Multilevel Preconditioners for a Quadrature Galerkin Solution of a Biharmonic Problem

Rakhim Aitbayev
Department of Mathematics, New Mexico Institute of Mining and Technology, Socorro, New Mexico 87801

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Efficient multilevel preconditioners are developed and analyzed for the quadrature finite element Galerkin approximation of the biharmonic Dirichlet problem. The quadrature scheme is formulated using the Bogner–Fox–Schmit rectangular element and the product two-point Gaussian quadrature. The proposed additive and multiplicative preconditioners are uniformly spectrally equivalent to the operator of the quadrature scheme. The preconditioners are implemented by optimal algorithms, and they are used to accelerate convergence of the preconditioned conjugate gradient method. Numerical results are presented demonstrating efficiency of the preconditioners. Numer Methods Partial Differential Eq 22: 000–000, 2006

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I. INTRODUCTION

The purpose of this article is to develop and analyze multilevel preconditioners for the quadrature finite element Galerkin approximation of a biharmonic problem. Efficient algorithms for the solution of biharmonic problems are important in several application areas; for example, in plane elasticity to model vertical displacements of thin plates and in fluid mechanics to model the incompressible flow with a small Reynolds number in a two-dimensional domain using the stream-function formulation of the Stokes problem. Some iterative numerical methods for solving the Navier–Stokes equation for the incompressible flow with a large Reynolds number also require efficient solution of biharmonic problems.

Let $\Omega \subset R^2$ be an open bounded rectangular polygonal region with the boundary $\partial\Omega$, and let the edges of $\partial\Omega$ be parallel to the coordinate axes. The Dirichlet boundary value problem for the biharmonic equation is formulated as follows:

$$\Delta^2 u = f \quad \text{in } \Omega \qquad \text{and} \qquad u = \partial_n u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

Correspondence to: Rakhim Aitbayev, Department of Mathematics, New Mexico Institute of Mining and Technology, Socorro, NM 87801 (email: aitbayev@nmt.edu)

where Δ is the Laplace operator and ∂_n denotes the outer normal derivative. Let $H_0^2(\Omega)$ be the standard Sobolev space, which is the closure in the H^2-norm of C^∞ functions with compact support in Ω. We assume that f is a continuous function on $\overline{\Omega}$ and consider the following variational form of problem (1.1): find $u \in H_0^2(\Omega)$ such that

$$a(u, v) = (f, v) \qquad \text{for all } v \in H_0^2(\Omega), \tag{1.2}$$

where a bilinear form $a(\cdot, \cdot)$ and a linear functional $(f, \cdot)$ are, respectively, defined by

$$a(w, v) = \int_\Omega \Delta w \Delta v \, dx \qquad \text{and} \qquad (f, v) = \int_\Omega f v \, dx. \tag{1.3}$$

Although multilevel methods for the solution of finite element Galerkin approximations of the biharmonic problem (1.2) have been extensively studied, applications of these methods for solving quadrature finite element Galerkin problems have received little attention. We discretize the variational problem (1.2) by a direct approach using the conforming Bogner–Fox–Schmit rectangle (see sections 2.2 and 4.1 in [1]). The discrete solution is a C^1 piecewise bicubic function with 16 degrees of freedom approximating the values of the solution, the first-order and the mixed second-order derivatives at the vertices of a rectangular element. The integrals in the stiffness matrix and in the load vector are approximated using the product two-point Gaussian quadrature. The spectral condition number of the corresponding stiffness matrix is of order $O(h^{-4})$, where h is the triangulation diameter (see Remark 3 in section 5.5 of [2]). An efficient solution of the corresponding linear system by an iterative method requires a preconditioner that can both significantly reduce the condition number of the preconditioned system and be implemented by a low computational cost algorithm.

Because of their efficiency and generality, multilevel and domain decomposition methods have become the state-of-art techniques for developing preconditioners for complex application problems involving partial differential equations [3,4]. In this article, we study multilevel preconditioning of the operator of the quadrature Galerkin problem and propose optimal cost additive and multiplicative preconditioning algorithms that are related to the multigrid V-cycle algorithms with Jacobi and Gauss–Seidel smoothers, respectively. Using an approach described in [5], we prove that the preconditioners are uniformly spectrally equivalent to the operator of the quadrature scheme. This result, in particular, implies that the number of iterations of the preconditioned conjugate gradient algorithm to compute the solution within some tolerance is bounded from above uniformly with respect to the discretization parameter h. Our numerical results demonstrate that both preconditioners perform well on test problems and that the performance of the multiplicative preconditioner is excellent.

The methods and results in this article are closely related to those in [5–10]. The quadrature Galerkin problem of the present article is studied in [6] for existence, uniqueness, and convergence of a solution. The optimal order error estimates in Sobolev norms are obtained under a sufficient regularity assumption using the fact that the quadrature Galerkin problem is equivalent to an orthogonal spline collocation scheme, which is related to that in [9]. Multilevel preconditioners are developed and analyzed in [7], which are similar to those in this work, for the solution of a Hermite orthogonal spline collocation discretization of a second-order boundary value problem on a rectangle with a non–self-adjoint, indefinite operator. Multilevel preconditioners are proposed in [8] for the solution of a second-order self-adjoint boundary value problem using the orthogonal spline collocation method with Hermite bicubics. Multilevel preconditioners for the finite element Galerkin solution of the biharmonic problem (1.2) using conforming and nonconforming rectangular elements are studied in [5]. Without assuming full regularity, optimal estimates on condition numbers are obtained using the equivalence between a special multilevel norm and the Sobolev H^2-norm. Similar results are proved in [10] using an abstract framework of Schwarz methods and the full elliptic regularity assumption.

Let us mention some other important, related works. Convergence analysis of multigrid methods is presented in [11] for solving second- and fourth-order elliptic boundary value problems using nonconforming finite elements, and uniform condition number estimates are obtained without the full elliptic regularity assumption. A multigrid preconditioning scheme for the mixed method of Ciarlet–Raviart is presented and analyzed in [12]. A conjugate-gradient method and a multigrid algorithm for the Morley finite element approximation of the biharmonic problem was studied in [13]. A fast Schur complement algorithm is developed in [14] for computing the piecewise Hermite bicubic orthogonal spline collocation solution of a biharmonic problem on a rectangle. A Gaussian quadrature Petrov–Galerkin scheme is analyzed in [15] for the solution of second-order boundary value problems.

The outline of the rest of the article is as follows. In section II, the quadrature Galerkin problem is formulated and existence, uniqueness, and convergence results are presented. The multilevel preconditioners are introduced in section III, and uniform spectral equivalence results are proved in section IV. A matrix-vector form of the problem is obtained in section V, and the algorithms implementing interpolation, restriction, and the multilevel preconditioners are presented in section VI. Numerical results are discussed in section VII, and the conclusion is given in section VIII.

II. QUADRATURE SCHEME

In this section, we introduce notation, formulate our quadrature finite element Galerkin problem, and present existence, uniqueness, and convergence results. For an open set $D \subset R^2$ and $x = (x_1, x_2)$, let $L^2(D)$ be the Hilbert space of square integrable functions with the norm $\|v\|_{L^2(D)} = \left(\int_D |v|^2 dx\right)^{1/2}$. Let $H^m(D)$ be the standard Sobolev space with the norm

$$\|v\|_{H^m(D)} = \left(\sum_{|\alpha|\le m} \|\partial^\alpha v\|^2_{L^2(D)}\right)^{1/2},$$

where $m \ge 0$ is an integer, $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2$, and $\partial^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1} \partial x_2^{\alpha_2})$.

