CONVERGENCE ANALYSIS OF A QUADRATURE FINITE ELEMENT GALERKIN SCHEME FOR A BIHARMONIC PROBLEM

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Abstract. A quadrature finite element Galerkin scheme for a Dirichlet boundary value problem for the biharmonic equation is analyzed for a solution existence, uniqueness, and convergence. Conforming finite element space of Bogner-Fox-Schmit rectangles and an integration rule based on the two-point Gaussian quadrature are used to formulate the discrete problem. An \(H^2\)-norm error estimate is obtained for the solution of the original finite element problem consistent with the solution regularity. A standard quadrature error analysis gives a suboptimal order error estimate. Optimal order error estimates under sufficient regularity assumptions are obtained using an alternative approach based on the equivalence of the quadrature problem with an orthogonal spline collocation problem.

Key words: biharmonic problem, finite elements, Galerkin method, Gaussian quadrature, orthogonal spline collocation

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1. Introduction. In this article, we analyze existence, uniqueness, and convergence of a quadrature finite element Galerkin approximation of a Dirichlet boundary value problem (BVP) with the biharmonic equation on a rectangular polygonal region. Problems with the biharmonic equation arise in many areas of applied mathematics, for example, plate problems in plane elasticity and problems for the stream function of steady-state Stokes flows in fluid mechanics.

Let \(\Omega \subset \mathbb{R}^2\) be an open rectangular polygon with the boundary \(\partial \Omega\). We consider the Dirichlet BVP

\[
\Delta^2 u = f \quad \text{in} \quad \Omega, \quad \text{and} \quad u = \partial_n u = 0 \quad \text{on} \quad \partial \Omega,
\]

where \(\Delta\) is the Laplace operator, \(f\) is a continuous function on \(\Omega\), and \(\partial_n\) denotes the outer normal derivative. In plane elasticity, problem (1.1) is known as a clamped plate problem.

Biharmonic problems can be solved numerically by a finite element Galerkin method using direct or mixed approaches. The direct approach is based on a direct discretization of the biharmonic equation (see see §2.2, Ch. 2, and Ch. 6 in [11]). Whereas in the mixed approach, the biharmonic equation is first replaced with a pair of coupled second-order differential equations by introducing a new dependent variable (see [11, 12, 14]). A review and a catalog of finite elements for plate problems is given in [15].

Forming a stiffness matrix and a load vector in a finite element Galerkin method for the biharmonic problem requires evaluation of integrals with polynomial integrands. An application of a numerical quadrature leads to a quadrature finite element Galerkin scheme. The main criteria in selecting a quadrature rule for a quadrature finite element Galerkin scheme is to preserve solution accuracy of the original finite element problem. For a fourth-order problem, the application of a numerical integration based on at least three-point Gaussian quadrature is suggested in Theorem 8.9 in [17] to preserve optimal error estimates.

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We apply a direct approach using the conforming Bogner-Fox-Schmit rectangle introduced in [9]. The discrete solution is a $C^1$ piecewise bicubic function with 16 degrees of freedom approximating the values of the solution, the first-order and the mixed second-order derivatives at the vertices of a rectangular element. The integrals in the stiffness matrix and in the load vector are approximated by a product two-point Gaussian quadrature. The resulting quadrature scheme demonstrates optimal order convergence in Sobolev norms and superconvergence in the maximum nodal norm.

In our convergence analysis, we use an auxiliary orthogonal spline collocation (OSC) scheme for the biharmonic problem. A review of OSC methods for solving partial differential equations is given in [7]. A mixed type OSC method for biharmonic problems on a rectangle and solution algorithms were studied in [5, 16], where two coupled second-order differential equations are collocated at the nodes of a product Gaussian quadrature with the solution in the finite element space of piecewise Hermite bicubics. Forming a matrix of OSC equations is fast. The OSC solution has optimal order error estimates and demonstrates superconvergence at partition nodes. Convergence analysis of OSC schemes requires higher than optimal regularity assumptions on the exact solution.

The main results of this work are as follows. We obtain an error estimate in the Sobolev $H^2$-norm for the solution of the original finite element problem consistent with the solution regularity. We prove existence and uniqueness of the solution of the quadrature scheme, and demonstrate that the standard error analysis based on the First Strang Lemma leads to a suboptimal order $H^2$-norm error estimate. We prove that the solution of the quadrature scheme is also the solution of an equivalent OSC problem, and obtain optimal order error estimates in the $H^1$- and $H^2$-norms. Results presented in this article are related to those in [16].

An outline of the rest of the article is as follows. In section 2, we give a variational formulation of BVP (1.1) and present solution regularity results. In section 3, we formulate the finite element Galerkin problem and obtain an $H^2$-norm error estimate consistent with the solution regularity. The quadrature finite element Galerkin scheme is defined in section 4, and existence and uniqueness of its solution is proved in section 5. In section 6, we prove that an error analysis of the quadrature scheme based on the First Strang Lemma gives a suboptimal order error estimate. Some auxiliary results are obtained in section 7 and, in section 8, we introduce an orthogonal spline collocation scheme, determine its relation with the quadrature Galerkin scheme, and obtain optimal order error estimates. Numerical results are presented in section 9, and our conclusions are summarized in section 10.

