \le C\,h^{-3}\,\bar\mu\,(C\,\lambda_v)\,\lambda_v\left(1+\lambda_v^4\right)\|w\|_{H^1(\tau)}. \tag{6.66}$$

Next we estimate $T_2$. Using (2.3a) for $m = 0$ and $|\beta| = 1$, (2.9), and (2.4), we get

$$\left\|\frac{\partial a_{ij}}{\partial s_\nu}(\cdot,\tilde v_h,\nabla\tilde v_h)\right\|_{C(\tau)} \le \tilde\mu\left(\|\tilde v_h\|_{C(\tau)},\|(\tilde v_h)_{x_1}\|_{C(\tau)},\|(\tilde v_h)_{x_2}\|_{C(\tau)}\right) \le \bar\mu(C\lambda_v). \tag{6.67}$$

For $1 \le n \le 3$, using (6.34) $(3-n)$-times, we have

$$\left\|\partial_k^{4-n} w_{x_\nu}\right\|_{L^2(\tau)} \le C\,h^{n-3}\,\|w_{x_\nu x_k}\|_{L^2(\tau)}. \tag{6.68}$$

Using (6.67), (6.68), and (6.64) for $l = 0$ and $1 \le n \le 3$, we obtain

$$T_2 \le C\,h^{-3}\,\bar\mu(C\,\lambda_v)\,\lambda_v\,\|w\|_{H^2(\tau)}. \tag{6.69}$$

Thus, from (6.62), (6.66), and (6.69), we have, for any $\tau \in \mathcal{T}_h$,

$$Q_{\nu,k,\tau}^{(ij)} \le C\,h^{-3}\,\bar\mu\,(C\,\lambda_v)\;\lambda_v\left(1+\lambda_v^4\right)\|w\|_{H^2(\tau)}, \qquad i,j,k = 1,2,\ \nu = 0,1,2. \tag{6.70}$$

Bounding $R_{\nu,k,\tau}$ of (6.55) in a way similar to that of $P_{k,\tau}^{(i,j)}$, for any $\tau \in \mathcal{T}_h$, we obtain

$$R_{\nu,k,\tau} \le C\,h^{-3}\,\bar\mu(C\,\lambda_v)\left(1+\lambda_v^4\right)\|w\|_{H^1(\tau)}, \qquad k = 1,2, \qquad \nu = 0,1,2. \tag{6.71}$$

Thus, from (6.52), (6.60), (6.70), and (6.71), we get

$$\left\|\partial_k^4 L_{\tilde v_h} w\right\|_{L^2(\tau)} \le C\,h^{-3}\,\bar\mu(C\,\lambda_v)\,(1+\lambda_v)\left(1+\lambda_v^4\right)\|w\|_{H^2(\tau)}. \tag{6.72}$$

Finally, (6.46) follows from (6.51), (6.72), (6.47), and $\mathcal{M}_h^0 \subset H^2(\Omega)$. ☐

**6.3. Bound on the inverse of the Fréchet derivative.** Using the first and the second auxiliary inequalities, we obtain a bound on $\|L_{h,\tilde v_h}^{-1}\|$ for $v \in H^4(\Omega)$.

LEMMA 6.4. *Under the conditions of Lemma 6.3, there exist positive constants $h_0 \le e^{-2}$ and $K_1$, both depending on $\|v\|_{H^4(\Omega)}$, such that, for any $h \in (0, h_0]$, $L_{h,\tilde v_h}^{-1}$ exists, and $\|L_{h,\tilde v_h}^{-1}\| \le K_1$.*

*Proof.* Substituting (6.36) into (6.1), we obtain

$$\|w\|_{H^2(\Omega)} \le (K_1/2)\,\|L_{h,\tilde v_h} w\|_h + h\,M\,\|w\|_{H^2(\Omega)}, \qquad w \in \mathcal{M}_h^0, \tag{6.73}$$

for some positive constants $K_1$ and $M$ which depend on $\lambda_v$ and $\nu$ but are independent of $h$. Setting $h_0 = \min\{e^{-2}, (2M)^{-1}\}$, we have

$$\|w\|_{H^2(\Omega)} \le K_1\,\|L_{h,\tilde v_h} w\|_h, \qquad w \in \mathcal{M}_h^0,\ h \in (0, h_0],$$

which implies that the linear operator $L_{h,\tilde v_h}$ has an inverse, and $\|L_{h,\tilde v_h}^{-1}\| \le K_1$. ☐

