

CONVERGENCE ANALYSIS OF A QUADRATURE FINITE ELEMENT GALERKIN SCHEME FOR A BIHARMONIC PROBLEM

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Abstract. A quadrature finite element Galerkin scheme for a Dirichlet boundary value problem for the biharmonic equation is analyzed for a solution existence, uniqueness, and convergence. Conforming finite element space of Bogner-Fox-Schmit rectangles and an integration rule based on the two-point Gaussian quadrature are used to formulate the discrete problem. An H^2 -norm error estimate is obtained for the solution of the original finite element problem consistent with the solution regularity. A standard quadrature error analysis gives a suboptimal order error estimate. Optimal order error estimates under sufficient regularity assumptions are obtained using an alternative approach based on the equivalence of the quadrature problem with an orthogonal spline collocation problem.

Key words: biharmonic problem, finite elements, Galerkin method, Gaussian quadrature, orthogonal spline collocation

AMS subject classification. 65N12, 65N15, 65N30, 65N35

1. Introduction. In this article, we analyze existence, uniqueness, and convergence of a quadrature finite element Galerkin approximation of a Dirichlet boundary value problem (BVP) with the biharmonic equation on a rectangular polygonal region. Problems with the biharmonic equation arise in many areas of applied mathematics, for example, plate problems in plane elasticity and problems for the stream function of steady-state Stokes flows in fluid mechanics.

Let $\Omega \subset R^2$ be an open rectangular polygon with the boundary $\partial\Omega$. We consider the Dirichlet BVP

$$(1.1) \quad \Delta^2 u = f \text{ in } \Omega, \quad \text{and } u = \partial_n u = 0 \text{ on } \partial\Omega,$$

where Δ is the Laplace operator, f is a continuous function on Ω , and ∂_n denotes the outer normal derivative. In plane elasticity, problem (1.1) is known as a clamped plate problem.

Biharmonic problems can be solved numerically by a finite element Galerkin method using direct or mixed approaches. The direct approach is based on a direct discretization of the biharmonic equation (see §2.2, Ch. 2, and Ch. 6 in [11]). Whereas in the mixed approach, the biharmonic equation is first replaced with a pair of coupled second-order differential equations by introducing a new dependent variable (see [11, 12, 14]). A review and a catalog of finite elements for plate problems is given in [15].

Forming a stiffness matrix and a load vector in a finite element Galerkin method for the biharmonic problem requires evaluation of integrals with polynomial integrands. An application of a numerical quadrature leads to a quadrature finite element Galerkin scheme. The main criteria in selecting a quadrature rule for a quadrature finite element Galerkin scheme is to preserve solution accuracy of the original finite element problem. For a fourth-order problem, the application of a numerical integration based on at least three-point Gaussian quadrature is suggested in Theorem 8.9 in [17] to preserve optimal error estimates.

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We apply a direct approach using the conforming Bogner-Fox-Schmit rectangle introduced in [9]. The discrete solution is a C^1 piecewise bicubic function with 16 degrees of freedom approximating the values of the solution, the first-order and the mixed second-order derivatives at the vertices of a rectangular element. The integrals in the stiffness matrix and in the load vector are approximated by a product two-point Gaussian quadrature. The resulting quadrature scheme demonstrates optimal order convergence in Sobolev norms and superconvergence in the maximum nodal norm.

In our convergence analysis, we use an auxiliary orthogonal spline collocation (OSC) scheme for the biharmonic problem. A review of OSC methods for solving partial differential equations is given in [7]. A mixed type OSC method for biharmonic problems on a rectangle and solution algorithms were studied in [5, 16], where two coupled second-order differential equations are collocated at the nodes of a product Gaussian quadrature with the solution in the finite element space of piecewise Hermite bicubics. Forming a matrix of OSC equations is fast. The OSC solution has optimal order error estimates and demonstrates superconvergence at partition nodes. Convergence analysis of OSC schemes requires higher than optimal regularity assumptions on the exact solution.

The main results of this work are as follows. We obtain an error estimate in the Sobolev H^2 -norm for the solution of the original finite element problem consistent with the solution regularity. We prove existence and uniqueness of the solution of the quadrature scheme, and demonstrate that the standard error analysis based on the First Strang Lemma leads to a suboptimal order H^2 -norm error estimate. We prove that the solution of the quadrature scheme is also the solution of an equivalent OSC problem, and obtain optimal order error estimates in the H^1 - and H^2 -norms. Results presented in this article are related to those in [16].

An outline of the rest of the article is as follows. In section 2, we give a variational formulation of BVP (1.1) and present solution regularity results. In section 3, we formulate the finite element Galerkin problem and obtain an H^2 -norm error estimate consistent with the solution regularity. The quadrature finite element Galerkin scheme is defined in section 4, and existence and uniqueness of its solution is proved in section 5. In section 6, we prove that an error analysis of the quadrature scheme based on the First Strang Lemma gives a suboptimal order error estimate. Some auxiliary results are obtained in section 7, and, in section 8, we introduce an orthogonal spline collocation scheme, determine its relation with the quadrature Galerkin scheme, and obtain optimal order error estimates. Numerical results are presented in section 9, and our conclusions are summarized in section 10.

2. Variational problem and solution regularity. In this section, we give a variational formulation on the problem and present solution regularity results. Let D be an open subset of R^2 , and let $m \geq 0$ be an integer. Let $C^m(\bar{D})$ be the space of functions that are m times continuously differentiable on \bar{D} with the norm and a seminorm, respectively, given by

$$\|v\|_{C^m(\bar{D})} = \max_{|\alpha| \leq m} \{ \sup_{x \in \bar{D}} |\partial^\alpha v(x)| \} \quad \text{and} \quad |v|_{C^m(\bar{D})} = \max_{|\alpha|=m} \{ \sup_{x \in \bar{D}} |\partial^\alpha v(x)| \},$$

where $x = (x_1, x_2)$, $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2$, and $\partial^\alpha = \partial^{|\alpha|} / (\partial x_1^{\alpha_1} \partial x_2^{\alpha_2})$. Let $L^2(D)$ be the Hilbert space of square integrable functions on D with the norm $\|v\|_{L^2(D)} = (\int_D v^2 dx)^{1/2}$. Let $H^m(D)$ be the standard Sobolev space

with the norm and a seminorm

$$\|v\|_{H^m(D)} = \left(\sum_{|\alpha| \leq m} \int_D |\partial^\alpha v|^2 dx \right)^{1/2} \quad \text{and} \quad |v|_{H^m(D)} = \left(\sum_{|\alpha|=m} \int_D |\partial^\alpha v|^2 dx \right)^{1/2},$$

respectively. For real $s > 0$, we also use a fractional order Sobolev space $H^s(\Omega)$ and its dual $H^{-s}(\Omega)$ (see §14.2 in [10]). Let $H_0^2(\Omega)$ be the closure in the H^2 -norm of the space of C^∞ functions with compact supports in Ω . It is known that $|\cdot|_{H^2(\Omega)}$ and $\|\cdot\|_{H^2(\Omega)}$ are equivalent norms on $H_0^2(\Omega)$. Throughout this article, C is a generic positive constant.

We consider the following variational form of BVP (1.1): find $u \in H_0^2(\Omega)$ such that

$$(2.1) \quad a(u, v) = (f, v) \quad \text{for all } v \in H_0^2(\Omega),$$

where the bilinear and linear forms $a(\cdot, \cdot)$ and (f, \cdot) are respectively defined by

$$(2.2) \quad a(w, v) = \int_\Omega \Delta w \Delta v dx \quad \text{and} \quad (f, v) = \int_\Omega f v dx,$$

(see §1.2 in [11]). Using the representation

$$a(v, v) = |v|_{H^2(\Omega)}^2, \quad v \in H_0^2(\Omega),$$

(see (1.2.8) in [11]) and the norm equivalence $\|\cdot\|_{H^2(\Omega)} \sim |\cdot|_{H^2(\Omega)}$, it is easy to see that the bilinear form $a(\cdot, \cdot)$ is V -elliptic and bounded on $H_0^2(\Omega)$; that is,

$$(2.3) \quad C|v|_{H^2(\Omega)}^2 \leq a(v, v), \quad v \in H_0^2(\Omega),$$

$$(2.4) \quad |a(v, w)| \leq C|v|_{H^2(\Omega)}|w|_{H^2(\Omega)}, \quad v, w \in H_0^2(\Omega).$$

It follows from (2.2)–(2.4), and the Riesz Representation Theorem (Theorem 2.4.2 in [10]) that problem (2.1) has a unique solution $u \in H_0^2(\Omega)$ such that

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{H^{-2}(\Omega)}.$$

If Ω has a re-entrant corner with angle $3\pi/2$, then

$$(2.5) \quad \|u\|_{H^{2+2s}(\Omega)} \leq C\|f\|_{H^{-2+2s}(\Omega)} \quad \text{for any } f \in H^{-2+2s}(\Omega), \quad 0 \leq s < 1/3,$$

(see [3]). If Ω is a rectangle then it follows from Theorem 2 in [8] that

$$\|u\|_{H^{4-s}(\Omega)} \leq C\|f\|_{H^{-s}(\Omega)}, \quad \text{for any } f \in H^{-s}(\Omega), \quad s = 0, 1.$$

3. Finite element Galerkin problem. In this section, we formulate a finite element Galerkin approximation of problem (2.1) and obtain an H^2 -norm error estimate consistent with the solution regularity. We assume that x_1 - and x_2 -axes are respectively directed to the right and up, and that a boundary edge of the rectangular polygonal domain Ω is parallel to a coordinate axis. Let Ω be equipped with a regular triangulation \mathcal{T}_h consisting of rectangular elements and satisfying the requirements of the finite element method (see properties (\mathcal{T}_h1) – (\mathcal{T}_h5) of the assumption (FEM 1) in §2.1 and §2.2 in [11]). Let h be the largest edge length of elements in \mathcal{T}_h .