2. Variational problem and solution regularity. In this section, we give a variational formulation on the problem and present solution regularity results. Let $D$ be an open subset of $\mathbb{R}^2$, and let $m \geq 0$ be an integer. Let $C^m(D)$ be the space of functions that are $m$ times continuously differentiable on $D$ with the norm and a seminorm, respectively, given by

$$
\|v\|_{C^m(D)} = \max_{|\alpha| \leq m} \{\sup_{x \in D} |\partial^\alpha v(x)|\} \quad \text{and} \quad |v|_{C^m(D)} = \max_{|\alpha| = m} \{\sup_{x \in D} |\partial^\alpha v(x)|\},
$$

where $x = (x_1, x_2)$, $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2$, and $\partial^{\alpha} = \partial^{\alpha_1}/(\partial x_1^{\alpha_1} \partial x_2^{\alpha_2})$. Let $L^2(D)$ be the Hilbert space of square integrable functions on $D$ with the norm $\|v\|_{L^2(D)} = (\int_D v^2 \, dx)^{1/2}$. Let $H^m(D)$ be the standard Sobolev space.
with the norm and a seminorm
\[ \| v \|_{H^m(D)} = \left( \sum_{|\alpha| \leq m} \int_D |\partial^\alpha v|^2 dx \right)^{1/2} \]
and
\[ |v|_{H^m(D)} = \left( \sum_{|\alpha| = m} \int_D |\partial^\alpha v|^2 dx \right)^{1/2}, \]
respectively. For real \( s > 0 \), we also use a fractional order Sobolev space \( H^s(\Omega) \) and its dual \( H^{-s}(\Omega) \) (see §14.2 in [10]). Let \( H^2_0(\Omega) \) be the closure in the \( H^2 \)-norm of the space of \( C^\infty \) functions with compact supports in \( \Omega \). It is known that \( | \cdot |_{H^2(\Omega)} \) and \( \| \cdot \|_{H^2(\Omega)} \) are equivalent norms on \( H^2_0(\Omega) \). Throughout this article, \( C \) is a generic positive constant.

We consider the following variational form of BVP (1.1): find \( u \in H^2_0(\Omega) \) such that
\[ a(u, v) = (f, v) \quad \text{for all } v \in H^2_0(\Omega), \]
where the bilinear and linear forms \( a(\cdot, \cdot) \) and \( (f, \cdot) \) are respectively defined by
\[ a(w, v) = \int_\Omega \Delta w \Delta v \, dx \quad \text{and} \quad (f, v) = \int_\Omega f v \, dx, \]
(see §1.2 in [11]). Using the representation
\[ a(v, v) = |v|_{H^2(\Omega)}^2, \quad v \in H^2_0(\Omega), \]
(see (1.2.8) in [11]) and the norm equivalence \( \| \cdot \|_{H^2(\Omega)} \sim | \cdot |_{H^2(\Omega)} \), it is easy to see that the bilinear form \( a(\cdot, \cdot) \) is \( V \)-elliptic and bounded on \( H^2_0(\Omega) \); that is,
\[ C\| v \|_{H^2(\Omega)}^2 \leq a(v, v), \quad v \in H^2_0(\Omega), \]
(2.3)
\[ |a(v, w)| \leq C|v|_{H^2(\Omega)}|w|_{H^2(\Omega)}, \quad v, w \in H^2_0(\Omega). \]
(2.4)
It follows from (2.2)–(2.4), and the Riesz Representation Theorem (Theorem 2.4.2 in [10]) that problem (2.1) has a unique solution \( u \in H^2_0(\Omega) \) such that
\[ \| u \|_{H^2(\Omega)} \leq C \| f \|_{H^{-2}(\Omega)}. \]

If \( \Omega \) has a re-entrant corner with angle \( 3\pi/2 \), then
\[ \| u \|_{H^2+2s(\Omega)} \leq C \| f \|_{H^{-2+2s}(\Omega)} \quad \text{for any } f \in H^{-2+2s}(\Omega), \quad 0 \leq s < 1/3, \]
(2.5)
(see [3]). If \( \Omega \) is a rectangle then it follows from Theorem 2 in [8] that
\[ \| u \|_{H^{s+1}(\Omega)} \leq C \| f \|_{H^{-s}(\Omega)}, \quad \text{for any } f \in H^{-s}(\Omega), \quad s = 0, 1. \]