Our multilevel preconditioning algorithms are formulated using a set of uniform nested triangulations of Ω constructed as follows. We assume that the edges of the rectangular polygon $\partial\Omega$ are parallel to the coordinate axes, and that domain Ω is equipped with a uniform coarse triangulation $\mathcal{T}_0$ consisting of congruent rectangles with the area A_0 satisfying the standard requirements of the finite element theory (see properties $(\mathcal{T}_h 1)$–$(\mathcal{T}_h 5)$ of the assumption (FEM 1) in sections 2.1 and 2.2 in [1]). For $l = 1, \dots, L$, triangulation $\mathcal{T}_l$ is obtained from triangulation T_{l-1} by dyadic partitioning as follows: every rectangular element of triangulation $\mathcal{T}_{l-1}$ is divided into four elements of triangulation $\mathcal{T}_l$ by connecting the midpoints of the opposite edges of the element. Let $h_l = 2^{-l}\sqrt{A_0}$ be the diameter of the triangulation $\mathcal{T}_l$, $l = 1, \dots, L$. In Figs. 1 and 2, we present examples of triangulations $\mathcal{T}_0$ and $\mathcal{T}_1$, respectively.

Based on the triangulations $\{\mathcal{T}_l\}_{l=1}^L$, we define a corresponding set of conforming finite element spaces of Bogner–Fox–Schmit elements:

$$V_l = \left\{v \in C^1(\overline{\Omega}) : v|_K \in Q_3|_K, \ \forall K \in \mathcal{T}_l; \ \ v = \partial_n v = 0 \ \text{ on } \ \partial\Omega\right\}, \qquad l = 1, \dots, L,$$

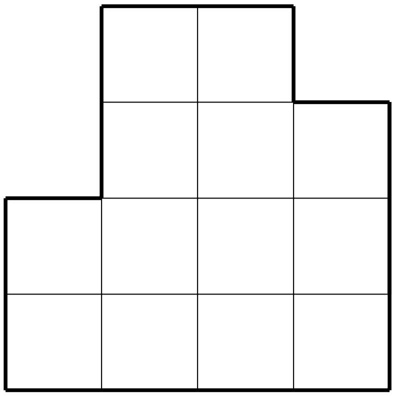

FIG. 1. Triangulation $\mathcal{T}_0$.

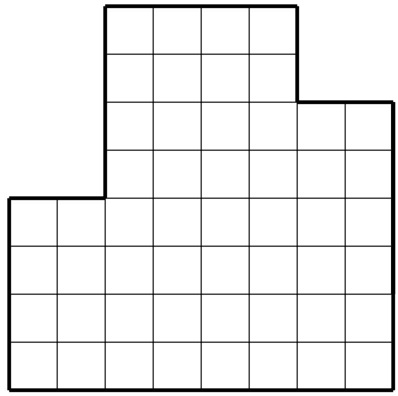

FIG. 2. Triangulation $\mathcal{T}_1$.

where Q_3 is the space of bicubic polynomials (see section 2.2 in [1]). We note that

$$V_1 \subset V_2 \subset \cdots \subset V_L \subset H_0^2(\Omega), \tag{2.1}$$

where the last inclusion is proved in Theorem 2.2.15 in [1]. The dimension N_l of the finite element space V_l equals 4 times the number of interior nodes in triangulation $\mathcal{T}_l$, and $v = v_{x_1} = v_{x_2} = v_{x_1 x_2} = 0$ on the boundary $\partial\Omega$ for any $v \in V_l, l = 1, \ldots, L$. In what follows, let $h = h_L$, $N = N_L$, $\mathcal{T}_h = \mathcal{T}_L$, and $V_h = V_L$, and let C, C_1, and C_2 denote generic positive constants independent of h.

A finite element Galerkin problem approximating the variational problem (1.2) is formulated as follows: find $U_h \in V_h$ such that

$$a(U_h, v) = (f, v) \qquad \text{for all } v \in V_h, \tag{2.2}$$

where the forms $a(\cdot,\cdot)$ and $(f,\cdot)$ are defined in (1.3). Using the Riesz Representation Theorem, it is easy to see that the finite element problem (2.2) has a unique solution. If $f \in H^{-2+2s}(\Omega)$, then

$$\|u - U_h\|_{H^2(\Omega)} \leq Ch^{2s}\|f\|_{H^{-2+2s}(\Omega)}, \qquad 0 \leq s < 1/3,$$

where u is the solution of problem (1.2), and $H^{-2+2s}(\Omega)$ is the dual of a fractional order Sobolev space (Theorem 3.1 in [6]).

To obtain a quadrature scheme for the finite element problem (2.2), we approximate the forms $a(\cdot,\cdot)$ and $(f,\cdot)$ using the product two-point Gaussian quadrature. For any interval $I = (a, b)$, let

$$\mathcal{G}_I = \left\{a + (b-a)(3-\sqrt{3})/6,\ a + (b-a)(3+\sqrt{3})/6\right\}$$

be the set of Gauss points in I, and, for any function v defined on $\mathcal{G}_I$, let

$$\sum_{\mathcal{G}_I} v = \frac{b-a}{2}\sum_{\xi\in\mathcal{G}_I} v(\xi)$$

be the two-point Gaussian quadrature on I. The product two-point Gaussian quadrature on a rectangle $K = I_1 \times I_2$ is defined by

$$\sum_{\mathcal{G}_K} v = \sum_{\mathcal{G}_{I_1}} \sum_{\mathcal{G}_{I_2}} v = \frac{|K|}{4} \sum_{\xi \in \mathcal{G}_K} v(\xi), \tag{2.3}$$

where $\mathcal{G}_K = \mathcal{G}_{I_1} \times \mathcal{G}_{I_2}$ is the set of Gauss points in K, $|K|$ is the area of K, and v is defined on $\mathcal{G}_K$. For any functions v and w defined on $\mathcal{T}_h$, let

$$(v, w)_h = \sum_{K \in \mathcal{T}_h} \sum_{\mathcal{G}_K} vw \qquad \text{and} \qquad \|v\|_h = \sqrt{(v, v)_h}, \tag{2.4}$$

and let

$$a_h(v, w) = (\Delta v, \Delta w)_h, \qquad v, w \in V_h, \tag{2.5}$$

be a symmetric bilinear form approximating $a(\cdot,\cdot)$ in (1.3). Our quadrature finite element Galerkin problem is formulated as follows: find $u_h \in V_h$ such that

$$a_h(u_h, w) = (f, w)_h \qquad \text{for all } w \in V_h. \tag{2.6}$$

In what follows, we refer to problem (2.6) and its solution as the quadrature problem and a quadrature solution, respectively.

The energy norm of the approximate bilinear form $a_h(\cdot,\cdot)$ is uniformly equivalent to the H^2-norm, that is,

$$C\|v\|^2_{H^2(\Omega)} \le a_h(v, v) \le C_2\|v\|^2_{H^2(\Omega)}, \qquad v \in V_h, \tag{2.7}$$

(Lemma 5.3 in [6]). Using the fact that $\sqrt{a(\cdot,\cdot)}$ is equivalent to the H^2-norm on $H_0^2(\Omega)$ (see (1.2.8) in [1]), (2.7), and $V_h \subset H_0^2(\Omega)$, we obtain the relations

$$C_1 a(v, v) \le a_h(v, v) \le C_2 a(v, v), \qquad v \in V_h. \tag{2.8}$$

It follows from (2.7) that $a_h(\cdot,\cdot)$ is an inner product on V_h, and, hence, the quadrature problem (2.6) has a unique solution $u_h \in V_h$ (Theorem 5.4 in [6]). We note that, if $u \in H^{8-k}(\Omega)$, then

$$\|u - u_h\|_{H^k(\Omega)} \le Ch^{4-k}\|u\|_{H^{8-k}(\Omega)}, \qquad k = 1, 2, \tag{2.9}$$

where u is the solution of the variational problem (1.2) (Theorem 8.2 in [6]).

III. MULTILEVEL PRECONDITIONERS

In this section, we present an operator form of the quadrature problem (2.6) and define our multilevel preconditioners. First, we establish the following result.

Lemma 3.1. *The bilinear form* $(\cdot,\cdot)_h$ *of* (2.4) *is an inner product on* V_h.