We consider a finite element space

$$X_h = \{v \in C^1(\bar{\Omega}) : v|_K \in Q_3(K), K \in \mathcal{T}_h\},$$

where $Q_3(K)$ denotes the space of bicubic polynomials restricted on K (see §2.2 in [11]). The subspace

$$V_h = \{v \in X_h : v = \partial_n v = 0 \text{ on } \partial\Omega\}$$

plays the major role in this article. Note that, for any $v \in V_h$,

$$v = v_{x_1} = v_{x_2} = v_{x_1 x_2} = 0 \text{ on } \partial\Omega,$$

and $V_h \subset H_0^2(\Omega)$ by Theorem 2.2.15 in [11]. The dimension of V_h equals four times the number of interior vertices of \mathcal{T}_h .

The finite element Galerkin problem approximating the variational problem (2.1) is formulated as follows: find $U_h \in V_h$ such that

$$(3.1) \quad a(U_h, v) = (f, v), \quad \text{for all } v \in V_h,$$

where the forms $a(\cdot, \cdot)$ and (f, \cdot) are defined in (2.2). It follows from (2.3) that V_h is a Hilbert space with the inner product $a(\cdot, \cdot)$. Using the Riesz Representation Theorem, it is easy to see that problem (3.1) has a unique solution.

In Theorem 6.1.6 in [11], it is proved that if $u \in H^4(\Omega) \cap H_0^2(\Omega)$ then

$$(3.2) \quad \|u - U_h\|_{H^2(\Omega)} \leq Ch^2 |u|_{H^4(\Omega)},$$

where u is the solution of problem (2.1). We note that the regularity assumption for (3.2) is satisfied for a rectangle but not for a domain that has a re-entrant corner with angle $3\pi/2$ (see [8]). The following convergence result is consistent with the solution regularity.

THEOREM 3.1. *Let u and U_h be the solutions of problems (2.1) and (3.1), respectively. If $f \in H^{-2+2s}(\Omega)$, then*

$$(3.3) \quad \|u - U_h\|_{H^2(\Omega)} \leq Ch^{2s} \|f\|_{H^{-2+2s}(\Omega)}, \quad 0 \leq s < 1/3.$$

Proof. To prove the theorem, we use results for the real method of interpolation of Sobolev spaces (see Chapter 14 in [10]). First, consider an orthogonal projection operator $P_h : H_0^2(\Omega) \rightarrow V_h$ such that, for any $v \in H_0^2(\Omega)$, $P_h v \in V_h$ satisfies

$$a(P_h v, \phi) = a(v, \phi) \quad \text{for all } \phi \in V_h,$$

and let $T_h = I - P_h$, where I is the identity operator on $H_0^2(\Omega)$. Since P_h is an orthogonal projection with respect to the inner product $a(\cdot, \cdot)$, using the triangle inequality and (2.4), we get

$$\|T_h v\|_{H^2(\Omega)} \leq C \|v\|_{H^2(\Omega)} \quad \text{for any } v \in H_0^2(\Omega).$$

From (3.2), we have

$$\|T_h v\|_{H^2(\Omega)} \leq Ch^2 \|v\|_{H^4(\Omega)}, \quad \text{for any } v \in H^4(\Omega) \cap H_0^2(\Omega).$$

Take $0 \leq s < 1/3$. It follows from an exact interpolation theorem (see Theorem 7.23 in [1]) with $\theta = 1 - s$ and $q = 2$, that operator T_h from $H^2(\Omega)$ to the interpolation space $[H^4(\Omega) \cap H_0^2(\Omega), H_0^2(\Omega)]_{1-s,2}$ is bounded with the corresponding operator norm satisfying $\|T_h\| \leq Ch^{2s}$; that is

$$(3.4) \quad \|T_h v\|_{H^2(\Omega)} \leq Ch^{2s} \|v\|_{[H^4(\Omega) \cap H_0^2(\Omega), H_0^2(\Omega)]_{1-s,2}}.$$

Let us prove that

$$(3.5) \quad \|u\|_{[H^4(\Omega) \cap H_0^2(\Omega), H_0^2(\Omega)]_{1-s,2}} \leq C \|f\|_{H^{-2+2s}(\Omega)}.$$

From

$$[H^2(\Omega), L^2(\Omega)]_{s,2} = H^{2-2s}(\Omega)$$

(see (14.2.5) in [10]), it follows that

$$[H^2(\Omega), L^2(\Omega)]'_{s,2} = H^{-2+2s}(\Omega),$$

where the prime denotes the dual space, which implies

$$(3.6) \quad \|f\|_{[H^2(\Omega), L^2(\Omega)]'_{s,2}} \leq C \|f\|_{H^{-2+2s}(\Omega)}.$$

By the duality theorem (Theorem 3.7.1 in [4]), we have

$$(3.7) \quad \|f\|_{[L^2(\Omega), H^{-2}(\Omega)]_{1-s,2}} \leq C \|f\|_{[H^2(\Omega), L^2(\Omega)]'_{s,2}}.$$

The following estimate was obtained in [3] (see estimate (4.8) and Theorem 4.1):

$$(3.8) \quad \|u\|_{[H^4(\Omega) \cap H_0^2(\Omega), H_0^2(\Omega)]_{1-s,2}} \leq \|f\|_{[L^2(\Omega), H^{-2}(\Omega)]_{1-s,2}},$$

where u is the solution of problem (2.1). Inequalities (3.6)–(3.8) imply (3.5).

Since $u \in [H^4(\Omega) \cap H_0^2(\Omega), H_0^2(\Omega)]_{1-s,2}$, replacing v in (3.4) by u and using (3.5), we get

$$\|T_h u\|_{H^2(\Omega)} \leq Ch^{2s} \|f\|_{H^{-2+2s}(\Omega)},$$

which, along with $T_h u = u - P_h u = u - U_h$, gives (3.3). \square

4. Quadrature scheme. In this section, we define our quadrature scheme that approximates the finite element Galerkin problem (3.1). First, we introduce notation and auxiliary facts.

For any interval $I = (a, b)$ with length $|I| = b - a$, let

$$\mathcal{G}_I = \{a + |I|(3 - \sqrt{3})/6, a + |I|(3 + \sqrt{3})/6\}$$

be the set of Gauss points in I , and let

$$\sum_{\mathcal{G}_I} v = \frac{b-a}{2} \sum_{\xi \in \mathcal{G}_I} v(\xi), \quad v \in C(I),$$

be the two-point Gaussian quadrature on I . Proved in [13], the following lemma provides an error formula for the Gaussian quadrature.

LEMMA 4.1. *Let $I = (a, b)$ be an interval and let $v \in C^4(I)$. Then*

$$(4.1) \quad \int_a^b v(t) dt = \sum_{\mathcal{G}_I} v + \frac{|I|^5}{4320} v^{(4)}(\xi), \quad \xi \in I.$$

For any $K = I_1 \times I_2 \in \mathcal{T}_h$, let

$$(4.2) \quad \sum_{\mathcal{G}_K} v = \sum_{\mathcal{G}_{I_1}} \sum_{\mathcal{G}_{I_2}} v = \frac{|K|}{4} \sum_{\xi \in \mathcal{G}_K} v(\xi), \quad v \in C(K),$$

be the product two-point Gaussian quadrature on K , where $\mathcal{G}_K = \mathcal{G}_{I_1} \times \mathcal{G}_{I_2}$ is the set of four quadrature nodes in K called the Gauss points, and $|K|$ is the area of K . For any functions v and w defined on \mathcal{T}_h , let

$$(4.3) \quad (v, w)_h = \sum_{K \in \mathcal{T}_h} \sum_{\mathcal{G}_K} vw \quad \text{and} \quad \|v\|_h = \sqrt{(v, v)_h}.$$

It is easy to verify that, for any v and w defined on \mathcal{T}_h ,

$$(4.4) \quad |(v, w)_h| \leq \|v\|_h \|w\|_h.$$

Let

$$(4.5) \quad a_h(w, v) = (\Delta w, \Delta v)_h, \quad w, v \in V_h,$$

be a bilinear form approximating $a(\cdot, \cdot)$ of (2.2). Our quadrature finite element Galerkin problem is formulated as follows: find $u_h \in V_h$ such that

$$(4.6) \quad a_h(u_h, v) = (f, v)_h \quad \text{for all } v \in V_h.$$

5. Existence and uniqueness. In this section, we prove that the quadrature Galerkin scheme (4.6) has a unique solution. First, we establish auxiliary results, formulated in the following two lemmas, which are used to prove V -ellipticity and boundedness of the approximate bilinear form $a_h(\cdot, \cdot)$ with respect to the H^2 -norm.