**7. Existence, uniqueness, and error estimates.** Let $L$ be given by (1.2), and $L_v^*$ be given by (6.27), (6.28), and (2.6). Let $u$ be a solution of (1.1).

THEOREM 7.1. *Suppose that* (2.3a) *holds for* $0 \le m \le 4$ *and all* $\beta$ *such that* $m + |\beta| \le 5$, *that* (2.3b) *holds for* $0 \le m \le 4$ *and all* $\beta$ *such that* $m + |\beta| \le 5$ *and* $|\beta| \ge 1$, *and that*

$$\tilde{\mu}\,(t_0,\, t_1,\, t_2) \;=\; \mu(t_0)\,(1 + t_1^q + t_2^q), \qquad q \ge 0, \tag{7.1}$$

*where* $\mu$ *is a continuous, positive, and nondecreasing function. Let* $u \in H^k(\Omega)$, $k = 4, 5$, *and suppose that operator* $L_u^*$ *is from* $H^{2,0}(\Omega)$ *onto* $L^2(\Omega)$ *and has a bounded inverse. Then there exist positive constants* $h_*$, $C_1$, $C_2$, *and* $M$ *that depend on* $\|u\|_{H^k(\Omega)}$ *such that, for any* $h \in (0,\, h_*]$ *and*

$$\rho \;=\; |\ln h|^{-2}(C_1 + C_2|\ln h|^q)^{-1}, \tag{7.2}$$

*the OSC problem* (3.3) *has a unique solution* $u_h \in B_h(\tilde{u}_h, \rho)$, *and*

$$\|u - u_h\|_{H^{6-k}(\Omega)} \;\le\; M\, h^{k-2}. \tag{7.3}$$

*Proof.* Lemma 4.1 with $v = u$ implies

$$\|Lu - L_h \tilde{u}_h\|_h \;\le\; K_3\, h^{k-2}, \tag{7.4}$$

where $K_3 \ge 0$ is independent of $h$ but depends on $\|u\|_{H^k(\Omega)}$. From Lemma 5.1 with $v = u$ and $\rho_0 = 1$ and (7.1), it follows that, for $h \in (0,\, e^{-2}]$, operator $L_h$ has a Fréchet derivative $L_{h,y}$ for any $y \in \mathcal{M}_h^0$, and

$$\|L_{h,z} - L_{h,y}\| \;\le\; K_2\,(h)\,\|z - y\|_{H^2(\Omega)}, \qquad y, z \in B_h(\tilde{u}_h, 1), \tag{7.5}$$

where

$$K_2\,(h) \;\equiv\; K_2\,(h, 1, u) \;=\; (C + \gamma)\,\mu(\gamma)\,(\ln^2\, h)(1 + 2\,(\gamma\,|\ln\, h|)^q), \tag{7.6}$$

and $\gamma = C\,(\|u\|_{H^4(\Omega)} + 1)$. We conclude from Lemma 6.4 with $v = u$ that there exist positive constants $h_0 \le e^{-2}$ and $K_1$, both depending on $\|u\|_{H^4(\Omega)}$, such that, for any $h \in (0, h_0]$, $L_{h,\tilde{u}_h}^{-1}$ exists, and (3.4) holds.