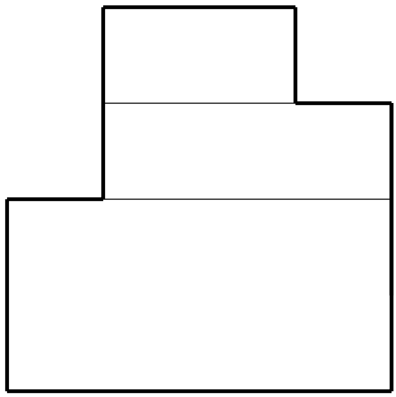

FIG. 3. Partition in R_V^n rectangles.

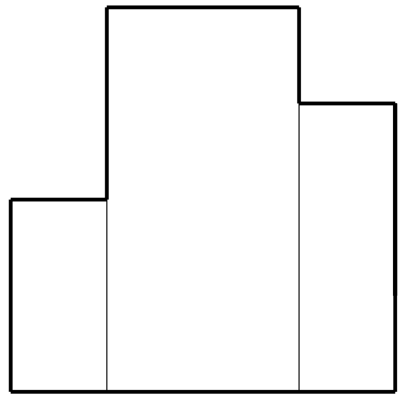

FIG. 4. Partition in R_H^n rectangles.

Proof. It suffices to prove the inequality

$$(v, v)_h \geq C\|v\|^2_{L^2(\Omega)}, \qquad v \in V_h, \tag{3.1}$$

since, if $v \in V_h$ and $(v, v)_h = 0$, then (3.1) implies $v = 0$. Domain $\overline{\Omega}$ can be decomposed in the following two unions of rectangles:

$$\overline{\Omega} = \bigcup_{n=1}^{N_V} \overline{R}_V^n = \bigcup_{n=1}^{N_H} \overline{R}_H^n, \tag{3.2}$$

where the sets $\{R_V^n\}_{n=1}^{N_V}$ and $\{R_H^n\}_{n=1}^{N_H}$ consist of open disjoint rectangles whose, respectively, vertical and horizontal edges are parts of the boundary $\partial\Omega$ (see Figs. 3 and 4).

Triangulation $\mathcal{T}_h$ determines the partition $\pi_V^n = \pi_{V,1}^n \times \pi_{V,2}^n$ of the rectangle R_V^n for $n = 1, \dots, N_V$, where one-dimensional partitions $\pi_{V,i}^n$, $i = 1, 2$, consist of subintervals. Similarly, for $n = 1, \dots, N_H$, $\pi_H^n = \pi_{H,1}^n \times \pi_{H,2}^n$ is the partition of the rectangle R_H^n determined by $\mathcal{T}_h$. It follows from (3.2) that, for any set of numbers $\{s_K\}_{K \in \mathcal{T}_h}$,

$$\sum_{n=1}^{N_V} \sum_{I_1 \in \pi_{V,1}^n} \sum_{I_2 \in \pi_{V,2}^n} s_{I_1 \times I_2} = \sum_{K \in \mathcal{T}_h} s_K = \sum_{n=1}^{N_H} \sum_{I_1 \in \pi_{H,1}^n} \sum_{I_2 \in \pi_{H,2}^n} s_{I_1 \times I_2}. \tag{3.3}$$

We take any $v \in V_h$ and note that

$$\sum_{I_1 \in \pi_{V,1}^n} \sum_{\mathcal{G}_{I_1}} v^2(\cdot, x_2) \geq C \sum_{I_1 \in \pi_{V,1}^n} \int_{I_1} v^2(x_1, x_2)\, dx_1, \qquad x_2 \in I_2^n, \quad n = 1, \dots, N_V, \tag{3.4}$$

$$\sum_{I_2 \in \pi_{H,2}^n} \sum_{\mathcal{G}_{I_2}} v^2(x_1, \cdot) \geq C \sum_{I_2 \in \pi_{H,2}^n} \int_{I_2} v^2(x_1, x_2)\, dx_2, \qquad x_1 \in I_1^n, \quad n = 1, \dots, N_H \tag{3.5}$$

(see (2.6) in [16]). Using (2.4), (3.3), (3.4), and (3.5), we obtain

$$\begin{aligned}
(v, v)_h &= \sum_{K\in\mathcal{T}_h} \sum_{\mathcal{G}_K} v^2 = \sum_{n=1}^{N_V} \sum_{I_2\in\pi_{V,2}^n} \sum_{\mathcal{G}_{I_2}} \sum_{I_1\in\pi_{V,1}^n} \sum_{\mathcal{G}_{I_1}} v^2 \\
&\geq C \sum_{n=1}^{N_V} \sum_{I_2\in\pi_{V,2}^n} \sum_{\mathcal{G}_{I_2}} \sum_{I_1\in\pi_{V,1}^n} \int_{I_1} v^2(x_1, \cdot)dx_1 \\
&= C \sum_{n=1}^{N_H} \sum_{I_1\in\pi_{H,1}^n} \int_{I_1} \sum_{I_2\in\pi_{H,2}^n} \sum_{\mathcal{G}_{I_2}} v^2(x_1, \cdot)dx_1 \\
&\geq C \sum_{n=1}^{N_H} \sum_{I_1\in\pi_{H,1}^n} \sum_{I_2\in\pi_{H,2}^n} \int_{I_1\times I_2} v^2 dx = C\|v\|_{L^2(\Omega)}^2,
\end{aligned}$$

which is (3.1). ■

Since V_h is a Hilbert space with the inner product $(\cdot,\cdot)_h$, we define an operator L_h from V_h to V_h by

$$(L_h v, w)_h = a_h(v, w) \qquad \text{for all } w \in V_h, \tag{3.6}$$

and a vector $f_h \in V_h$ by

$$(f_h, w)_h = (f, w)_h \qquad \text{for all } w \in V_h, \tag{3.7}$$

and consider the following operator form the quadrature problem (2.6): find $u_h \in V_h$ such that

$$L_h u_h = f_h. \tag{3.8}$$

It follows from (3.6) and the first inequality in (2.7) that operator L_h is self-adjoint, positive definite on V_h in the $(\cdot,\cdot)_h$ inner product; hence, the equation (3.8) can be solved by the preconditioned conjugate gradient method.

We now define our multilevel preconditioners. Let $\{\psi_i^l\}_{i=1}^{N_l}$ be the standard finite element basis of V_l, and let

$$V_i^l = \operatorname{span}(\psi_i^l), \qquad i = 1, \ldots, N_l, \quad l = 1, \ldots, L, \tag{3.9}$$

be one-dimensional subspaces of V_l. Let T_i^l be a projection operator from V_h to V_i^l defined by

$$a_h(T_i^l v, w) = a_h(v, w) \qquad \text{for all } w \in V_i^l, \quad i = 1, \ldots, N_l, \quad l = 1, \ldots, L. \tag{3.10}$$

The multilevel additive and multiplicative preconditioners are defined by

$$B_A = L_h T_A^{-1}, \tag{3.11}$$

$$B_M = L_h T_M^{-1}, \tag{3.12}$$

respectively, where

$$T_A = \sum_{l=1}^{L} \sum_{i=1}^{N_l} T_i^l, \tag{3.13}$$

$$T_M = I_h - \left[\prod_{l=L}^{1} \prod_{i=1}^{N_l} \left(I_h - T_i^l\right) \right] \left[\prod_{l=1}^{L} \prod_{i=N_l}^{1} \left(I_h - T_i^l\right) \right], \tag{3.14}$$

and I_h is the identity operator on V_h (see Examples 1 and 2 in section 5.1 in [17]).