Domain $\bar{\Omega}$ can be decomposed in the following two unions of rectangles:

$$(5.1) \quad \bar{\Omega} = \bigcup_{l=1}^{L_V} \bar{R}_V^l = \bigcup_{l=1}^{L_H} \bar{R}_H^l,$$

where the sets $\{R_V^l\}_{l=1}^{L_V}$ and $\{R_H^l\}_{l=1}^{L_H}$ consist of open disjoint rectangles whose, respectively, vertical and horizontal edges are parts of the boundary $\partial\Omega$. Triangulation \mathcal{T}_h determines triangulations $\{\pi_V^l\}_{l=1}^{L_V}$ and $\{\pi_H^l\}_{l=1}^{L_H}$ of rectangles $\{R_V^l\}_{l=1}^{L_V}$ and $\{R_H^l\}_{l=1}^{L_H}$, respectively. Let $\pi_V^l = \pi_{V,1}^l \times \pi_{V,2}^l$ for $l = 1, \dots, L_V$, and $\pi_H^l = \pi_{H,1}^l \times \pi_{H,2}^l$ for $l = 1, \dots, L_H$, where one-dimensional partitions $\pi_{V,i}^l$ and $\pi_{H,i}^l$, $i = 1, 2$, consist of subintervals. It follows from (5.1) that, for any set of numbers $\{s_K\}_{K \in \mathcal{T}_h}$,

$$(5.2) \quad \sum_{l=1}^{L_V} \sum_{I_1 \in \pi_{V,1}^l} \sum_{I_2 \in \pi_{V,2}^l} s_{I_1 \times I_2} = \sum_{K \in \mathcal{T}_h} s_K = \sum_{l=1}^{L_H} \sum_{I_1 \in \pi_{H,1}^l} \sum_{I_2 \in \pi_{H,2}^l} s_{I_1 \times I_2}.$$

LEMMA 5.1. For any $v \in X_h$ such that $v = 0$ on $\partial\Omega$,

$$(5.3) \quad \|v_{x_i x_i}\|_h \geq C \|v_{x_i x_i}\|_{L^2(\Omega)}, \quad i = 1, 2,$$

$$(5.4) \quad (v_{x_1 x_1}, v_{x_2 x_2})_h \geq \|v_{x_1 x_2}\|_{L^2(\Omega)}^2.$$

Proof. Take any $v \in X_h$ that vanishes on $\partial\Omega$. Let us prove (5.3) for $i = 1$ since the proof for the other case is similar. Fix $l = 1, \dots, L_H$, let $R_H^l = I_1^l \times I_2^l$, and take any $x_1 \in I_1^l$. Restricted on the vertical line segment $\{x_1\} \times I_2^l$, $v_{x_1 x_1}$ is a piecewise Hermite cubic polynomial function vanishing at the end points of I_2^l . The key relation in proving (5.3) is

$$(5.5) \quad \sum_{I_2 \in \pi_{H,2}^l} \sum_{\mathcal{G}_{I_2}} v_{x_1 x_1}^2(x_1, \cdot) \geq C \sum_{I_2 \in \pi_{H,2}^l} \int_{I_2} v_{x_1 x_1}^2(x_1, x_2) dx_2,$$

which follows directly from (2.6) in [18].

Using (4.3), the second identity in (5.2), the fact that $v_{x_1 x_1}^2$ is a polynomial of degree ≤ 2 in x_1 -variable and the exactness of the two-point Gaussian quadrature, changing orders of summations and the integration, and applying (5.5) and the second identity in (5.2), we obtain

$$\begin{aligned} \|v_{x_1 x_1}\|_h^2 &= \sum_{K \in \mathcal{T}_h} \sum_{\mathcal{G}_K} v_{x_1 x_1}^2 = \sum_{l=1}^{L_H} \sum_{I_1 \in \pi_{H,1}^l} \sum_{I_2 \in \pi_{H,2}^l} \sum_{\mathcal{G}_{I_1}} \sum_{\mathcal{G}_{I_2}} v_{x_1 x_1}^2 \\ &= \sum_{l=1}^{L_H} \sum_{I_1 \in \pi_{H,1}^l} \sum_{I_2 \in \pi_{H,2}^l} \sum_{\mathcal{G}_{I_2}} \int_{I_1} v_{x_1 x_1}^2(x_1, \cdot) dx_1 \\ &\geq C \sum_{l=1}^{L_H} \sum_{I_1 \in \pi_{H,1}^l} \sum_{I_2 \in \pi_{H,2}^l} \int_{I_1 \times I_2} v_{x_1 x_1}^2 dx = C \sum_{K \in \mathcal{T}_h} \int_K v_{x_1 x_1}^2 dx = C \|v_{x_1 x_1}\|_{L^2(\Omega)}^2, \end{aligned}$$

which is (5.3) for $i = 1$.

We now prove (5.4). Using (4.3), the first identity in (5.2), and (4.1) in x_1 -direction, we obtain

$$(5.6) \quad \begin{aligned} (v_{x_1 x_1}, v_{x_2 x_2})_h &= \sum_{K \in \mathcal{T}_h} \sum_{\mathcal{G}_K} v_{x_1 x_1} v_{x_2 x_2} = \sum_{l=1}^{L_V} \sum_{I_1 \in \pi_{V,1}^l} \sum_{I_2 \in \pi_{V,2}^l} \sum_{\mathcal{G}_{I_1}} \sum_{\mathcal{G}_{I_2}} v_{x_1 x_1} v_{x_2 x_2} \\ &= S_1 + C S_2, \end{aligned}$$

where

$$(5.7) \quad S_1 = \sum_{l=1}^{L_V} \sum_{I_1 \in \pi_{V,1}^l} \sum_{I_2 \in \pi_{V,2}^l} \sum_{\mathcal{G}_{I_2}} \int_{I_1} (v_{x_1 x_1} v_{x_2 x_2})(x_1, \cdot) dx_1,$$

$$(5.8) \quad S_2 = - \sum_{l=1}^{L_V} \sum_{I_1 \in \pi_{V,1}^l} |I_1|^5 \sum_{I_2 \in \pi_{V,2}^l} \sum_{\mathcal{G}_{I_2}} (\partial^{(3,0)} v \partial^{(3,2)} v)(t_{I_1}(\cdot), \cdot),$$

with some $\{t_{I_1}(\xi_2)\}_{\xi_2 \in \mathcal{G}_{I_2}}$.

Let us prove

$$(5.9) \quad S_1 \geq \|v_{x_1 x_2}\|_{L^2(\Omega)}^2,$$

$$(5.10) \quad S_2 \geq 0,$$

which, along with (5.6), imply estimate (5.4). First, we obtain (5.9). Applying (4.1) in x_2 -direction on S_1 in (5.7), we get

$$(5.11) \quad S_1 = S_{11} + CS_{12},$$

where

$$(5.12) \quad S_{11} = \sum_{l=1}^{L_V} \sum_{I_1 \in \pi_{V,1}^l} \sum_{I_2 \in \pi_{V,2}^l} \int_{I_1 \times I_2} v_{x_1 x_1} v_{x_2 x_2} dx = \sum_{l=1}^{L_V} \int_{R_V^l} v_{x_1 x_1} v_{x_2 x_2} dx,$$

$$(5.13) \quad S_{12} = - \sum_{l=1}^{L_V} \sum_{I_1 \in \pi_{V,1}^l} \sum_{I_2 \in \pi_{V,2}^l} |I_2|^5 \int_{I_1} (\partial^{(2,3)} v \partial^{(0,3)} v)(x_1, t_{I_2}(x_1)) dx_1,$$

with some $t_{I_2}(x_1) \in I_2$ for any $x_1 \in I_1$. Since function $\partial^{(2,3)} v \partial^{(0,3)} v$ is constant in x_2 -variable on $I_1 \times I_2$, we set

$$(5.14) \quad t_{I_2}(x_1) = t_{I_1 \times I_2} = \text{const}, \quad \text{for all } x_1 \in I_1.$$