3. Finite element Galerkin problem. In this section, we formulate a finite element Galerkin approximation of problem (2.1) and obtain an \( H^2 \)-norm error estimate consistent with the solution regularity. We assume that \( x_1 \) and \( x_2 \)-axes are respectively directed to the right and up, and that a boundary edge of the rectangular polygonal domain \( \Omega \) is parallel to a coordinate axis. Let \( \Omega \) be equipped with a regular triangulation \( T_h \) consisting of rectangular elements and satisfying the requirements of the finite element method (see properties (T_h1)–(T_h5) of the assumption (FEM 1) in §2.1 and §2.2 in [11]). Let \( h \) be the largest edge length of elements in \( T_h \).
We consider a finite element space

\[ X_h = \{ v \in C^1(\Omega) : v|_K \in Q_3(K), \ K \in T_h \}, \]

where \( Q_3(K) \) denotes the space of bicubic polynomials restricted on \( K \) (see §2.2 in [11]). The subspace

\[ V_h = \{ v \in X_h : v = \partial_n v = 0 \text{ on } \partial \Omega \} \]

plays the major role in this article. Note that, for any \( v \in V_h \),

\[ v = v_{x_1} = v_{x_2} = v_{x_1x_2} = 0 \text{ on } \partial \Omega, \]

and \( V_h \subset H^2_0(\Omega) \) by Theorem 2.2.15 in [11]. The dimension of \( V_h \) equals four times the number of interior vertices of \( T_h \).

The finite element Galerkin problem approximating the variational problem (2.1) is formulated as follows: find \( U_h \in V_h \) such that

\[ a(U_h, v) = (f, v), \quad \text{for all } v \in V_h, \]

where the forms \( a(\cdot, \cdot) \) and \( (f, \cdot) \) are defined in (2.2). It follows from (2.3) that \( V_h \) is a Hilbert space with the inner product \( a(\cdot, \cdot) \). Using the Riesz Representation Theorem, it is easy to see that problem (3.1) has a unique solution.

In Theorem 6.1.6 in [11], it is proved that if \( u \in H^4(\Omega) \cap H^2_0(\Omega) \) then

\[ \| u - U_h \|_{H^2(\Omega)} \leq Ch^2 \| u \|_{H^4(\Omega)}, \]

(3.2)

where \( u \) is the solution of problem (2.1). We note that the regularity assumption for (3.2) is satisfied for a rectangle but not for a domain that has a re-entrant corner with angle \( 3\pi/2 \) (see [8]). The following convergence result is consistent with the solution regularity.

**Theorem 3.1.** Let \( u \) and \( U_h \) be the solutions of problems (2.1) and (3.1), respectively. If \( f \in H^{-2+2s}(\Omega) \), then

\[ \| u - U_h \|_{H^2(\Omega)} \leq Ch^2 \| f \|_{H^{-2+2s}(\Omega)}, \quad 0 \leq s < 1/3. \]

**Proof.** To prove the theorem, we use results for the real method of interpolation of Sobolev spaces (see Chapter 14 in [10]). First, consider an orthogonal projection operator \( P_h : H^2_0(\Omega) \to V_h \) such that, for any \( v \in H^2_0(\Omega) \), \( P_h v \in V_h \) satisfies

\[ a(P_h v, \phi) = a(v, \phi) \quad \text{for all } \phi \in V_h, \]

and let \( T_h = I - P_h \), where \( I \) is the identity operator on \( H^2_0(\Omega) \). Since \( P_h \) is an orthogonal projection with respect to the inner product \( a(\cdot, \cdot) \), using the triangle inequality and (2.4), we get

\[ \| T_h v \|_{H^2(\Omega)} \leq C \| v \|_{H^2(\Omega)} \quad \text{for any } v \in H^2_0(\Omega). \]

From (3.2), we have

\[ \| T_h v \|_{H^2(\Omega)} \leq Ch^2 \| v \|_{H^4(\Omega)}, \quad \text{for any } v \in H^4(\Omega) \cap H^2_0(\Omega). \]
Take $0 \leq s < 1/3$. It follows from an exact interpolation theorem (see Theorem 7.23 in [1]) with $\theta = 1 - s$ and $q = 2$, that operator $T_h$ from $H^2(\Omega)$ to the interpolation space $[H^4(\Omega) \cap H^2_0(\Omega), H^2_0(\Omega)]_{1-s,2}$ is bounded with the corresponding operator norm satisfying $\|T_h\| \leq Ch^{2s}$; that is
\[
(3.4) \quad \|T_h v\|_{H^2(\Omega)} \leq C h^{2s} v\|_{[H^4(\Omega) \cap H^2_0(\Omega), H^2_0(\Omega)]_{1-s,2}}.
\]

Let us prove that
\[
(3.5) \quad \|u\|_{[H^4(\Omega) \cap H^2_0(\Omega), H^2_0(\Omega)]_{1-s,2}} \leq C \|f\|_{H^{-2+2s}(\Omega)}.
\]

From $[H^2(\Omega), L^2(\Omega)]_{s,2} = H^{2-2s}(\Omega)$ (see (14.2.5) in [10]), it follows that
\[
[H^2(\Omega), L^2(\Omega)]_{s,2} = H^{-2+2s}(\Omega),
\]
where the prime denotes the dual space, which implies
\[
(3.6) \quad \|f\|_{[H^2(\Omega), L^2(\Omega)]_{s,2}} \leq C \|f\|_{H^{-2+2s}(\Omega)}.
\]

By the duality theorem (Theorem 3.7.1 in [4]), we have
\[
(3.7) \quad \|f\|_{L^2(\Omega), H^{-2}(\Omega)]_{1-s,2}} \leq C \|f\|_{[H^2(\Omega), L^2(\Omega)]_{s,2}}.
\]

The following estimate was obtained in [3] (see estimate (4.8) and Theorem 4.1):
\[
(3.8) \quad \|u\|_{[H^4(\Omega) \cap H^2_0(\Omega), H^2_0(\Omega)]_{1-s,2}} \leq \|f\|_{[L^2(\Omega), H^{-2}(\Omega)]_{1-s,2}},
\]

where $u$ is the solution of problem (2.1). Inequalities (3.6)–(3.8) imply (3.5).