IV. UNIFORM SPECTRAL EQUIVALENCE

In this section, we prove that our multilevel preconditioners are uniformly spectrally equivalent to the operator of the quadrature problem. These results imply that the condition numbers of the preconditioned systems are uniformly bounded from above. First, we introduce additional notation. Based on (2.1) and (3.9), we consider the following two sum representations of V_h:

$$\sum_{l=1}^{L} V_l = V_h \qquad \text{and} \qquad \sum_{l=1}^{L} \sum_{i=1}^{N_l} V_i^l = V_h.$$

For any $v \in V_h$, let

$$\mathcal{V}_1(v) = \left\{ \{v_l\}_{l=1}^{L} : \sum_{l=1}^{L} v_l = v,\ v_l \in V_l,\ l = 1, \ldots, L \right\},$$

$$\mathcal{V}_2(v) = \left\{ \{v_i^l\}_{i,l=1}^{N_l,L} : \sum_{l=1}^{L} \sum_{i=1}^{N_l} v_i^l = v,\ v_i^l \in V_i^l,\ i = 1 \ldots, N_l,\ l = 1, \ldots, L \right\},$$

be sets of sum representations for v. For any $v \in V_h$, let

$$\|v\|_{*,h} = \left(\inf_{\mathcal{V}_1(v)} \sum_{l=1}^{L} (h_l)^{-4} \|v_l\|_{L^2(\Omega)}^2 \right)^{1/2}, \tag{4.1}$$

$$\|v\|_{\Sigma} = \left(\inf_{\mathcal{V}_2(v)} \sum_{l=1}^{L} \sum_{i=1}^{N_l} a(v_i^l, v_i^l) \right)^{1/2}, \tag{4.2}$$

$$\|v\|_{\Sigma,h} = \left(\inf_{\mathcal{V}_2(v)} \sum_{l=1}^{L} \sum_{i=1}^{N_l} a_h(v_i^l, v_i^l) \right)^{1/2}, \tag{4.3}$$

where $\inf_{\mathcal{V}_1(v)}$ and $\inf_{\mathcal{V}_2(v)}$ denote the infimums with respect to all sum representations $\{v_l\}_{l=1}^{L} \in \mathcal{V}_1(v)$ and $\{v_i^l\}_{l,i=1}^{L,N_l} \in \mathcal{V}_2(v)$, respectively.

Theorem 4.1. *Linear operator B_A of* (3.11) *is self-adjoint, positive definite on V_h in the $(\cdot,\cdot)_h$ inner product, and*

$$C_1 (B_A v, v)_h \le (L_h v, v)_h \le C_2 (B_A v, v)_h, \qquad v \in V_h. \tag{4.4}$$

Proof. It is stated in Corollary 2.1 in [5] that

$$C_1\|v\|_{H^2(\Omega)} \le \|v\|_{*,h} \le C_2\|v\|_{H^2(\Omega)}, \qquad v \in V_h,$$

where the functional $\|\cdot\|_{*,h}$ is defined by (4.1). Using this result, it is proved in Lemma 3.5 of [7] that

$$C_1\|v\|_{H^2(\Omega)} \le \|v\|_{\Sigma} \le C_2\|v\|_{H^2(\Omega)}, \qquad v \in V_h, \tag{4.5}$$

where $\|\cdot\|_{\Sigma}$ is defined by (4.2). From (4.5), using (2.8), (2.7), and (4.3), we obtain the following important relations:

$$C_1 a_h(v,v) \le \|v\|^2_{\Sigma,h} \le C_2 a_h(v,v), \qquad v \in V_h. \tag{4.6}$$

We note that the second inequality in (4.6) is one of the key assumptions in the theory of abstract Schwarz methods (Assumption 1, section 5.2, [17]).

Lemma 2.5 in [3] implies that operator T_A is positive definite in the $a_h(\cdot,\cdot)$ inner product and

$$a_h(T_A^{-1}v, v) = \|v\|^2_{\Sigma,h}, \qquad v \in V_h,$$

which, by (3.6) and $B_A = L_h T_A^{-1}$, gives

$$(B_A v, v)_h = \|v\|^2_{\Sigma,h}, \qquad v \in V_h. \tag{4.7}$$

From (4.6), using (4.7) and (3.6), we obtain (4.4).

Since L_h is positive definite in the $(\cdot,\cdot)_h$-inner product, the second inequality in (4.4) implies that operator B_A is positive definite in the $(\cdot,\cdot)_h$-inner product. Operator T_A is self-adjoint in the $a_h(\cdot,\cdot)$ inner product since $\{T_i^l\}$ are self-adjoint (Lemma 2, section 5.2, [17]). Hence, T_A^{-1} is self-adjoint in the $a_h(\cdot,\cdot)$ inner product, and B_A is self-adjoint in the $(\cdot,\cdot)_h$-inner product. ■

A proof of a similar result for the multiplicative preconditioner requires so-called strengthened Cauchy–Schwarz inequalities (Assumption 2, section 5.2, [17]).

Lemma 4.2. *Let the set of all basis functions* $\{\{\psi_j^l\}_{j=1}^{N_l}\}_{l=1}^{L}$ *be ordered, let* $\mathcal{E}$ *be a corresponding matrix with the entries*

$$|a_h(\psi_i^k,\psi_j^l)|/\big(a_h(\psi_i^k,\psi_i^k)a_h(\psi_j^l,\psi_j^l)\big)^{1/2}, \qquad i=1,\ldots,N_k, \quad j=1,\ldots,N_l, \quad k,l=1,\ldots,L,$$

and let $\rho(\mathcal{E})$ *be the spectral radius of matrix* $\mathcal{E}$. *Then* $\rho(\mathcal{E}) \le C$.

Proof. The statement follows from a similar result for the bilinear form $a(\cdot,\cdot)$ proved in Lemma 6.1 in [10] and (2.8). ■

Theorem 4.3. *Linear operator* B_M *is self-adjoint, positive definite on* V_h *in the* $(\cdot,\cdot)_h$ *inner product, and*

$$C_1(B_M v, v)_h \le (L_h v, v)_h \le C_2(B_M v, v)_h, \qquad v \in V_h. \tag{4.8}$$

Proof. Inequalities in (4.8) follow from the second inequality in (4.6), Lemma 4.2, and Lemma 4 in section 5.2 of [17] with $\omega = 1$. Since operators $\{T_i^l\}$ are self-adjoint on V_h in the

$a_h(\cdot,\cdot)$ inner product, it is easy to see that T_M is self-adjoint, positive definite in the $a_h(\cdot,\cdot)$ inner product. Hence, $B_M = L_h T_M^{-1}$ is self-adjoint, positive definite in the $(\cdot,\cdot)_h$ inner product. ■

V. A MATRIX-VECTOR FORM OF THE PROBLEM

In this section, we obtain the matrix-vector form of the quadrature problem using the standard basis for V_h. In particular, we determine a relation between the element stiffness matrix and an auxiliary orthogonal spline collocation matrix corresponding to the Laplace operator.

The Hermite Finite Element Basis

Fix $l = 1,\ldots,L$, and let $\hat{\mathcal{T}}_l$ denote the set of all interior nodes in the triangulation $\mathcal{T}_l$, and let $h_{l,1}$ and $h_{l,2}$ be the lengths of the horizontal and the vertical sides of the elements in $\mathcal{T}_l$, respectively. Let

$$\mathcal{D} = \{(0,0),(0,1),(1,0),(1,1)\} \tag{5.1}$$

be the set of derivative multi-indices that correspond to the degrees of freedom of an element in the finite element space V_l. For $\alpha \in \mathcal{D}$ and $z \in \hat{\mathcal{T}}_l$, the standard Hermite finite element basis function $\psi^l_{\alpha,z} \in V_l$ is defined by

$$\partial^\beta \psi^l_{\alpha,z}(x) = \begin{cases} h_{l,1}^{-\alpha_1} h_{l,2}^{-\alpha_2}, & \text{if } x = z \text{ and } \beta = \alpha, \\ & \forall \beta \in \mathcal{D} \text{ and } \forall x \in \hat{\mathcal{T}}_l, \\ 0, & \text{otherwise,} \end{cases}$$

and it is expressed as the product

$$\psi^l_{\alpha,z}(x) = H^l_{\alpha_1,z_1}(x_1) H^l_{\alpha_2,z_2}(x_2) \tag{5.2}$$

of one-dimensional basis functions

$$H^l_{\alpha_i,z_i}(x_i) = H_{\alpha_i}\left(\frac{x_i - z_i}{h_{l,i}}\right), \qquad i = 1,2, \tag{5.3}$$

where

$$H_0(t) = \begin{cases} (t+1)^2(1-2t), & t \in [-1,0], \\ 2t^3 - 3t^2 + 1, & t \in [0,1], \\ 0, & t \notin [-1,1], \end{cases} \qquad H_1(t) = \begin{cases} t(t+1)^2, & t \in [-1,0], \\ t(t-1)^2, & t \in [0,1], \\ 0, & t \notin [-1,1], \end{cases} \tag{5.4}$$

are the standard Hermite basis functions on the interval $(-1,1)$ (see [18]). For $l = 1,\ldots,L$, let $\{\psi^l_j\}_{j=1}^{N_l}$ be the basis $\{\psi^l_{\alpha,z} : \alpha \in \mathcal{D},\ z \in \hat{\mathcal{T}}_l\}$ ordered in some way, and let $[v]_{\psi,l}$ be the corresponding vector representation of $v \in V_l$. In what follows, we denote $\psi_j = \psi^L_j$, $\psi_{\alpha,z} = \psi^L_{\alpha,z}$, and $[v]_\psi = [v]_{\psi,L}$.