From (5.13), using the integration by parts in x_1 -variable, (5.14), continuity in x_1 -variable of $(\partial^{(1,3)} v \partial^{(0,3)} v)$, and the fact that $\partial^{(0,3)} v$ vanishes on the vertical edges of R_V^l , we obtain

$$(5.15) \quad S_{12} = \sum_{l=1}^{L_V} \sum_{I_1 \in \pi_{V,1}^l} \sum_{I_2 \in \pi_{V,2}^l} |I_2|^5 \int_{I_1} (\partial^{(1,3)} v)^2(x_1, t_{I_1 \times I_2}) dx_1 \geq 0.$$

We now prove

$$(5.16) \quad S_{11} = \|v_{x_1 x_2}\|_{L^2(\Omega)}^2,$$

where S_{11} is defined by (5.12). Using the integration by parts in x_1 -variable, continuity of $v_{x_1} v_{x_2 x_2}$ in x_1 -direction, the fact that $v_{x_2 x_2}$ vanishes on the vertical edges of R_V^l , and the first representation of Ω in (5.1), we obtain

$$S_{11} = - \sum_{l=1}^{L_V} \sum_{R_V^l} \int_{R_V^l} v_{x_1} v_{x_1 x_2 x_2} dx = - \int_{\Omega} v_{x_1} v_{x_1 x_2 x_2} dx.$$

Similarly, using the second representation of Ω in (5.1), the integration by parts in x_2 -variable, continuity of $v_{x_1} v_{x_1 x_2}$, and the fact that v_{x_1} vanishes on the horizontal edges of R_H^l , we get (5.16). Relations (5.11), (5.16), and (5.15), imply (5.9).

It remains to prove (5.10), where S_2 is defined by (5.8) and used in (5.6). Since function $(\partial^{(3,0)} v \partial^{(3,2)} v)$ is constant in x_1 -variable on $I_1 \times I_2$, in (5.8), we set

$$(5.17) \quad t_{I_1}(x_2) = t_{I_1 \times I_2} = \text{const}, \quad \text{for all } x_2 \in I_2.$$

From (5.8), using (5.17), (5.2), and (4.1) in x_2 -direction, we obtain

$$(5.18) \quad S_2 = - \sum_{l=1}^{L_H} \sum_{I_1 \in \pi_{H,1}^l} |I_1|^5 \sum_{I_2 \in \pi_{H,2}^l} \sum_{\mathcal{G}_{I_2}} (\partial^{(3,0)} v \partial^{(3,2)} v)(t_{I_1 \times I_2}, \cdot) = S_{21} + CS_{22},$$

where

$$(5.19) \quad S_{21} = - \sum_{l=1}^{L_H} \sum_{I_1 \in \pi_{H,1}^l} |I_1|^5 \sum_{I_2 \in \pi_{H,2}^l} \int_{I_2} (\partial^{(3,0)} v \partial^{(3,2)} v)(t_{I_1 \times I_2}, x_2) dx_2,$$

$$(5.20) \quad S_{22} = \sum_{l=1}^{L_H} \sum_{I_1 \in \pi_{H,1}^l} |I_1|^5 \sum_{I_2 \in \pi_{H,2}^l} |I_2|^5 (\partial^{(3,3)} v)^2|_{I_1 \times I_2} \geq 0.$$

We note that, in (5.20), $\partial^{(3,3)} v$ is constant on $I_1 \times I_2$.

From (5.19), using the integration by parts in x_2 -direction, continuity in x_2 -variable of $(\partial^{(3,0)} v \partial^{(3,1)} v)$, and the fact that $\partial^{(3,0)}$ vanishes on the horizontal boundaries of R_H^l , we obtain

$$(5.21) \quad S_{21} = \sum_{l=1}^{L_H} \sum_{I_1 \in \pi_{H,1}^l} |I_1|^5 \sum_{I_2 \in \pi_{H,2}^l} \int_{I_2} (\partial^{(3,1)} v)^2(t_{I_1 \times I_2}, x_2) dx_2 \geq 0.$$

Identity (5.18) and the inequalities (5.21) and (5.20) imply (5.10). \square

LEMMA 5.2.

$$(5.22) \quad \|\partial^\alpha v\|_h \leq C \|\partial^\alpha v\|_{L^2(\Omega)}, \quad v \in X_h, \quad |\alpha| \leq 2.$$

Proof. Take any $v \in X_h$. Applying the inverse inequality

$$\|w\|_{C(K)} \leq Ch^{-1} \|w\|_{L^2(K)}, \quad K \in \mathcal{T}_h, \quad w \in X_h,$$

(see (3.2.33) in [11]) and the inclusion $X_h \subset H^2(\Omega)$, we obtain

$$\|\partial^\alpha v\|_h^2 = \sum_{K \in \mathcal{T}_h} \frac{|K|}{4} \sum_{\xi \in \mathcal{G}_K} |(\partial^\alpha v)(\xi)|^2 \leq C \sum_{K \in \mathcal{T}_h} \|\partial^\alpha v\|_{L^2(K)}^2 = C \|\partial^\alpha v\|_{L^2(\Omega)}^2. \quad \square$$

The following lemma states that the approximate bilinear form $a_h(\cdot, \cdot)$ is uniformly V -elliptic and bounded relative to the H^2 -norm.

LEMMA 5.3.

$$(5.23) \quad C \|v\|_{H^2(\Omega)}^2 \leq a_h(v, v), \quad v \in V_h,$$

$$(5.24) \quad |a_h(v, w)| \leq C |v|_{H^2(\Omega)} |w|_{H^2(\Omega)}, \quad v, w \in V_h.$$

Proof. Inequality (5.23) follows from the representation

$$a_h(v, v) = \|v_{x_1 x_1}\|_h^2 + 2(v_{x_1 x_1}, v_{x_2 x_2})_h + \|v_{x_2 x_2}\|_h^2, \quad v \in V_h,$$

Lemma 5.1, and the fact that $|\cdot|_{H^2(\Omega)}$ is equivalent to $\|\cdot\|_{H^2(\Omega)}$ on $H_0^2(\Omega)$. Using (4.5), (4.4), the Cauchy-Schwarz inequality in R^4 , and (5.22), we obtain, for any $v, w \in V_h$,

$$\begin{aligned} |a_h(v, w)| &\leq \sum_{i,j=1}^2 \|v_{x_i x_i}\|_h \|w_{x_j x_j}\|_h \\ &\leq 2(\|v_{x_1 x_1}\|_h^2 + \|v_{x_2 x_2}\|_h^2)^{1/2} (\|w_{x_1 x_1}\|_h^2 + \|w_{x_2 x_2}\|_h^2)^{1/2} \\ &\leq C|v|_{H^2(\Omega)}|w|_{H^2(\Omega)}, \end{aligned}$$

which is (5.24). \square

THEOREM 5.4. *For any $f \in C(\Omega)$, the quadrature finite element Galerkin scheme (4.6) has a unique solution.*

Proof. It follows from (4.5) and Lemma 5.3 that V_h is a Hilbert space with the inner product $a_h(\cdot, \cdot)$. A linear functional $(f, \cdot)_h$ is bounded on V_h . Therefore, the statement of the theorem follows from the Riesz Representation Theorem. \square

6. An H^2 -norm error estimate. In this section, we prove that an error analysis of the quadrature scheme based on the First Strang Lemma (Theorem 4.1.1 in [11]) gives an H^2 -norm error estimate with suboptimal order $O(h)$.

To begin with, we introduce a piecewise Hermite bicubic interpolant and state some of its properties. For $K \in \mathcal{T}_h$, let $\{a_i\}_{i=1}^4$ be the vertices of K . For any $v \in C^2(K)$, let $\Pi_K v \in Q_3(K)$ be the bicubic Hermite interpolant of v defined by the following relations:

$$(6.1) \quad \partial^\alpha(\Pi_K v)(a_i) = \partial^\alpha v(a_i), \quad 1 \leq i \leq 4, \quad |\alpha_j| \leq 1, \quad j = 1, 2.$$

For $v \in C^2(\Omega)$, let $\Pi_h v \in X_h$ be the piecewise Hermite bicubic interpolant of v defined by

$$(6.2) \quad (\Pi_h v)|_K = \Pi_K(v|_K), \quad \text{for all } K \in \mathcal{T}_h.$$

Theorem 3.1.6 in [11] (also, see estimate (6.1.7)) implies

$$(6.3) \quad \|v - \Pi_K v\|_{H^m(K)} \leq Ch^{4-m}|v|_{H^4(K)}, \quad 0 \leq m \leq 4, \quad K \in \mathcal{T}_h, \quad v \in H^4(\Omega).$$

Using the triangle inequality and (6.3), it is easy to obtain

$$(6.4) \quad \sum_{K \in \mathcal{T}_h} \|\Pi_K v\|_{H^m(K)}^2 \leq C\|v\|_{H^4(\Omega)}^2, \quad 0 \leq m \leq 4, \quad v \in H^4(\Omega).$$

In the following lemma, we prove that the bilinear form $a_h(\cdot, \cdot)$ approximates $a(\cdot, \cdot)$ with order $O(h)$.