Since $u \in [H^4(\Omega) \cap H^2_0(\Omega), H^2_0(\Omega)]_{1-s,2}$, replacing $v$ in (3.4) by $u$ and using (3.5), we get
\[
\|T_h u\|_{H^2(\Omega)} \leq C h^{2s} \|f\|_{H^{-2+2s}(\Omega)},
\]
which, along with $T_h u = u - P_h u = u - U_h$, gives (3.3). \hfill \Box

4. Quadrature scheme. In this section, we define our quadrature scheme that approximates the finite element Galerkin problem (3.1). First, we introduce notation and auxiliary facts.

For any interval $I = (a, b)$ with length $|I| = b - a$, let
\[
G_I = \{a + |I|(3 - \sqrt{3})/6, a + |I|(3 + \sqrt{3})/6\}
\]
be the set of Gauss points in $I$, and let
\[
\sum_{\xi \in G_I} v = \frac{b - a}{2} \sum_{\xi \in G_I} v(\xi), \quad v \in C(I),
\]
be the two-point Gaussian quadrature on $I$. Proved in [13], the following lemma provides an error formula for the Gaussian quadrature.
Lemma 4.1. Let \( I = (a, b) \) be an interval and let \( v \in C^4(I) \). Then

\[
\int_a^b v(t) dt = \sum_{\varphi_i} v + \frac{|I|^5}{4320} v^{(4)}(\xi), \quad \xi \in I.
\]

For any \( K = I_1 \times I_2 \in \mathcal{T}_h \), let

\[
\sum_{\varphi_k} v = \sum_{\varphi_i \in \mathcal{G}_{I_1}} \sum_{\varphi_i \in \mathcal{G}_{I_2}} v = \frac{|K|}{4} \sum_{\xi \in \varphi_k} v(\xi), \quad v \in C(K),
\]

be the product two-point Gaussian quadrature on \( K \), where \( \mathcal{G}_K = \mathcal{G}_{I_1} \times \mathcal{G}_{I_2} \) is the set of four quadrature nodes in \( K \) called the Gauss points, and \( |K| \) is the area of \( K \). For any functions \( v \) and \( w \) defined on \( \mathcal{T}_h \), let

\[
(v, w)_h = \sum_{K \in \mathcal{T}_h} \sum_{\varphi_k} vw \quad \text{and} \quad \|v\|_h = \sqrt{(v, v)_h}.
\]

It is easy to verify that, for any \( v \) and \( w \) defined on \( \mathcal{T}_h \),

\[
|(v, w)_h| \leq \|v\|_h \|w\|_h.
\]

Let

\[
a_h(w, v) = (\Delta w, \Delta v)_h, \quad w, v \in V_h,
\]

be a bilinear form approximating \( a(\cdot, \cdot) \) of (2.2). Our quadrature finite element Galerkin problem is formulated as follows: find \( u_h \in V_h \) such that

\[
a_h(u_h, v) = (f, v)_h \quad \text{for all } v \in V_h.
\]

5. Existence and uniqueness. In this section, we prove that the quadrature Galerkin scheme (4.6) has a unique solution. First, we establish auxiliary results, formulated in the following two lemmas, which are used to prove \( V \)-ellipticity and boundedness of the approximate bilinear form \( a_h(\cdot, \cdot) \) with respect to the \( H^2 \)-norm.

Domain \( \bar{\Omega} \) can be decomposed in the following two unions of rectangles:

\[
\bar{\Omega} = \bigcup_{i=1}^{L_V} R^L_V = \bigcup_{i=1}^{L_H} R^L_H,
\]

where the sets \( \{R^L_V\}_{i=1}^{L_V} \) and \( \{R^L_H\}_{i=1}^{L_H} \) consist of open disjoint rectangles whose, respectively, vertical and horizontal edges are parts of the boundary \( \partial \Omega \). Triangulation \( \mathcal{T}_h \) determines triangulations \( \{\pi^L_V\}_{i=1}^{L_V} \) and \( \{\pi^L_H\}_{i=1}^{L_H} \) of rectangles \( \{R^L_V\}_{i=1}^{L_V} \) and \( \{R^L_H\}_{i=1}^{L_H} \), respectively. Let \( \pi^l_V = \pi^l_{V,1} \times \pi^l_{V,2} \) for \( l = 1, \ldots, L_V \), and \( \pi^l_H = \pi^l_{H,1} \times \pi^l_{H,2} \) for \( l = 1, \ldots, L_H \), where one-dimensional partitions \( \pi^l_{V,1} \) and \( \pi^l_{H,i} \), \( i = 1, 2 \), consist of subintervals. It follows from (5.1) that, for any set of numbers \( \{s_K\}_{K \in \mathcal{T}_h} \),