Matrix-Vector Form

Let $N_{\mathcal{G}}$ be the number of the composite Gaussian quadrature nodes $\mathcal{G}_h = \{\mathcal{G}_K\}_{K\in\mathcal{T}_h}$, called the Gauss points, in the triangulation $\mathcal{T}_h$ and assume that the set $\mathcal{G}_h = \{\xi_i\}_{i=1}^{N_{\mathcal{G}}}$ is ordered. Note that $N_{\mathcal{G}} > N = \dim(V_h)$. Besides $[v]_\psi$, we consider the following two coordinate representations of a function $v \in V_h$:

$$[v]_{\mathcal{G}} = [v(\xi_1), \ldots, v(\xi_{N_{\mathcal{G}}})]^t \in R^{N_{\mathcal{G}}}, \tag{5.5}$$

$$[v] = (4/h^2)[(v, \psi_1)_h, \ldots, (v, \psi_N)_h]^t \in R^N. \tag{5.6}$$

We note that the relation (5.5) is valid for any function v defined on $\mathcal{G}_h$, and that, by (2.4) and (5.5),

$$(v, w)_h = (h^2/4)[w]_{\mathcal{G}}^t [v]_{\mathcal{G}}, \tag{5.7}$$

for any functions v and w defined on $\mathcal{G}_h$.

With the row index i, let

$$[\Delta]_\psi^{\mathcal{G}} = (\Delta\psi_j(\xi_i))_{N_{\mathcal{G}}\times N} \qquad \text{and} \qquad [I_h]_\psi^{\mathcal{G}} = (\psi_j(\xi_i))_{N_{\mathcal{G}}\times N} \tag{5.8}$$

be orthogonal spline collocation matrices corresponding to the Laplacian Δ and the identity operator I_h on V_h, respectively. Using (5.8), we obtain

$$[v]_{\mathcal{G}} = [I_h]_\psi^{\mathcal{G}}[v]_\psi \qquad \text{and} \qquad [\Delta v]_{\mathcal{G}} = [\Delta]_\psi^{\mathcal{G}}[v]_\psi, \quad v \in V_h. \tag{5.9}$$

The following lemma states that the collocation matrices $[\Delta]_\psi^{\mathcal{G}}$ and $[I_h]_\psi^{\mathcal{G}}$ can be used to form the stiffness matrix and the load vector of the quadrature problem (2.6).

Lemma 5.1. *The quadrature problem* (2.6) *has the matrix-vector form*

$$A[u_h]_\psi = [f], \qquad \text{where } A = ([\Delta]_\psi^{\mathcal{G}})^t[\Delta]_\psi^{\mathcal{G}} \quad \text{and} \quad [f] = ([I_h]_\psi^{\mathcal{G}})^t[f]_{\mathcal{G}}. \tag{5.10}$$

Proof. It follows from the formulation of the quadrature problem (2.6) that, defined by (5.6), vector $[f]$ is the load vector, and

$$A \in R^{N\times N}, \qquad (A)_{ij} = (4/h^2)a_h(\psi_j, \psi_i). \qquad i, j = 1, \ldots, N \tag{5.11}$$

is the stiffness the matrix. Representations (2.5) and (5.7), and the second relation in (5.9) imply

$$a_h(v, w) = (\Delta v, \Delta w)_h = (h^2/4)[\Delta w]_{\mathcal{G}}^t[\Delta v]_{\mathcal{G}} = (h^2/4)[w]_\psi^t([\Delta]_\psi^{\mathcal{G}})^t[\Delta]_\psi^{\mathcal{G}}[v]_\psi \tag{5.12}$$

for any v and w in V_h. Replacing v and w in (5.12) by ψ_j and ψ_i, respectively, and using (5.11), we obtain

$$(A)_{ij} = [\psi_j]_\psi^t([\Delta]_\psi^{\mathcal{G}})^t[\Delta]_\psi^{\mathcal{G}}[\psi_i]_\psi. \tag{5.13}$$

Since $[\psi_j]_\psi = \vec{e}_j$, $j = 1, \ldots, N$, where $\vec{e}_j$ is the j standard basis vector in R^N, representation (5.13) implies $A = ([\Delta]_\psi^{\mathcal{G}})^t[\Delta]_\psi^{\mathcal{G}}$. Similarly, using (5.6), (3.7), (5.7), and the first relation in (5.9), we get

$$\begin{aligned}([f])_i = (4/h^2)(f_h, \psi_i)_h = (4/h^2)(f, \psi_i)_h = [f]_{\mathcal{G}}^t[\psi_i]_{\mathcal{G}} \\ = [f]_{\mathcal{G}}^t[I_h]_\psi^{\mathcal{G}}[\psi_i]_\psi = (\vec{e}_i)^t([I_h]_\psi^{\mathcal{G}})^t[f]_{\mathcal{G}}, \qquad i = 1, \ldots, N,\end{aligned}$$

which gives $[f] = ([I_h]_\psi^{\mathcal{G}})^t[f]_{\mathcal{G}}$. ■

The Element Stiffness Matrix

In the standard way, the global stiffness matrix can be formed using the element stiffness matrix. Let $\hat{\psi}_{\alpha,z}$ be the standard basis function corresponding to the vertex z of the reference element $\hat{K} = (0,1) \times (0,1)$ and the derivative index $\alpha \in \mathcal{D}$ of (5.1). Consider the block function matrix

$$\Psi = \begin{bmatrix} \Psi_{(0,0)} & \Psi_{(1,0)} \\ \Psi_{(0,1)} & \Psi_{(1,1)} \end{bmatrix}, \qquad \text{where } \Psi_z = \begin{bmatrix} \hat{\psi}_{z,(0,0)} & \hat{\psi}_{z,(1,0)} \\ \hat{\psi}_{z,(0,1)} & \hat{\psi}_{z,(1,1)} \end{bmatrix}.$$

A local ordering of basis functions is obtained by reshaping matrix Ψ into a vector by columns. Let

$$\xi_1 = (3 - \sqrt{3})/6 \quad \text{and} \quad \xi_2 = (3 + \sqrt{3})/6 \tag{5.14}$$

be the Gauss points in the interval $(0, 1)$, and let

$$\hat{\mathcal{G}} = \{(\xi_1, \xi_1),\ (\xi_1, \xi_2),\ (\xi_2, \xi_1),\ (\xi_2, \xi_2)\}$$

be an ordered set of the Gauss points in the reference square $\hat{K}$. Let $\sigma = h_{0,1}/h_{0,2}$ be the aspect ratio of the elements in the coarse triangulation $\mathcal{T}_0$ with the horizontal and the vertical side lengths $h_{0,1}$ and $h_{0,2}$, respectively.