LEMMA 6.1.

$$(6.5) \quad |a(\Pi_h u, w) - a_h(\Pi_h u, w)| \leq Ch\|u\|_{H^4(\Omega)}|w|_{H^2(\Omega)}, \quad u \in H^4(\Omega), \quad w \in V_h.$$

Proof. Take any v and w in V_h . Using (2.2), (4.5), and (4.3), we obtain

$$(6.6) \quad a(v, w) - a_h(v, w) = \sum_{K \in \mathcal{T}_h} E_K(\Delta v \Delta w),$$

where

$$(6.7) \quad E_K(g) = \int_K g(x) dx - \sum_{\mathcal{G}_K} g, \quad g \in C(K).$$

Take any $K \in \mathcal{T}_h$ and let $F : \hat{K} \rightarrow K$ be an invertible affine mapping, where $\hat{K} = (0, 1)^2$. Changing the variables with the transformation F , we obtain

$$(6.8) \quad |E_K(\Delta v \Delta w)| \leq Ch^{-2} |E_{\hat{K}}(\Delta \hat{v} \Delta \hat{w})|,$$

where $\hat{v} = v|_K \circ F$ and $\hat{w} = w|_K \circ F$.

Consider a linear functional

$$(6.9) \quad l(\hat{\phi}) = E_{\hat{K}}(\hat{\phi} \Delta \hat{w}), \quad \hat{\phi} \in C^1(\hat{K}).$$

Since $\hat{w} \in Q_3(\hat{K})$ (and, hence, $\Delta \hat{w} \in Q_3(\hat{K})$), using the equivalence of norms $\|\cdot\|_{C(\hat{K})}$ and $\|\cdot\|_{L^2(\hat{K})}$ on the finite-dimensional vector space $Q_3(\hat{K})$, we get, for any $\hat{\phi} \in C^1(\hat{K})$,

$$|l(\hat{\phi})| \leq C \|\hat{\phi}\|_{C(\hat{K})} \|\Delta \hat{w}\|_{C(\hat{K})} \leq (C \|\Delta \hat{w}\|_{L^2(\hat{K})}) \|\hat{\phi}\|_{C(\hat{K})} \leq (C |\hat{w}|_{H^2(\hat{K})}) \|\hat{\phi}\|_{C^1(\hat{K})};$$

that is, the linear functional $l(\cdot)$ is bounded on $C^1(\hat{K})$. Since the product two-point Gaussian quadrature is exact for polynomials in Q_3 and $\Delta \hat{w} \in Q_3(\hat{K})$, the linear functional $l(\cdot)$ vanishes on polynomials of degree zero. Applying the Bramble-Hilbert Lemma (Theorem 4.1.3 in [11]), we obtain

$$|l(\hat{\phi})| \leq \left(C |\hat{w}|_{H^2(\hat{K})} \right) |\hat{\phi}|_{C^1(\hat{K})}.$$

In the last estimate, replacing $\hat{\phi}$ by $\Delta \hat{v} \in Q_3(\hat{K})$, using (6.9) and the norm equivalence in $Q_3(\hat{K})$, we get

$$(6.10) \quad |E_{\hat{K}}(\Delta \hat{v} \Delta \hat{w})| \leq C |\hat{w}|_{H^2(\hat{K})} |\Delta \hat{v}|_{C^1(\hat{K})} \leq \left(C |\hat{w}|_{H^2(\hat{K})} \right) |\hat{v}|_{H^3(\hat{K})}.$$

From (6.8), using (6.10) and changing the variables with the transformation F^{-1} , we obtain

$$|E_K(\Delta v \Delta w)| \leq Ch^{-2} (h^2 h^{-1} |w|_{H^2(K)}) (h^3 h^{-1} |v|_{H^3(K)}) = Ch |v|_{H^3(K)} |w|_{H^2(K)}.$$

Substituting this estimate in (6.6) and using the Cauchy-Schwarz inequality, we get

$$|a(v, w) - a_h(v, w)| \leq Ch \sum_{K \in \mathcal{T}_h} |v|_{H^3(K)} |w|_{H^2(K)} \leq Ch \left(\sum_{K \in \mathcal{T}_h} |v|_{H^3(K)}^2 \right)^{1/2} |w|_{H^2(\Omega)}.$$

Finally, in the last estimate, replacing v by $\Pi_h u$ and using (6.4) with $m = 3$, we obtain estimate (6.5). \square

A similar analysis with a numerical integration based on the three-point Gaussian quadrature gives an optimal consistence error estimate of order $O(h^2)$. The following result gives the consistency error estimate of the approximate linear form $(f, \cdot)_h$ defined by (4.3) and (4.2).

LEMMA 6.2. *If $f \in C^2(\Omega)$ then*

$$(6.11) \quad |(f, v) - (f, v)_h| \leq Ch^2 \|f\|_{C^2(\Omega)} \|v\|_{H^1(\Omega)}, \quad v \in V_h.$$

Proof. The proof is similar to that of Theorem 4.1.5 in [11]. Take any $v \in V_h$. Using (2.2), (4.3), (4.2), and (6.7), we get

$$(6.12) \quad (f, v) - (f, v)_h = \sum_{K \in \mathcal{T}_h} E_K(fv).$$

Take any $K \in \mathcal{T}_h$, and let $F : \hat{K} = (0, 1)^2 \rightarrow K$ be an invertible affine mapping. Changing variables by the transformation F , we obtain

$$(6.13) \quad E_K(fv) = h^2 E_{\hat{K}}(\hat{f}\hat{v}),$$

where $\hat{f} = f|_K \circ F \in C^2(\hat{K})$ and $\hat{v} = v|_K \circ F \in Q_3(\hat{K})$. Linearity of $E_{\hat{K}}(\cdot)$ implies

$$(6.14) \quad E_{\hat{K}}(\hat{f}\hat{v}) = \hat{v}_a \hat{E}(\hat{f}) + \hat{E}(\hat{f}(\hat{v} - \hat{v}_a)),$$

where $\hat{v}_a = \int_{\hat{K}} \hat{v}(s_1, s_2) ds_1 ds_2$ is the average value of \hat{v} on \hat{K} . Using the Cauchy-Schwarz inequality on $L^2(\hat{K})$, we obtain

$$(6.15) \quad |\hat{v}_a| \leq \|\hat{v}\|_{L^2(\hat{K})}.$$

Applying the Bramble-Hilbert Lemma and using the norm equivalence on $Q(\hat{K})$, we get

$$(6.16) \quad \|\hat{v} - \hat{v}_a\|_{C(\hat{K})} \leq C|\hat{v}|_{C^1(\hat{K})} \leq C|\hat{v}|_{H^1(\hat{K})}.$$

Let us estimate the first term in (6.14). A linear functional

$$(6.17) \quad l(\hat{\phi}) = \hat{v}_a E_{\hat{K}}(\hat{\phi}), \quad \hat{\phi} \in C^2(\hat{K}),$$

is bounded on $C^2(\hat{K})$ since, by (6.15),

$$|l(\hat{\phi})| \leq C|\hat{v}_a| \|\hat{\phi}\|_{C(\hat{K})} \leq (C\|\hat{v}\|_{L^2(\hat{K})}) \|\hat{\phi}\|_{C^2(\hat{K})}, \quad \hat{\phi} \in C^2(\hat{K}),$$

and $l(\cdot)$ vanishes on polynomials of degree ≤ 1 . Applying the Bramble-Hilbert Lemma, we obtain

$$|l(\hat{\phi})| \leq C\|\hat{v}\|_{L^2(\hat{K})} |\hat{\phi}|_{C^2(\hat{K})}, \quad \hat{\phi} \in C^2(\hat{K}).$$

In the last estimate, replacing $\hat{\phi}$ by \hat{f} and using (6.17), we obtain

$$(6.18) \quad |\hat{v}_a E_{\hat{K}}(\hat{f})| \leq C|\hat{f}|_{C^2(\hat{K})} \|\hat{v}\|_{L^2(\hat{K})}.$$

Let us estimate the second term in (6.14). Consider now a linear functional

$$(6.19) \quad l(\hat{\phi}) = E_{\hat{K}}(\hat{\phi}(\hat{v} - \hat{v}_a)), \quad \hat{\phi} \in C^1(\hat{K}).$$

Using (6.16), we obtain boundedness of $l(\cdot)$:

$$|l(\hat{\phi})| \leq \|\hat{\phi}\|_{C(\hat{K})} \|\hat{v} - \hat{v}_a\|_{C(\hat{K})} \leq (C|\hat{v}|_{H^1(\hat{K})}) \|\hat{\phi}\|_{C^1(\hat{K})}, \quad \hat{\phi} \in C^1(\hat{K}).$$