\[
\sum_{l=1}^{L_V} \sum_{I_1 \in \pi^l_{V,1}} \sum_{I_2 \in \pi^l_{V,2}} s_{I_1 \times I_2} + \sum_{l=1}^{L_H} \sum_{I_1 \in \pi^l_{H,1}} \sum_{I_2 \in \pi^l_{H,2}} s_{I_1 \times I_2} = \sum_{K \in \mathcal{T}_h} s_K.
\]
Lemma 5.1. For any $v \in X_h$ such that $v = 0$ on $\partial \Omega$,

\begin{align}
(5.3) & \quad \|v \cdot x_i\|_h \geq C \|v \cdot x_i\|_{L^2(\Omega)}, \quad i = 1, 2, \\
(5.4) & \quad \langle v \cdot x_{i1}, v \cdot x_{i2} \rangle_h \geq \|v \cdot x_{i1}\|_{L^2(\Omega)}^2.
\end{align}

Proof. Take any $v \in X_h$ that vanishes on $\partial \Omega$. Let us prove (5.3) for $i = 1$ since the proof for the other case is similar. Fix $l = 1, \ldots, L_H$, let $\mathcal{H}_l = I_1 \times I_2^l$, and take any $x_1 \in I_1$. Restricted on the vertical line segment $\{x_1\} \times I_2^l$, $v \cdot x_{i1}$ is a piecewise Hermite cubic polynomial function vanishing at the end points of $I_2^l$. The key relation in proving (5.3) is

\begin{equation}
(5.5) \quad \sum_{l_x \in \mathcal{H}_{x_2}} \sum_{l_z \in \mathcal{H}_{x_2}} \mathcal{V}_{l_x}^2(x_1, \cdot) \geq C \sum_{l_x \in \mathcal{H}_{x_2}} \int_{I_2} v_{z_{x1}}^2(x_1, x_2) dx_2,
\end{equation}

which follows directly from (2.6) in [18].

Using (4.3), the second identity in (5.2), the fact that $v_{z_{x1}}^2$ is a polynomial of degree $\leq 2$ in $x_1$-variable and the exactness of the two-point Gaussian quadrature, changing orders of summations and the integration, and applying (5.5) and the second identity in (5.2), we obtain

\begin{equation*}
\|v \cdot x_{i1}\|_h^2 = \sum_{K \in \mathcal{T}_h} \sum_{G_K} v_{z_{x1}}^2 = \sum_{l = 1}^{L_H} \sum_{l_1 \in \mathcal{I}_{H,1}} \sum_{l_2 \in \mathcal{I}_{H,2}} \sum_{l_z \in \mathcal{H}_{x_2}} v_{z_{x1}}^2
\end{equation*}

\begin{equation*}
= \sum_{l = 1}^{L_H} \sum_{l_1 \in \mathcal{I}_{H,1}} \sum_{l_2 \in \mathcal{I}_{H,2}} \sum_{l_z \in \mathcal{H}_{x_2}} \int_{I_1} v_{z_{x1}}^2(x_1, \cdot) dx_1
\end{equation*}

\begin{equation*}
\geq C \sum_{l = 1}^{L_H} \sum_{l_1 \in \mathcal{I}_{H,1}} \sum_{l_2 \in \mathcal{I}_{H,2}} \sum_{l_z \in \mathcal{H}_{x_2}} \int_{I_1 \times I_2} v_{z_{x1}}^2(x_1, x_2) dx = C \sum_{K \in \mathcal{T}_h} \int_K v_{z_{x1}}^2(x_1, x_2) dx = \|v \cdot x_{i1}\|_{L^2(\Omega)}^2,
\end{equation*}

which is (5.3) for $i = 1$.

We now prove (5.4). Using (4.3), the first identity in (5.2), and (4.1) in $x_1$-direction, we obtain

\begin{equation}
\langle v \cdot x_{i1}, v \cdot x_{i2} \rangle_h = \sum_{K \in \mathcal{T}_h} \sum_{G_K} v_{z_{x1}} v_{z_{x2}} = \sum_{l = 1}^{L_V} \sum_{l_1 \in \mathcal{I}_{V,1}} \sum_{l_2 \in \mathcal{I}_{V,2}} \sum_{l_z \in \mathcal{H}_{x_2}} v_{z_{x1}} v_{z_{x2}},
\end{equation}

\begin{equation}
= S_1 + CS_2,
\end{equation}

where

\begin{equation}
S_1 = \sum_{l = 1}^{L_V} \sum_{l_1 \in \mathcal{I}_{V,1}} \sum_{l_2 \in \mathcal{I}_{V,2}} \sum_{l_z \in \mathcal{H}_{x_2}} \int_{I_1} (v \cdot x_{i1}, v \cdot x_{i2})(x_1, \cdot) dx_1,
\end{equation}

\begin{equation}
S_2 = -\sum_{l = 1}^{L_V} \sum_{l_1 \in \mathcal{I}_{V,1}} \sum_{l_2 \in \mathcal{I}_{V,2}} \sum_{l_z \in \mathcal{H}_{x_2}} (\partial f_{3,0}^2 \partial f_{3,2}^2) \xi_1(\cdot),
\end{equation}

with some $\{I_1(\xi_2)\}_{\xi_2 \in \mathcal{H}_{x_2}}$. 