The element stiffness matrix of the quadrature problem (2.6) can be computed using a reference orthogonal spline collocation matrix $[\Delta]_{\hat{\psi}}^{\hat{\mathcal{G}}} \in R^{4\times 16}$ corresponding to the Laplacian Δ and the described local orderings of the basis functions and the Gauss points. The first relation in (5.8), (5.2), and (5.3) give

$$[\Delta]_{\hat{\psi}}^{\hat{\mathcal{G}}} = \left(B_2 \otimes B_0 + \sigma^2 B_0 \otimes B_2\right), \tag{5.15}$$

$$B_k = \begin{pmatrix} H_0^{(k)}(\xi_1) & H_1^{(k)}(\xi_1) & H_0^{(k)}(\xi_1 - 1) & H_1^{(k)}(\xi_1 - 1) \\ H_0^{(k)}(\xi_2) & H_1^{(k)}(\xi_2) & H_0^{(k)}(\xi_2 - 1) & H_1^{(k)}(\xi_2 - 1) \end{pmatrix}, \qquad k = 0, 2, \tag{5.16}$$

where the symbol $\otimes$ denotes the matrix tensor product, the standard Hermite basis functions $H_0(t)$ and $H_1(t)$ are defined in (5.4), and the superscript (k) denotes the derivative of order k. Applying (5.16), (5.4), and (5.14), we obtain

$$B_2 = \begin{pmatrix} -a_1 & -a_2 & a_1 & -a_3 \\ a_1 & a_3 & -a_1 & a_2 \end{pmatrix} \quad \text{and} \quad B_0 = \begin{pmatrix} b_1 & b_2 & b_3 & -b_4 \\ b_3 & b_4 & b_1 & -b_2 \end{pmatrix},$$

where

$$a_1 = 2\sqrt{3}, \quad a_2 = \sqrt{3} + 1, \quad a_3 = \sqrt{3} - 1,$$
$$b_1 = \left(9 + 4\sqrt{3}\right)/18, \quad b_2 = \left(3 + \sqrt{3}\right)/36, \quad b_3 = \left(9 - 4\sqrt{3}\right)/18, \quad b_4 = \left(3 - \sqrt{3}\right)/36.$$

The element collocation matrix is equal to the reference collocation matrix $[\Delta]_{\hat{\psi}}^{\hat{\mathcal{G}}}$ scaled by the factor $(\sigma h^2)^{-1}$. By Lemma 5.1, the global stiffness matrix has the representation $A = ([\Delta]_{\psi}^{\mathcal{G}})^t [\Delta]_{\psi}^{\mathcal{G}}$. Hence,

$$A_{\mathrm{e}} = \sigma^{-2} h^{-4} ([\Delta]_{\hat{\psi}}^{\hat{\mathcal{G}}})^t [\Delta]_{\hat{\psi}}^{\hat{\mathcal{G}}} \in R^{16\times 16}$$

is the element stiffness matrix of the quadrature problem (2.6), where $[\Delta]_{\hat{\psi}}^{\hat{\mathcal{G}}}$ is defined by (5.15).

VI. MULTILEVEL ALGORITHMS

In this section, using the Hermite bicubic interpolation, we obtain algorithms implementing interpolation and restriction, and present the algorithms for the multilevel preconditioners.

Interpolation and Restriction Operators

Fix $l = 2, 3, \ldots, L$ and let $v \in V_{l-1}$. Let us show that vector $[v]_{\psi,l}$ can be computed using 36 entries in vector $[v]_{\psi,l-1}$ applying the bicubic Hermite interpolation. Defined by (5.3), the one-dimensional standard basis functions on level $l-1$ are uniquely expressed as linear combinations of six standard basis functions on level l as follows:

$$H_{0,z_i}^{l-1} = \frac{1}{2}H_{0,z_i-h_{l,i}}^{l} + \frac{3}{4}H_{1,z_i-h_{l,i}}^{l} + H_{0,z_i}^{l} + \frac{1}{2}H_{0,z_i+h_{l,i}}^{l} - \frac{3}{4}H_{1,z_i+h_{l,i}}^{l},$$
$$H_{1,z_i}^{l-1} = -\frac{1}{8}H_{0,z_i-h_{l,i}}^{l} - \frac{1}{8}H_{1,z_i-h_{l,i}}^{l} + \frac{1}{2}H_{1,z_i}^{l,i} + \frac{1}{8}H_{0,z_i+h_{l,i}}^{l} - \frac{1}{8}H_{1,z_i+h_{l,i}}^{l}, \tag{6.1}$$

for any $z \in \mathcal{T}_{l-1}$ and $i = 1, 2$. It follows from (5.2) and (6.1) that a function in the two-dimensional basis $\{\psi_{\alpha,z}^{l-1}\}$ is uniquely expressed as a linear combination of 25 functions in the basis $\{\psi_{\alpha,z}^{l}\}$.

Using the representations in (6.1), we formulate our interpolation and restriction procedures. Let

$$P^{(2)} = P^{(1)} \otimes P^{(1)} \in R^{36\times 4} \tag{6.2}$$

be the local two-dimensional interpolation matrix corresponding to the Hermite bicubic interpolation from V_{l-1} to V_l, where

$$P^{(1)} = \frac{1}{8}\begin{pmatrix} 4 & 6 & 8 & 0 & 4 & -6 \\ -1 & -1 & 0 & 4 & 1 & -1 \end{pmatrix}^t \tag{6.3}$$

is the one-dimensional local interpolation matrix determined from (6.1). For $w \in V_{l-1}$ and $x \in \hat{\mathcal{T}}_{l-1}$, let

$$W_{ij} = \begin{pmatrix} w & w_{x_1} \\ w_{x_2} & w_{x_1x_2} \end{pmatrix}\Bigg|_{(x_1+(i-2)h_{l,1},x_2+(j-2)h_{l,2})}, \qquad i, j = 1, 2, 3,$$

and let $W = (W_{ij})_{3\times 3}$ be a block matrix with values of w and its derivatives at x and the nodes of $\hat{\mathcal{T}}_l$ that surround x. We define injection matrices $R_{c,x} \in R^{4\times N_{l-1}}$ and $R_{f,x} \in R^{36\times N_l}$ by

$$R_{c,x}[w]_{\psi}^{l-1} = [w(x), w_{x_2}(x), w_{x_1}(x), w_{x_1x_2}(x)]^t,$$
$$R_{f,x}[w]_{\psi}^{l} = [W_1^t, \ldots, W_6^t]^t,$$

respectively, where W_j is the j-column of matrix W for $j = 1, \ldots, 6$. The interpolation and the restriction algorithms are presented in Figs. 5 and 6.

Level Stiffness Matrices

Let us show that the level stiffness matrices, required by the multilevel algorithms, can be computed using the recurrence relation

$$A_{l-1} = P_{l-1}^t A_l P_{l-1} \qquad \text{for } l = L, L-1, \ldots, 2, \tag{6.4}$$

input: l, $[v]_{\psi,l-1}$
output: $[v]_{\psi,l}$
$[v]_{\psi,l} \leftarrow 0$
for all $x \in \hat{\mathcal{T}}_{l-1}$, **compute**
  $[v]_{\psi,l} \leftarrow R^t_{f,x} P^{(2)} R_{c,x} [v]_{\psi,l-1}$
end

FIG. 5. Interpolation algorithm.

input: l, $[v]_{\psi,l}$
output: $[v]_{\psi,l-1}$
$[v]_{\psi,l-1} \leftarrow 0$
for all $x \in \hat{\mathcal{T}}_{l-1}$, **compute**
  $[v]_{\psi,l-1} \leftarrow R^t_{c,x} (P^{(2)})^t R_{f,x} [v]_{\psi,l}$
end

FIG. 6. Restriction algorithm.

where $A_l = (a^l_{ij}) \in R^{N_l \times N_l}$ is the level stiffness matrix with entries

$$a^l_{ij} = (4/h^2) a_h(\psi^l_j, \psi^l_i), \qquad i, j = 1, \ldots, N_l, \qquad l = 1, \ldots, L, \tag{6.5}$$

and $P_{l-1} \in R^{N_l \times N_{l-1}}$ is the global interpolation matrix such that

$$P_{l-1}[v]_{\psi,l-1} = [v]_{\psi,l}, \qquad l = 2, \ldots, L. \tag{6.6}$$

We fix $l = 2, \ldots, L$ and apply formula (6.6) recurrently to get

$$[v]_\psi = P_{L-1} \cdots P_l [v]_{\psi,l}, \qquad v \in V_l. \tag{6.7}$$

In particular, replacing v in (6.7) by ψ^l_j, we obtain

$$[\psi^l_j]_\psi = P_{L-1} \cdots P_l \vec{e}^{\,l}_j, \qquad j = 1, \ldots, N_l, \tag{6.8}$$

where $\vec{e}^{\,l}_j$ is the j standard basis vector in R^{N_l}. Similarly to (5.13), we get

$$a^l_{ij} = [\psi^l_j]^t_\psi \Delta^t_h \Delta_h [\psi^l_i]_\psi, \qquad i, j = 1, \ldots, N_l, \tag{6.9}$$

which, by (6.8), give (6.4).