Since $\hat{v} - \hat{v}_a \in Q_3(\hat{K})$, functional $l(\cdot)$ vanishes on polynomials of degree zero. Applying the Bramble-Hilbert Lemma, we get

$$|l(\hat{\phi})| \leq C|\hat{v}|_{H^1(\hat{K})} |\hat{\phi}|_{C^1(\hat{K})}.$$

In the last estimate, replacing $\hat{\phi}$ by \hat{f} and using (6.19), we obtain

$$(6.20) \quad |E_{\hat{K}}(\hat{f}(\hat{v} - \hat{v}_a))| \leq C|\hat{f}|_{C^1(\hat{K})} |\hat{v}|_{H^1(\hat{K})}.$$

Relations (6.14), (6.18), and (6.20) imply

$$|E_{\hat{K}}(\hat{f}\hat{v})| \leq C|\hat{f}|_{C^2(\hat{K})} \|\hat{v}\|_{L^2(\hat{K})} + C|\hat{f}|_{C^1(\hat{K})} |\hat{v}|_{H^1(\hat{K})}.$$

Changing variables by the transformation F^{-1} , we get

$$(6.21) \quad \begin{aligned} |\hat{E}(\hat{f}\hat{v})| &\leq Ch^2|f|_{C^2(K)}h^{-1}\|v\|_{L^2(K)} + Ch|f|_{C^1(K)}|v|_{H^1(K)} \\ &\leq Ch\|f\|_{C^2(K)}\|v\|_{H^1(K)}. \end{aligned}$$

From (6.12), (6.13), and (6.21), using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |(f, v) - (f, v)_h| &\leq Ch^3 \sum_{K \in \mathcal{T}_h} \|f\|_{C^2(K)} \|v\|_{H^1(K)} \leq Ch^3 \|f\|_{C^2(\Omega)} \sum_{K \in \mathcal{T}_h} \|v\|_{H^1(K)} \\ &\leq Ch^3 \|f\|_{C^2(\Omega)} \|v\|_{H^1(\Omega)} \left(\sum_{K \in \mathcal{T}_h} 1 \right)^{1/2} \leq Ch^2 \|f\|_{C^2(\Omega)} \|v\|_{H^1(\Omega)}, \end{aligned}$$

which is (6.11). \square

THEOREM 6.3. *Let u and u_h be the solutions of problems (2.1) and (4.6), respectively. If $u \in H^4(\Omega)$ and $f \in C^2(\Omega)$ then*

$$(6.22) \quad \|u - u_h\|_{H^2(\Omega)} \leq Ch^2 \|f\|_{C^2(\Omega)} + Ch \|u\|_{H^4(\Omega)}.$$

Proof. The statement follows from the First Strang Lemma (Theorem 4.1.1 in [11]), V -ellipticity and boundedness of the approximate bilinear form $a_h(\cdot, \cdot)$ proved in Lemma 5.3, and the consistency results obtained in Lemma 6.1 and Lemma 6.2. \square

The error estimate (6.22) has suboptimal order $O(h)$. In section 8, using an auxiliary orthogonal spline collocation problem, we obtain optimal order error estimates in H^1 - and H^2 -norms under higher than optimal regularity assumptions on the solution of the variational problem. An analysis in [17] shows that the solution of the quadrature scheme based on the three-point Gaussian quadrature has an optimal H^2 -norm error estimate (see Theorem 8.9).

7. Additional auxiliary results. In this section, we obtain auxiliary results that will be used to analyze an auxiliary orthogonal spline collocation scheme for problem (2.1). The following lemma is a generalization of Lemma 4.2 in [6].

LEMMA 7.1. *If $v \in H^4(\Omega)$ then*

$$(7.1) \quad \|\partial^\alpha(v - \Pi_h v)\|_h \leq Ch^{4-|\alpha|} \|v\|_{H^4(\Omega)}, \quad |\alpha| \leq 2.$$

Moreover, if $v \in H^5(\Omega)$ then

$$(7.2) \quad \|\partial^\alpha(v - \Pi_h v)\|_h \leq Ch^3 \|v\|_{H^5(\Omega)}, \quad |\alpha| = 2.$$

Proof. We prove (7.1) first. Take any $v \in H^4(\Omega)$ and let α be a multi-index such that $|\alpha| \leq 2$. By (4.3),

$$(7.3) \quad \|\partial^\alpha(v - \Pi_h v)\|_h^2 = \sum_{K \in \mathcal{T}_h} \sum_{\mathcal{G}_K} [\partial^\alpha(v - \Pi_K v)]^2.$$

Take any $K \in \mathcal{T}_h$, let $F : \hat{K} = (0, 1)^2 \rightarrow K$ be an invertible affine mapping, and let $\hat{v} = v|_K \circ F$. Changing variables by the transformation F , we obtain

$$(7.4) \quad \sum_{\mathcal{G}_K} |\partial^\alpha(v - \Pi_K v)|^2 \leq Ch^{2-2|\alpha|} \sum_{\xi \in \mathcal{G}_{\hat{K}}} |l_\xi(\hat{v})|^2,$$

where, for any $\xi \in \mathcal{G}_{\hat{K}}$,

$$(7.5) \quad l_\xi(\hat{w}) = \partial^\alpha(\hat{w} - \Pi_{\hat{K}} \hat{w})(\xi), \quad \text{for all } \hat{w} \in C^2(\hat{K}).$$

Let us show that

$$(7.6) \quad |l_\xi(\hat{v})| \leq C|\hat{v}|_{H^4(\hat{K})}, \quad \xi \in \mathcal{G}_{\hat{K}}.$$

Take $\xi \in \mathcal{G}_{\hat{K}}$. Linear functional $l_\xi(\cdot)$ is bounded on $H^4(\hat{K})$ since $H^4(\hat{K}) \subset C^2(\hat{K})$ and $|\alpha| \leq 2$, and it vanishes on polynomials of degree ≤ 3 . Applying the Bramble-Hilbert Lemma, we obtain

$$|l_\xi(\hat{w})| \leq C|\hat{w}|_{H^4(\hat{K})}, \quad \hat{w} \in H^4(\hat{K}),$$

which implies (7.6).

Substituting (7.6) in (7.4) and changing the variables by the transformation F^{-1} , we get

$$\sum_{\mathcal{G}_K} |\partial^\alpha(v - \Pi_K v)|^2 \leq Ch^{2(4-|\alpha|)} |v|_{H^4(K)}^2.$$

The last estimate along with (7.3) gives

$$\|\partial^\alpha(v - \Pi_h v)\|_h^2 \leq Ch^{2(4-|\alpha|)} \sum_{K \in \mathcal{T}_h} |v|_{H^4(K)}^2 = Ch^{2(4-|\alpha|)} |v|_{H^4(\Omega)}^2,$$

which implies (7.1). Similarly, we obtain estimate (7.2) using the fact that $l_\xi(\cdot)$ in (7.5) vanishes on polynomials of degree ≤ 4 when $|\alpha| = 2$. \square

The following is a discrete form of Green's formula.

LEMMA 7.2.

$$(7.7) \quad (\Delta v, w)_h = (v, \Delta w)_h, \quad v \in V_h, \quad w \in X_h.$$

Proof. Take any $v \in V_h$ and $w \in X_h$. We prove

$$(7.8) \quad (v_{x_1 x_1}, w)_h = (v, w_{x_1 x_1})_h,$$

and the proof of $(v_{x_2 x_2}, w)_h = (v, w_{x_2 x_2})_h$ is similar.

Using (4.3), (4.2), (4.1) in x_1 -direction, the integration by parts in x_1 -variable, continuity of $v_{x_1}w$, and the fact $v_{x_1}|_{\partial\Omega} = 0$, we get

$$(7.9) \quad \begin{aligned} (v_{x_1x_1}, w)_h &= \sum_{I_1 \times I_2 \in \mathcal{T}_h} \sum_{\mathcal{G}_{I_2}} \sum_{\mathcal{G}_{I_1}} v_{x_1x_1} w \\ &= \sum_{I_1 \times I_2 \in \mathcal{T}_h} \sum_{\mathcal{G}_{I_2}} \left(\int_{I_1} v_{x_1x_1} w(x_1, \cdot) dx_1 - C|I_1|^5 (\partial^{(3,0)} v \partial^{(3,0)} w)(t_{I_1 \times I_2}, \cdot) \right) = S, \end{aligned}$$

where

$$S = - \sum_{I_1 \times I_2 \in \mathcal{T}_h} \left(\sum_{\mathcal{G}_{I_2}} \int_{I_1} v_{x_1} w_{x_1}(x_1, \cdot) dx_1 + C|I_1|^5 (\partial^{(3,0)} v \partial^{(3,0)} w)(t_{I_1 \times I_2}, \cdot) \right).$$

Similarly, using continuity of $w_{x_1}v$, $v|_{\partial\Omega} = 0$, and the fact that $(\partial^{(3,0)} v \partial^{(3,0)} w)|_{I_1 \times I_2}$ is constant in x_1 -variable, we obtain $(w_{x_1x_1}, v)_h = S$, which, along with (7.9), implies (7.8). \square

The following lemma states that a bicubic polynomial p is uniquely determined by the values of p and Δp at the Gauss points in a square and the values of p at any two connected vertices of the square.