Since $\mu(\gamma) > 0$ and $q \ge 0$, it follows from (7.6) that $K_2(h) \to \infty$ as $h \to 0$. Hence, there exists $h_1 \in (0,\, e^{-2}]$ such that

$$[2\,K_1\,K_2(h)]^{-1} \;\le\; 1, \qquad h \in (0,\, h_1]. \tag{7.7}$$

Next we prove that there exists $h_2 \in (0,\, e^{-2}]$ such that

$$K_1\,K_3\,h^{k-2} \;\le\; [4\,K_1\,K_2(h)]^{-1}, \qquad h \in (0, h_2]. \tag{7.8}$$

We take any $h \in (0,\, e^{-2}]$ and set $t = \gamma\,|\ln\, h|$. Using $h^{k-2} \;=\; e^{-(k-2)t/\gamma}$, (7.6), and $\ln^2\, h = (t/\gamma)^2$, we see that the inequality in (7.8) is equivalent to

$$t^2 + 2\,t^{2+q} \;\le\; \gamma^2\left(4\,K_1^2\;K_3\,(C + \gamma)\mu\,(\gamma)\right)^{-1}\, e^{(k-2)t/\gamma}. \tag{7.9}$$

Clearly there exists $t_* > 0$ such that (7.9) holds for all $t \ge t_*$. Therefore we conclude that (7.8) holds with $h_2 = \min\{e^{-t_*/\gamma},\, e^{-2}\}$.

We set $h_* = \min\{h_0,\, h_1,\, h_2\}$ and

$$\rho \;=\; [2\,K_1\,K_2(h)]^{-1}, \qquad h \in (0,\, h_*]. \tag{7.10}$$

Using (7.10) and (7.6), for $h \in (0,\, h_*]$, it is easy to obtain (7.2) with $C_1 = 2K_1(C + \gamma)\mu(\gamma)$ and $C_2 = 2\gamma^q C_1$.

We fix any $h \in (0,\, h_*]$ and consider $\rho$ given by (7.10). We have proved that the operator $L_h$ has a Fréchet derivative $L_{h,y}$ for any $y \in \mathcal{M}_h^0$, $L_{h,\tilde u_h}^{-1}$ exists and (3.4) holds. It follows from (7.10) and (7.7) that $\rho \le 1$. Hence, (7.5) implies (3.5), and (7.4) implies (3.6) with $p = k-2$. Finally, (7.10) and (7.8) imply (3.7) and (3.8) with $p = k-2$. Thus it follows from Theorem 3.1 that (3.3) has a unique solution $u_h \in B_h(\tilde u_h, \rho)$. Moreover, using the triangle inequality, (2.10) and (3.9) with $p = k-2$, we obtain

$$\|u - u_h\|_{H^{6-k}(\Omega)} \;\le\; \|u - \tilde u_h\|_{H^{6-k}(\Omega)} + \|\tilde u_h - u_h\|_{H^2(\Omega)} \;\le\; (C\,\|u\|_{H^4(\Omega)} + 2K_1K_3)\; h^{k-2},$$

which gives (7.3) with $M$ depending on $\|u\|_{H^k(\Omega)}$. ∎

In (7.3), the $H^2$ error estimate is optimal, whereas the $H^1$ error estimate has optimal order.

Corollary 7.1. *Under the conditions of Theorem* 7.1, *there exists* $h_{**} > 0$ *such that, for any* $h \in (0,\, h_{**}]$, $L_{h,\tilde u_h}^{-1}$ *exists and* (3.4) *holds with* $K_1 > 0$ *independent of* $h$, $L_h$ *has a Fréchet derivative* $L_{h,y}$ *for any* $y \in \mathcal{M}_h^0$, (3.5) *holds with* $\rho = [2\,K_1\,K_2(h)]^{-1}$ *and* $K_2 = K_2(h)$ *given by* (7.6), *and the OSC problem* (3.3) *has a unique solution* $u_h \in B_h(\tilde u_h, \rho/2)$.