Additive Algorithm

Let us present the additive preconditioning algorithm which computes $w \in V_h$ such that

$$B_A w = v \qquad \text{for } v \in V_h.$$

Using (3.11) and (3.13), we get

$$w = T_A L_h^{-1} v = \left(\sum_{l=1}^{L} \sum_{i=1}^{N_l} T^l_i \right) L_h^{-1} v = \sum_{l=1}^{L} w_l, \qquad \text{where} \quad w_l = \sum_{i=1}^{N_l} T^l_i L_h^{-1} v.$$

Therefore, to obtain w, we need to compute and sum w_l for $l = 1, 2, \ldots, L$. The additive preconditioning algorithm is presented in Fig. 7, and it is related to the V-cycle algorithm with a Jacobi smoothing (see [7] for details). The computational cost of the additive algorithm is of order $O(h^{-2})$.

input: L, $[v]$, $\{\text{diag}(A_l)\}_{l=1}^{L}$
output: $[w]_\psi$
$\vec{v}_L \leftarrow [v]$
for $l = L, L-1, \ldots, 1$
  if $(l < L)$ $\vec{v}_l = P_l^t \vec{v}_{l+1}$ **end**
  solve $\text{diag}(A_l)\vec{w}_l = \vec{v}_l$
end
for $l = 1, 2, \ldots, L-1$
  $\vec{w}_{l+1} \leftarrow \vec{w}_{l+1} + P_l \vec{w}_l$
end
$[w]_\psi \leftarrow \vec{w}_L$

FIG. 7. Additive preconditioning algorithm.

Multiplicative Algorithm

We now consider the computation of w such that

$$B_M w = v \qquad \text{for } v \in V_h,$$

where B_M is the multiplicative preconditioner. Using (3.12) and $u = L_h^{-1} v$, we get

$$w = B_M^{-1} v = T_M L_h^{-1} v = T_M u,$$

which by (3.14) implies

$$u - w = \left[\prod_{l=L}^{1} \prod_{i=1}^{N_l} \left(I_h - T_i^l\right)\right] \left[\prod_{l=1}^{L} \prod_{i=N_l}^{1} \left(I_h - T_i^l\right)\right] u. \tag{6.10}$$

Let S be the set of index pairs (l, i) ordered according to the sequence of the factors in (6.10) from right to left. Setting $y = u - w$, we see that $u - w$ can be computed using the algorithm

$$y \leftarrow u; \;\; y \leftarrow (I_h - T_i^l) y \qquad \text{for } (l, i) \in S;$$

which is equivalent to

$$w \leftarrow 0; \;\; w \leftarrow w + T_i^l (u - w) \qquad \text{for } (l, i) \in S.$$

The multiplicative preconditioning algorithm is presented in Fig. 8, where, for $l = 1, \ldots, L$, the matrix L_l contains the lower triangular part of the level stiffness matrix A_l (see [7] for details). A lower triangular linear system with matrix L_l is solved in the descend phase followed by the ascend phase, where an upper triangular linear system with matrix L_l^t is solved. With the computational cost of order $O(h^{-2})$, the multiplicative preconditioning algorithm is related to the V-cycle multigrid algorithm with a Gauss–Seidel smoothing.

Iterative Method

The linear system (5.10) corresponding to the quadrature Galerkin problem (2.6) has a symmetric, positive definite coefficient matrix, and it can be solved using the preconditioned conjugate

input: L, $[v]$, $\{A_l\}_{l=1}^{L}$
output: $[w]_\psi$
$\vec{g}_L \leftarrow [v]$
for $l = L, \ldots, 1$
  solve $L_l \vec{w}_l = \vec{g}_l$
  if $(l > 1)$ $\vec{g}_{l-1} \leftarrow P_{l-1}^t(\vec{g}_l - A_l \vec{w}_l)$ **end**
end
for $l = 1, \ldots, L$
  if $(l > 1)$ $\vec{w}_l \leftarrow \vec{w}_l + P_{l-1}\vec{w}_{l-1}$ **end**
  solve $L_l^t \vec{w} = \vec{g}_l - A_l \vec{w}_l$
  $\vec{w}_l \leftarrow \vec{w}_l + \vec{w}$
end
$[w]_\psi \leftarrow \vec{w}_L$

FIG. 8. Multiplicative preconditioning algorithm.

gradient (Algorithm 9.4.14 in [19]). The convergence rate of the PCG algorithm with a preconditioner M for solving a linear system $Ax = b$ is bounded from above by $\rho = \big(\sqrt{\kappa} - 1\big)/\big(\sqrt{\kappa} + 1\big)$, where κ is the spectral condition number of the preconditioned matrix $\tilde{A} = M^{-1/2}AM^{-1/2}$ (Theorem 9.4.14 in [19]). We note that $\kappa = \lambda_{\max}/\lambda_{\min}$, where $\lambda_{\min}$ and $\lambda_{\max}$ are respectively the smallest and the largest eigenvalues of matrix $\tilde{A}$. Using this bound, it is easy to see that the number of PCG iterations to reduce the relative error to within the tolerance $\varepsilon > 0$ is bounded from above by $|\log \varepsilon|\sqrt{\kappa}/2$.

Let $\lambda_{\min,h}$ and $\lambda_{\max,h}$ denote correspondingly the smallest and the largest eigenvalues of the preconditioned operator

$$\tilde{A}_h = M_h^{-1/2} L_h M_h^{-1/2}, \tag{6.11}$$

where L_h is the operator of the equation (3.8), and M_h is either the additive or the multiplicative preconditioner defined by (3.11) and (3.12), respectively. Let κ_h be the spectral condition number of the operator $\tilde{A}_h$. Theorems 4.1 and 4.3 imply that $\kappa_h \leq C_2/C_1$ uniformly with respect to h; therefore, the number of PCG iterations to reduce the relative error to within the tolerance $\varepsilon > 0$ is of order $\mathrm{O}(|\log \varepsilon|)$ as $h \to 0+$.

VII. NUMERICAL RESULTS

In this section, we present numerical results that demonstrate efficiency of the proposed multilevel preconditioners. The computations were performed on a computer with Intel Xeon 2.8 GHz CPU with a cache size 512 kb. The size of the coarsest triangulation was set to 2×2 and 7 nested triangulations were constructed by dyadic partitioning with the finest triangulation of size 256×256. The convergence tolerance was set to $\varepsilon = 10^{-10}$. The extreme eigenvalues of the preconditioned operator $\tilde{A}_h$ of (6.11) provide valuable information on the convergence rate of the PCG method, and they were approximated using certain PCG coefficients that are related to the Lanczos iterations.

Example I. First, we tested our preconditioners on a model problem with a smooth load function f corresponding to the the exact solution $u(x) = (1 - \cos 2\pi x_1)(1 - \cos 2\pi x_2)$ on the

TABLE I. Example I: spectral constants and PCG iteration numbers.