LEMMA 7.3. *Let $\hat{K} = (-1, 1)^2$, and let η_1 and η_2 be connected vertices of \hat{K} . Let $p(x_1, x_2)$ be a bicubic polynomial such that*

$$(7.10) \quad \begin{aligned} p(\xi) = \Delta p(\xi) &= 0, \quad \text{for all } \xi \in \mathcal{G}_{\hat{K}}, \\ p(\eta_i) = p_{x_1}(\eta_i) = p_{x_2}(\eta_i) &= p_{x_1x_2}(\eta_i) = 0, \quad i = 1, 2. \end{aligned}$$

Then, $p = 0$.

Proof. Let the following set of 16 collocation points

$$\{\mathcal{G}_{\hat{K}}, \mathcal{G}_{\hat{K}}, \eta_1, \eta_1, \eta_1, \eta_1, \eta_2, \eta_2, \eta_2, \eta_2\}$$

and the set of the coefficients of the polynomial p be ordered in a certain way, and let $M \in R^{16 \times 16}$ be the matrix corresponding to the equations in (7.10). Using a computer algebra system, we computed

$$\det(M) = \pm 268435456/59049,$$

which implies $p = 0$. \square

We note that, if the vertices η_1 and η_2 are not connected, then matrix M is singular.

8. An equivalent OSC problem. In this section, we introduce an auxiliary orthogonal spline collocation scheme, determine its relation with the quadrature scheme, and obtain optimal order error estimates. As in [16], we consider the following coupled form of problem (1.1):

$$(8.1) \quad \Delta u = v, \quad \Delta v = f \quad \text{in } \Omega, \quad \text{and } u = \partial_n u = 0 \quad \text{on } \partial\Omega.$$

Let $\mathcal{G}_h = \cup_{K \in \mathcal{T}_h} \mathcal{G}_K$ be the set of all Gauss points in Ω . We note that the number of points in \mathcal{G}_h is greater than the dimension of V_h since the number of elements in \mathcal{T}_h is greater than the number of interior vertices of \mathcal{T}_h . We consider the following

orthogonal spline collocation scheme for problem (8.1): find $u_h \in V_h$ and $v_h \in X_h$, such that

$$(8.2a) \quad \Delta u_h(\xi) = v_h(\xi), \quad \text{for all } \xi \in \mathcal{G}_h,$$

$$(8.2b) \quad \Delta v_h(\xi) = f(\xi), \quad \text{for all } \xi \in \mathcal{G}_h.$$

LEMMA 8.1. *Let $f \in C(\Omega)$ and let $\{\eta_1, \eta_2\}$ be connected vertices of any element $K_\eta \in \mathcal{T}_h$. The system of OSC equations (8.2) is under-determined by eight constraints. The system (8.2) with the additional equations*

$$(8.3) \quad v_h(\eta_i) = (v_h)_{x_1}(\eta_i) = (v_h)_{x_2}(\eta_i) = (v_h)_{x_1 x_2}(\eta_i) = 0, \quad i = 1, 2,$$

has a unique solution $\{u_h, v_h\}$, where u_h is the solution of the quadrature scheme (4.6).

Proof. Let us prove that the OSC system (8.2) is under-determined by eight constraints. Let N_e , N_i , and N_b denote, respectively, the numbers of elements, internal nodes, and boundary nodes in the triangulation \mathcal{T}_h . The set \mathcal{G}_h has $4N_e$ points; hence, there are $8N_e$ equations in (8.2). Using a mathematical induction argument, we verified that $N_b = 2(N_e - N_i + 1)$. Thus, the system (8.2) has

$$8N_e = 8N_i + 4N_b - 8 \quad \text{constraints.}$$

Since $u_h \in V_h$ and $v_h \in X_h$, the system (8.2) has

$$4N_i + 4(N_i + N_b) = 8N_i + 4N_b \quad \text{degrees of freedom;}$$

that is, the OSC system (8.2) is under-determined by eight constraints.

Let us prove that the problem (8.2)–(8.3) has a unique solution $\{u_h, v_h\}$. Since the numbers of degrees of freedom and constraints are equal, it suffices to show that the homogeneous system consisting of the equations (8.2a),

$$(8.4) \quad \Delta v_h(\xi) = 0, \quad \text{for all } \xi \in \mathcal{G}_h,$$

and (8.3) has only the zero solution. Let u_h and v_h be solutions of $\{(8.2a), (8.4), (8.3)\}$. Using (4.5), (8.2a), Lemma 7.2, and (8.4), we obtain

$$a_h(u_h, u_h) = (\Delta u_h, \Delta u_h)_h = (v_h, \Delta u_h)_h = (\Delta v_h, u_h)_h = 0.$$

Hence, by (5.23), we obtain $u_h = 0$.

Let us prove that $v_h = 0$. Using (8.2a) with $u_h = 0$, (8.4), (8.3), and Lemma 7.3, we obtain $v_h|_{K_\eta} = 0$. By a similar argument, $v_h|_K = 0$ for any element $K \in \mathcal{T}_h$ adjacent to K_η . By recursion, $v_h|_K = 0$ for any $K \in \mathcal{T}_h$, and, hence, $v_h = 0$.

Thus, problem (8.2)–(8.3) has a unique solution $\{u_h, v_h\}$. Using (4.5), (8.2a), Lemma 7.2, and (8.2b), we obtain, for any $w \in V_h$,

$$a_h(u_h, w) = (\Delta u_h, \Delta w)_h = (v_h, \Delta w)_h = (\Delta v_h, w)_h = (f, w)_h,$$

that is, u_h is a solution of problem (4.6), which is unique by Theorem 5.4. \square

In [16], for $\Omega = (0, 1)^2$, problem (8.1) was approximated by the OSC scheme (8.2) with additional eight constraints

$$(8.5) \quad v_h(a, b) = \frac{\partial v_h}{\partial x_2}(a, b) = 0, \quad \text{for } a, b = 0, 1.$$

The authors proved existence and uniqueness of the solution and obtained optimal order error estimates in the Sobolev H^k -norms for $k = 0, 1, 2$. In the following theorem, we state optimal order convergence estimates for the quadrature scheme (4.6) in the H^1 - and H^2 -norms.

THEOREM 8.2. *Let $f \in C(\Omega)$ and let u and u_h be the solutions of the problems (2.1) and (4.6), respectively. If $u \in H^{s-k}(\Omega)$, then*

$$(8.6) \quad \|u - u_h\|_{H^k(\Omega)} \leq Ch^{4-k} \|u\|_{H^{s-k}(\Omega)}, \quad k = 1, 2.$$

Proof. It follows from Theorem 5.4 and Lemma 8.1, that the quadrature scheme (4.6) has a unique solution u_h , which is also the solution of the OSC scheme (8.2)–(8.3). The proof of (8.6) is similar to that of Theorem 3.1 in [16]. Since $u \in H^6(\Omega)$, u and $v = \Delta u \in C^2(\Omega)$ are solutions of the coupled problem (8.1). Let

$$(8.7) \quad U = u_h - \Pi_h u \quad \text{and} \quad V = v_h - \Pi_h v,$$

where v_h is the solution of the OSC scheme (8.2)–(8.3). We note that

$$(8.8) \quad U \in V_h \quad \text{and} \quad V \in X_h.$$

It follows from (8.2a) and $\Delta u = v$ that

$$(8.9) \quad \begin{aligned} \Delta U(\xi) - V(\xi) &= -\Delta(\Pi_h u)(\xi) + (\Pi_h v)(\xi) \\ &= \Delta(u - \Pi_h u)(\xi) - (v - \Pi_h v)(\xi), \quad \text{for any } \xi \in \mathcal{G}_h. \end{aligned}$$

Similarly, using (8.2b) and $\Delta v = f$, we obtain

$$(8.10) \quad \Delta V(\xi) = \Delta(v - \Pi_h v)(\xi), \quad \text{for any } \xi \in \mathcal{G}_h.$$

Taking the product $(\cdot, \cdot)_h$ of (8.9) and (8.10) with ΔU and U , respectively, we obtain

$$\begin{aligned} \|\Delta U\|_h^2 - (V, \Delta U)_h &= (\Delta(u - \Pi_h u), \Delta U)_h - (v - \Pi_h v, \Delta U)_h, \\ (\Delta V, U)_h &= (\Delta(v - \Pi_h v), U)_h. \end{aligned}$$

Summing the last two identities and using (8.8) and Lemma 7.2, we obtain

$$\|\Delta U\|_h^2 = (\Delta(u - \Pi_h u), \Delta U)_h - (v - \Pi_h v, \Delta U)_h + (\Delta(v - \Pi_h v), U)_h.$$