## 8. Newton's method.

Newton's method for the iterative solution of (3.3) is formulated as follows. Given $u_0 \in \mathcal{M}_h^0$, compute $u_{k+1}$ from

$$L_{h,u_k}(u_{k+1} - u_k) \;=\; -(L_h u_k - f_h), \qquad k = 0,\, 1,\, \ldots\, . \tag{8.1}$$

Theorem 8.1. *Suppose that the conditions of Theorem* 7.1 *hold, and let* $h_{**}$, $K_1$, $K_2(h)$, $\rho$, *and* $u_h$ *be as in Corollary* 7.1. *If* $h \in (0, h_{**}]$ *and* $u_0 \in B_h(u_h,\, \rho/2)$, *then Newton's method* (8.1) *generates a sequence* $\{u_k\}_{k=0}^{\infty} \subset B_h(u_h,\, \rho/2)$ *and*

$$\|u_k - u_h\|_{H^2(\Omega)} \;\le\; \rho\,(1/2)^{2^k-1}, \qquad k = 1, 2, \ldots\, . \tag{8.2}$$

*Proof.* We set $r = \rho/2$, $\kappa_1 = 2K_1$, $\kappa_2 = K_2(h)$, and take any $y \in B_h(u_h, r)$. According to Corollary 7.1, there exists $L_{h,\tilde u_h}^{-1}$, and (3.4) holds. Using (3.5), the triangle inequality, $u_h \in B_h(\tilde u_h,\, \rho/2)$, $y \in B_h(u_h, \rho/2)$, and $\rho = [2\,K_1\,K_2(h)]^{-1}$, we obtain

$$\begin{aligned} \|L_{h,\tilde u_h} - L_{h,y}\| &\le K_2(h)(\|\tilde u_h - u_h\|_{H^2(\Omega)} + \|u_h - y\|_{H^2(\Omega)}) \\ &\le K_2(h)\rho = 1/(2K_1), \end{aligned} \tag{8.3}$$

which implies $\|L_{h,\tilde u_h} - L_{h,y}\| < \|L_{h,\tilde u_h}^{-1}\|^{-1}$ by (3.4). Applying the generalization of Banach's theorem [13, Ch. V, section 4, Theorem 4], we conclude that there exists $L_{h,y}^{-1}$ and

$$\|L_{h,y}^{-1}\| \;\le\; \|L_{h,\tilde u_h}^{-1}\|/(1 - \|L_{h,\tilde u_h}^{-1}\|\,\|L_{h,y} - L_{h,\tilde u_h}\|),$$

which, along with (3.4) and (8.3), implies $\|L_{h,y}^{-1}\| \le \kappa_1$ for all $y \in B_h(u_h, r)$.

Since $u_h \in B_h(\tilde{u}_h, \rho/2)$, we have $B_h(u_h, \rho/2) \subset B_h(\tilde{u}_h, \rho)$. By Corollary 7.1, $L_h$ has a Fréchet derivative $L_{h,y}$ at any $y \in \mathcal{M}_h^0$, (3.5) holds, and

$$\|L_{h,z} - L_{h,y}\| \le \kappa_2 \|z - y\|_{H^2(\Omega)}, \qquad y, z \in B_h(u_h, r).$$

Using $\rho = [2K_1 K_2(h)]^{-1} = (\kappa_1 \kappa_2)^{-1}$, we obtain $r = \rho/2 < (\kappa_1 \kappa_2)^{-1}$. Thus, using Theorem 9.1 in [1] which is a modification of Theorem 4 in [22, p. 359], we conclude that Newton's method (8.1) generates the sequence $\{u_k\}_{k=0}^{\infty} \subset B_h(u_h, \rho/2)$, and (8.2) holds. ∎

It follows from (7.10) and the statement following the proof of Lemma 5.1 that $\rho$ is independent of $h$ if the coefficients $a_{ij}$ of the differential operator $L$ do not depend on $\nabla u$.

**9. Conclusions.** We have shown that the nonlinear OSC problem (3.3) locally has a unique solution $u_h$ with the same convergence properties as the solution of the corresponding linear OSC problem considered in [2]. That is, for both the linear and the nonlinear OSC solutions, we have an optimal $H^2$ error estimate, and an optimal order $H^1$ error estimate if $u \in H^5(\Omega)$. We have also shown that Newton's method converges quadratically provided that the initial approximation lies sufficiently close to the OSC solution.

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