A quadrature scheme for a biharmonic problem
Let us prove

\[(5.9) \quad S_1 \geq \|v_{x_1x_2}\|^2_{L^2(\Omega)},\]
\[(5.10) \quad S_2 \geq 0,\]

which, along with (5.6), imply estimate (5.4). First, we obtain (5.9). Applying (4.1) in \(x_2\)-direction on \(S_1\) in (5.7), we get

\[(5.11) \quad S_1 = S_{11} + CS_{12},\]

where

\[(5.12) \quad S_{11} = \sum_{l=1}^{L_1} \sum_{l_1 \in \pi_1^{l_1}, l_2 \in \pi_2} \int_{I_{l_1} \times I_{l_2}} v_{x_1x_1}v_{x_2x_2} \, dx = \sum_{l=1}^{L_1} \int_{R_{l}^1} v_{x_1x_1}v_{x_2x_2} \, dx,\]

\[(5.13) \quad S_{12} = -\sum_{l=1}^{L_1} \sum_{l_1 \in \pi_1^{l_1}, l_2 \in \pi_2} |I_2|^5 \int_{I_1} (\partial^{(2,3)}_v \partial^{(0,3)}_v)(x_1, t_{I_2}(x_1)) \, dx_1,

with some \(t_{I_2}(x_1) \in I_2\) for any \(x_1 \in I_1\). Since function \(\partial^{(2,3)}_v \partial^{(0,3)}_v\) is constant in \(x_2\)-variable on \(I_1 \times I_2\), we set

\[(5.14) \quad t_{I_2}(x_1) = t_{I_1 \times I_2} = \text{const}, \quad \text{for all} \quad x_1 \in I_1.\]

From (5.13), using the integration by parts in \(x_1\)-variable, (5.14), continuity in \(x_1\)-variable of \((\partial^{(1,3)}_v \partial^{(0,3)}_v)\), and the fact that \(\partial^{(0,3)}_v\) vanishes on the vertical edges of \(R_{l}^1\), we obtain

\[(5.15) \quad S_{12} = \sum_{l=1}^{L_1} \sum_{l_1 \in \pi_1^{l_1}, l_2 \in \pi_2} |I_2|^5 \int_{I_1} (\partial^{(1,3)}_v)^2(x_1, t_{I_1 \times I_2}) \, dx_1 \geq 0.\]

We now prove

\[(5.16) \quad S_{11} = \|v_{x_1x_2}\|^2_{L^2(\Omega)},\]

where \(S_{11}\) is defined by (5.12). Using the integration by parts in \(x_1\)-variable, continuity of \(v_{x_1}, v_{x_1x_2}\) in \(x_1\)-direction, the fact that \(v_{x_2x_2}\) vanishes on the vertical edges of \(R_{l}^1\), and the first representation of \(\Omega\) in (5.1), we obtain

\[S_{11} = -\sum_{l=1}^{L_1} \sum_{l_1} \int_{R_{l}^1} v_{x_1}v_{x_1x_2} \, dx = -\int_{\Omega} v_{x_1}v_{x_1x_2} \, dx.\]

Similarly, using the second representation of \(\Omega\) in (5.1), the integration by parts in \(x_2\)-variable, continuity of \(v_{x_1}, v_{x_1x_2}\), and the fact that \(v_{x_1}\) vanishes on the horizontal edges of \(R_{l}^1\), we get (5.16). Relations (5.11), (5.16), and (5.15), imply (5.9).

It remains to prove (5.10), where \(S_2\) is defined by (5.8) and used in (5.6). Since function \((\partial^{(3,0)}_v \partial^{(3,2)}_v)\) is constant in \(x_1\)-variable on \(I_1 \times I_2\), in (5.8), we set

\[(5.17) \quad t_{I_1}(x_2) = t_{I_1 \times I_2} = \text{const}, \quad \text{for all} \quad x_2 \in I_2.\]
From (5.8), using (5.17), (5.2), and (4.1) in \(x_2\)-direction, we obtain

\[
S_2 = -\sum_{l=1}^{L_H} \sum_{l_1 \in \pi_{H,1}^l} |I_1|^5 \sum_{l_2 \in \pi_{H,2}^l} (\partial^{(3,0)}v \partial^{(3,2)}v)(t_{l_1 \times l_2}, \cdot) = S_{21} + CS_{22},
\]

where

\[
S_{21} = -\sum_{l=1}^{L_H} \sum_{l_1 \in \pi_{H,1}^l} |I_1|^5 \sum_{l_2 \in \pi_{H,2}^l} \int_{I_2} (\partial^{(3,0)}v \partial^{(3,2)}v)(t_{l_1 \times l_2}, x_2) dx_2,
\]

\[
S_{22} = \sum_{l=1}^{L_H} \sum_{l_1 \in \pi_{H,1}^l} |I_1|^5 \sum_{l_2 \in \pi_{H,2}^l} |I_2|^5 (\partial^{(3,3)}v)^2 l_{1 \times l_2} \geq 0.
\]

We note that, in (5.20), \(\partial^{(3,3)}v\) is constant on \(I_1 \times I_2\).