Using the Cauchy-Schwarz inequality for $(\cdot, \cdot)_h$, estimate (5.22) with $\alpha = (0, 0)$ and (5.23), we get

$$(8.11) \quad \|\Delta U\|_h \leq \|\Delta(u - \Pi_h u)\|_h + \|v - \Pi_h v\|_h + C\|\Delta(v - \Pi_h v)\|_h.$$

Using (5.23), (8.11), Lemma 7.1, and $v = \Delta u$, we obtain

$$(8.12) \quad \begin{aligned} \|U\|_{H^2(\Omega)} &\leq C\|\Delta U\|_h \leq Ch^{4-k} \|u\|_{H^{6-k}(\Omega)} + Ch^4 \|v\|_{H^4(\Omega)} + Ch^{4-k} \|v\|_{H^{6-k}(\Omega)} \\ &\leq Ch^{4-k} \|u\|_{H^{s-k}(\Omega)}, \quad k = 1, 2. \end{aligned}$$

Finally, using the triangle inequality for the H^k -norm, (6.3) with $m = 1, 2$, (8.7), and (8.12), we get

$$\|u - u_h\|_{H^k(\Omega)} \leq \|u - \Pi_h u\|_{H^k(\Omega)} + \|\Pi_h u - u_h\|_{H^2(\Omega)} \leq Ch^{4-k} \|u\|_{H^{s-k}(\Omega)}, \quad k = 1, 2,$$

which is (8.6). \square

TABLE 9.1
Maximum nodal errors and approximate convergence orders.

h	e_h		$(e_h)_{x_1}$		$(e_h)_{x_2}$		$(e_h)_{x_1x_2}$	
1/8	1.3E-3	4.3	7.8E-4	4.6	7.8E-4	4.6	1.7E-2	4.4
1/16	7.9E-5	4.1	4.2E-5	4.2	4.2E-5	4.2	9.4E-4	4.1
1/32	4.9E-6	4.0	2.5E-6	4.1	2.5E-6	4.1	5.8E-5	4.0
1/64	3.0E-7	4.0	1.5E-7	4.0	1.6E-7	4.0	3.6E-6	4.0
1/128	1.1E-8	4.8	3.3E-8	2.2	3.5E-8	2.1	3.1E-7	3.6

TABLE 9.2
Sobolev norm errors and convergence orders.

h	L^2		H^1		H^2	
1/8	6.9E-4	4.1	2.2E-2	3.1	1.1E+0	2.0
1/16	4.2E-5	4.0	2.7E-3	3.0	2.8E-1	2.0
1/32	2.6E-6	4.0	3.4E-4	3.0	6.9E-2	2.0
1/64	1.6E-7	4.0	4.2E-5	3.0	1.7E-2	2.0
1/128	1.1E-8	3.9	5.2E-6	3.0	4.3E-3	2.0

9. Numerical results. In this section, we present numerical results for a test problem, which was used by several authors, in particular, as Example 1 in [5] and as Problem 2 in [2]. A test problem is considered on $\Omega = (0, 1)^2$ with the exact solution $u(x) = (1 - \cos 2\pi x_1)(1 - \cos 2\pi x_2)$. We carried out a series of computations for decreasing values of h obtained by halving to determine convergence orders of the numerical solution. Linear systems were solved by an LU decomposition.

Let \tilde{e}_h be an error defined on the set N_h of the interior nodes of the triangulation \mathcal{T}_h . Let $\|\tilde{e}\|_{C_h} = \max_{x \in N_h} |\tilde{e}_h(x)|$ be the maximal nodal error of \tilde{e}_h . If $\|\tilde{e}_h\|_{C_h} = O(h^p)$ for $p > 0$, then the quantity $\log_2(\|\tilde{e}_{2h}\|_{C_{2h}}/\|\tilde{e}_h\|_{C_h})$, which we call an approximate convergence order, approximates p .

In Table 9.1, we present the maximum nodal errors and the corresponding approximate convergence orders of the quadrature Galerkin solution and its first-order and mixed second-order derivatives. We observe that all approximate convergence orders are close to 4. The higher convergence orders for the derivatives indicate to a superconvergence property of the quadrature solution. The approximate convergence orders for $h = 1/128$ decrease due to the effect of round-off errors. We note that the condition number of the stiffness matrix is of order $O(h^{-4})$, and for $h = 1/128$, the condition number $\approx 10^7$.

In Table 9.2, we present errors in Sobolev norms and their approximate convergence orders. The approximate convergence orders of the error in the H^k -norm, $k = 0, 1, 2$, are close to the optimal values 4, 3, and 2, respectively. We note that, for $h = 1/128$, in contrast with the maximum nodal norm, the convergence order corresponding to the L^2 -norm deteriorated insignificantly due to the average nature of the norm.

10. Conclusion. A quadrature finite element Galerkin scheme for a biharmonic Dirichlet problem on a rectangular polygonal domain is proposed and analyzed. A finite element theory suggests to use a Gaussian quadrature with at least three points. Our quadrature scheme uses a numerical integration based on the product two-point Gaussian quadrature and the finite element space of Bogner-Fox-Schmit rectangles. Forming a stiffness matrix for the proposed scheme is faster since fewer function evalu-

ations are required. The scheme has a unique solution, which converges to the solution of the boundary value problem with optimal rates in the Sobolev H^1 - and H^2 -norms. The solution of the quadrature scheme is also the solution of an equivalent orthogonal spline collocation problem. Numerical results confirm our theoretical estimates and demonstrate a superconvergence property of the quadrature Galerkin solution.

REFERENCES

- [1] R. ADAMS AND J. F. FOURNIER, *Sobolev Spaces*, Academic Press, New York, 2003.
- [2] I. ALTAS, J. DYM, M. M. GUPTA, AND R. P. MANOHAR, *Multigrid solution of automatically generated high-order discretizations for the biharmonic equation*, SIAM J. Sci. Comput., 19 (1998), pp. 1575–1585.
- [3] C. BACUTA, J. H. BRAMBLE, AND J. E. PASCIAK, *Shift theorems for the biharmonic Dirichlet problem*, in Recent progress in computational and applied PDEs (Zhangjiajie, 2001), Kluwer/Plenum, New York, 2002, pp. 1–26.
- [4] J. BERGH AND J. LÖFSTRÖM, *Interpolation Spaces, an Introduction*, Springer-Verlag, New York, 1976.
- [5] B. BIALECKI, *A fast solver for the orthogonal spline collocation solution of the biharmonic Dirichlet problem on rectangles*, J. Comp. Physics, 191 (2003), pp. 601–621.
- [6] B. BIALECKI AND X.-C. CAI, *H^1 -norm error bounds for piecewise Hermite bicubic orthogonal spline collocation schemes for elliptic boundary value problems*, SIAM J. Numer. Anal., 31 (1994), pp. 1128–1146.
- [7] B. BIALECKI AND G. FAIRWEATHER, *Orthogonal spline collocation methods for partial differential equations*, J. Comput. Appl. Math., 128 (2001), pp. 55–82.
- [8] H. BLUM AND R. RANNACHER, *On the boundary value problem of the biharmonic operator on domains with angular corners*, Math. Meth. in the Appl. Sci., 2 (1980), pp. 556–581.
- [9] F. BOGNER, R. L. FOX, AND J. L.A. SCHMIT, *The generalization of inter-element-compatible stiffness and mass matrices by the use of interpolation formulas*, in Proc. First Conference on Matrix Methods in Structural Mechanics, Write Patterson A.F.B., Ohio, 1965, pp. 397–443.
- [10] S. BRENNER AND L. SCOTT, *The Mathematical Theory of Finite Element Methods*, Springer, New York, 2002.
- [11] P. CIARLET, *The Finite Element Method for Elliptic Problems*, SIAM, Philadelphia, 2002.
- [12] P. CIARLET AND P. RAVIART, *A mixed finite element method for the biharmonic equation*, in Mathematical Aspects of Finite Elements in Partial Differential Equations, C. de Boor, ed., New York, 1974, Academic Press, pp. 125–143.
- [13] P. J. DAVIS AND P. ROBINOWITZ, *Methods of Numerical Integration*, Computer Science and Applied Mathematics, Academic Press, New York, 1984.
- [14] R. GLOWINSKI AND O. PIRONNEAU, *Numerical methods for the first biharmonic equation and for the two-dimensional Stokes problem*, SIAM Review, 21 (1979), pp. 167–212.
- [15] M. HRABOK AND T. HRUDEY, *A review and catalogue of plate bending finite elements*, Comput. & Structures, 19 (1984), pp. 479–495.
- [16] Z.-M. LOU, B. BIALECKI, AND G. FAIRWEATHER, *Orthogonal spline collocation methods for biharmonic problems*, Numer. Math., 80 (1998), pp. 267–303.
- [17] J. ODEN AND J. REDDY, *An Introduction to the Mathematical Theory of Finite Elements*, John Wiley & Sons, New York, 1976.
- [18] P. PERCELL AND M. F. WHEELER, *A C^1 finite element collocation method for elliptic equations*, SIAM J. Numer. Anal., 17 (1980), pp. 605–622.