			Additive				Multiplicative			
h	L^2-error	$\kappa(L_h)$	$\lambda_{h,\min}$	$\lambda_{h,\max}$	κ_h	*iter.*	$\lambda_{h,\min}$	$\lambda_{h,\max}$	κ_h	*iter.*
1/4	1.2e−02	2.6e+2	0.624	1.923	3.082	6	0.754	0.999	1.326	9
1/8	6.9e−04	1.6e+3	0.586	2.320	3.957	19	0.741	0.997	1.345	10
1/16	4.2e−05	1.8e+4	0.569	2.837	4.983	24	0.739	0.994	1.345	10
1/32	2.6e−06	2.3e+5	0.571	3.416	5.986	28	0.738	0.994	1.348	11
1/64	1.6e−07	3.3e+6	0.565	3.907	6.917	33	0.736	0.991	1.346	11
1/128	1.1e−08	5.0e+7	0.562	4.312	7.672	37	0.735	0.990	1.346	12
1/256	6.4e−08	7.8e+8	0.562	4.648	8.276	40	0.735	0.986	1.342	12

unit square $\Omega = (0,1) \times (0,1)$ (see Example 1 in [14] and Problem 2 in [20]). Numerical results are presented in Table I, where

- h the triangulation parameter,
- $\kappa(L_h)$ a condition number estimate for operator L_h,
- $\lambda_{\min,h}$ an approximation of the smallest eigenvalue of $\tilde{A}_h$,
- $\lambda_{\max,h}$ an approximation of the largest eigenvalue of $\tilde{A}_h$,
- κ_h an approximation of the condition number of $\tilde{A}_h$,
- *iter* the number of PCG iterations to reduce the relative residual norm to within the tolerance.

Simple calculations show that the approximate convergence rates of the presented L^2-errors are close to 4. The multiplicative preconditioner performed significantly better that the additive preconditioner. For the values of $h \leq 1/64$, the number of PCG iterations is more than 3 times less with the multiplicative than with the additive preconditioner. The multiplicative preconditioner improves the conditioning of the linear system to an almost perfect $\kappa_h \approx 1.34$. For $h \leq 1/16$, the smallest eigenvalue $\lambda_{\min,h}$ is accurately approximated using both preconditioners. It is interesting to note that the approximation to the largest eigenvalue $\lambda_{\max,h}$ monotonically increases for the additive preconditioner and monotonically decreases at a much lower rate for the multiplicative preconditioner.

In Fig. 9, we present plots of the relative residual norm against the iteration number in the logarithmic scale for $h = 1/256$. The dashed and the solid lines represent the additive and the multiplicative preconditioners, respectively. The residual curve for the multiplicative preconditioner decreases monotonically with an almost constant slope for larger iteration numbers, whereas

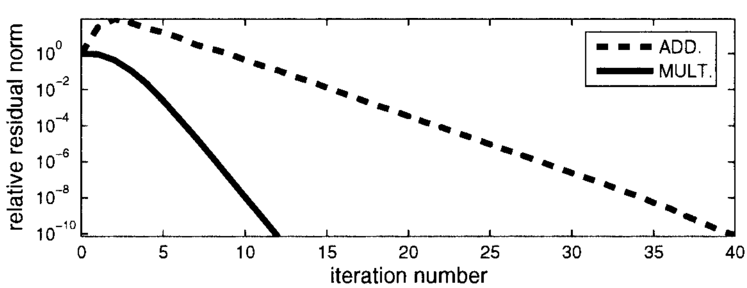

FIG. 9. Relative residual norm as a function of an iteration number, 256 × 256 grid.

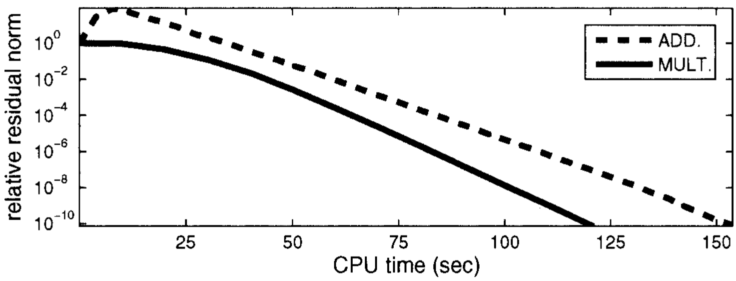

FIG. 10. Relative residual norm as a function of CPU time, 256 × 256 grid.

the curve for the additive preconditioner increases at the first few iterations and then monotonically decreases also with an almost constant slope. In Fig. 10, we present the residual curves plotted against the CPU time and note that qualitative behaviors of the residual curves remain the same as in Fig. 9. The advantage of the multiplicative preconditioner is now less spectacular than in Fig. 9.

To compare the PCG algorithm with Matlab's sparse LU solver for solving the linear system in (5.10), we plot the CPU time against the number of degrees of freedom in Fig. 11. Since the uniform triangulation $\mathcal{T}_h$ has $(1/h-1)^2$ interior nodes, there are $4(1/h-1)^2$ degrees of freedom in the corresponding linear system. We note that the CPU time for the multiplicative preconditioner was smaller than that for the additive preconditioner and that both multilevel preconditioners performed significantly better than the LU solver.

Example II. To test our preconditioners on an example with a discontinuous load function f, we computed deflections of a square clamped plate under the unit load concentrated at the center of the plate (see Example 4 in [14]). In this example, $\Omega = (0,1) \times (0,1)$,

$$f(x) = \begin{cases} 1/(4h)^2 & \text{if } |x_i - 0.5| < h, \quad i = 1, 2, \\ 0 & \text{otherwise,} \end{cases}$$

and, accurate to 4 digits, the exact solution at the center is $u(0.5, 0.5) = 0.0056$. The numerical results are presented in Table II. The iteration numbers in Tables I and II show that the preconditioners performed slightly better in Example II than in Example I.

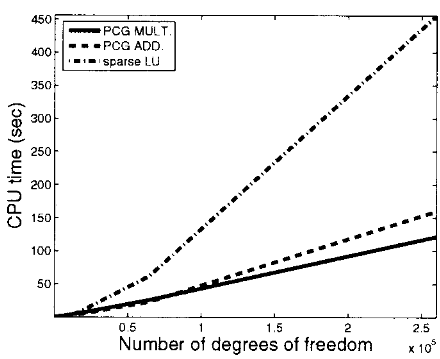

FIG. 11. CPU time comparison of a sparse LU and the multilevel PCG solvers.

TABLE II. Example II: spectral constants, iteration numbers, and plate deflections.

	Additive				Multiplicative				
h	$\lambda_{h,\min}$	$\lambda_{h,\max}$	κ_h	*iter.*	$\lambda_{h,\min}$	$\lambda_{h,\max}$	κ_h	*iter.*	$u(0.5, 0.5)$
1/4	0.624	1.923	3.082	6	0.754	0.999	1.326	9	0.003386715611
1/8	0.586	2.320	3.957	19	0.741	0.996	1.345	10	0.004768317859
1/16	0.569	2.837	4.983	24	0.739	0.995	1.346	10	0.005329303836
1/32	0.563	3.416	6.065	28	0.737	0.994	1.349	11	0.005523392879
1/64	0.562	3.907	6.954	32	0.736	0.992	1.347	11	0.005585377711
1/128	0.561	4.312	7.685	34	0.735	0.988	1.343	11	0.005604240240
1/256	0.561	4.648	8.283	37	0.735	0.986	1.342	12	0.005609797325

VIII. CONCLUSION

In this article, we developed and analyzed multilevel preconditioners for the quadrature finite element Galerkin approximation of the biharmonic problem. The proposed additive and multiplicative preconditioners are self-adjoint, positive definite operators that are uniformly spectrally equivalent to the operator of the quadrature problem. The preconditioners are implemented using algorithms with the optimal order computational cost. Numerical tests confirmed the theoretical findings and demonstrated the efficiency of the preconditioners. In particular, both preconditioners performed significantly better than a sparse LU solver, and the multilevel preconditioner was more efficient than the additive preconditioner.

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