From (5.19), using the integration by parts in \(x_2\)-direction, continuity in \(x_2\)-variable of \((\partial^{(3,0)}v \partial^{(3,1)}v)\), and the fact that \(\partial^{(3,0)}\) vanishes on the horizontal boundaries of \(R_H\), we obtain

\[
S_{21} = \sum_{l=1}^{L_H} \sum_{l_1 \in \pi_{H,1}^l} |I_1|^5 \sum_{l_2 \in \pi_{H,2}^l} \int_{I_2} (\partial^{(3,1)}v)^2 (t_{l_1 \times l_2}, x_2) dx_2 \geq 0.
\]

Identity (5.18) and the inequalities (5.21) and (5.20) imply (5.10).

**Proof.** Take any \(v \in X_h\). Applying the inverse inequality

\[
\|w\|_{C(K)} \leq Ch^{-1}\|w\|_{L^2(K)}, \quad K \in T_h, \quad w \in X_h,
\]

(see (3.2.33) in [11]) and the inclusion \(X_h \subset H^2(\Omega)\), we obtain

\[
\|\partial^\alpha v\|_{h}^2 = \sum_{K \in T_h} \frac{|K|}{4} \sum_{\xi \in \partial K} |(\partial^\alpha v)(\xi)|^2 \leq C \sum_{K \in T_h} \|\partial^\alpha v\|_{L^2(K)}^2 = C \|\partial^\alpha v\|_{L^2(\Omega)}^2.
\]

The following lemma states that the approximate bilinear form \(a_h(\cdot, \cdot)\) is uniformly \(V\)-elliptic and bounded relative to the \(H^2\)-norm.

**Lemma 5.3.**

\[
C\|v\|_{H^2(\Omega)}^2 \leq a_h(v, v), \quad v \in V_h,
\]

\[
|a_h(v, w)| \leq C\|v\|_{H^2(\Omega)} \|w\|_{H^2(\Omega)}, \quad v, w \in V_h.
\]

**Proof.** Inequality (5.23) follows from the representation

\[
a_h(v, v) = \|v_{x_1 x_1}\|_h^2 + 2(v_{x_1 x_1}, v_{x_2 x_2})_h + \|v_{x_2 x_2}\|_h^2, \quad v \in V_h,
\]
Lemma 5.1, and the fact that $|\cdot|_{H^2(\Omega)}$ is equivalent to $\|\cdot\|_{H^2(\Omega)}$ on $H^2_0(\Omega)$. Using (4.5), (4.4), the Cauchy-Schwarz inequality in $H^4$, and (5.22), we obtain, for any $v, w \in V_h$,

$$|a_h(v, w)| \leq \sum_{i,j=1}^{2} \|v_{x_i x_j}\|_h \|w_{x_i x_j}\|_h \leq 2(\|v_{x_1 x_1}\|_h^2 + \|v_{x_2 x_2}\|_h^2)^{1/2}(\|w_{x_1 x_1}\|_h^2 + \|w_{x_2 x_2}\|_h^2)^{1/2} \leq C|v|_{H^2(\Omega)}|w|_{H^2(\Omega)},$$

which is (5.24).

**Theorem 5.4.** For any $f \in C(\Omega)$, the quadrature finite element Galerkin scheme (4.6) has a unique solution.

**Proof.** It follows from (4.5) and Lemma 5.3 that $V_h$ is a Hilbert space with the inner product $a_h(\cdot, \cdot)$. A linear functional $(f, \cdot)_h$ is bounded on $V_h$. Therefore, the statement of the theorem follows from the Riesz Representation Theorem.

**6. An $H^2$-norm error estimate.** In this section, we prove that an error analysis of the quadrature scheme based on the First Strang Lemma (Theorem 4.1.1 in [11]) gives an $H^2$-norm error estimate with suboptimal order $O(h)$.

To begin with, we introduce a piecewise Hermite bicubic interpolant and state some of its properties. For $K \in T_h$, let $\{a_i\}_{i=1}^4$ be the vertices of $K$. For any $v \in C^2(K)$, let $\Pi_K v \in Q_3(K)$ be the bicubic Hermite interpolant of $v$ defined by the following relations:

\[ \partial^\alpha (\Pi_K v)(a_i) = \partial^\alpha v(a_i), \quad 1 \leq i \leq 4, \quad |\alpha_j| \leq 1, \quad j = 1, 2. \]

For $v \in C^2(\Omega)$, let $\Pi_K v \in X_h$ be the piecewise Hermite bicubic interpolant of $v$ defined by

\[ (\Pi_K v)|_K = \Pi_K v|_K, \quad \text{for all} \quad K \in T_h. \]

Theorem 3.1.6 in [11] (also, see estimate (6.1.7)) implies

\[ \|v - \Pi_K v\|_{H^m(K)} \leq C h^{4-m} |v|_{H^4(K)}, \quad 0 \leq m \leq 4, \quad K \in T_h, \quad v \in H^4(\Omega). \]

Using the triangle inequality and (6.3), it is easy to obtain

\[ \sum_{K \in T_h} \|\Pi_K v\|_{H^m(K)}^2 \leq C \|v\|_{H^4(\Omega)}^2, \quad 0 \leq m \leq 4, \quad v \in H^4(\Omega). \]

In the following lemma, we prove that the bilinear form $a_h(\cdot, \cdot)$ approximates $a(\cdot, \cdot)$ with order $O(h)$.

**Lemma 6.1.**

\[ |a(\Pi_h u, w) - a_h(\Pi_h u, w)| \leq C h \|u\|_{H^4(\Omega)} \|w\|_{H^2(\Omega)}, \quad u \in H^4(\Omega), \quad w \in V_h. \]

**Proof.** Take any $v$ and $w$ in $V_h$. Using (2.2), (4.5), and (4.3), we obtain

\[ a(v, w) - a_h(v, w) = \sum_{K \in T_h} E_K(\Delta v \Delta w), \]

where $E_K(\Delta v \Delta w)$ is the error in approximating $\Delta v \Delta w$ by the piecewise bilinear form $a_h(\cdot, \cdot)$ on the element $K$.
Finally, in the last estimate, replacing $v$ by $\tilde{v}$ we obtain

\begin{equation}
E_K(\tilde{v}) = \int_K g(x)dx - \sum_{\delta_K} g, \quad g \in C(K).
\end{equation}

Take any $K \in T_h$ and let $F : \tilde{K} \to K$ be an invertible affine mapping, where $\tilde{K} = (0,1)^2$. Changing the variables with the transformation $F$, we obtain

\begin{equation}
|E_K(\Delta v \Delta w)| \leq Ch^{-2}|E_{\tilde{K}}(\Delta \tilde{v} \Delta \tilde{w})|,
\end{equation}

where $\tilde{v} = v|_K \circ F$ and $\tilde{w} = w|_K \circ F$.

Consider a linear functional

\begin{equation}
l(\cdot) = E_K(\Delta w), \quad \tilde{w} \in C^1(\tilde{K}).
\end{equation}

Since $\tilde{w} \in Q_3(\tilde{K})$ (and, hence, $\Delta \tilde{w} \in Q_3(\tilde{K})$), using the equivalence of norms $\| \cdot \|_{C(\tilde{K})}$ and $\| \cdot \|_{L^2(\tilde{K})}$ on the finite-dimensional vector space $Q(\tilde{K})$, we get, for any $\tilde{w} \in C^1(\tilde{K})$,

\begin{equation}
|l(\Delta \tilde{w})| \leq C\|\Delta \tilde{w}\|_{C(\tilde{K})} \|\Delta \tilde{w}\|_{L^2(\tilde{K})} \|\Delta \tilde{w}\|_{C(\tilde{K})} \leq (C\|\Delta \tilde{w}\|_{H^1(\tilde{K})}) |\tilde{w}|_{C^1(\tilde{K})},
\end{equation}

that is, the linear functional $l(\cdot)$ is bounded on $C^1(\tilde{K})$. Since the product two-point Gaussian quadrature is exact for polynomials in $Q_3$ and $\Delta \tilde{w} \in Q_3(\tilde{K})$, the linear functional $l(\cdot)$ vanishes on polynomials of degree zero. Applying the Bramble-Hilbert Lemma (Theorem 4.1.3 in [11]), we obtain

\begin{equation}
|l(\cdot)| \leq (C\|\tilde{w}\|_{H^2(\tilde{K})}) |\tilde{w}|_{C^1(\tilde{K})}.
\end{equation}

In the last estimate, replacing $\tilde{w}$ by $\Delta \tilde{w} \in Q_3(\tilde{K})$, using (6.9) and the norm equivalence in $Q_3(\tilde{K})$, we get

\begin{equation}
|E_K(\Delta v \Delta w)| \leq C\|\Delta \tilde{w}\|_{H^2(\tilde{K})} |\Delta \tilde{w}|_{C^1(\tilde{K})} \leq (C\|\tilde{w}\|_{H^2(\tilde{K})} |\tilde{w}|_{H^2(\tilde{K})}).
\end{equation}

From (6.8), using (6.10) and changing the variables with the transformation $F^{-1}$, we obtain

\begin{equation}
|E_K(\Delta v \Delta w)| \leq Ch^{-2}(h^2h^{-1}|w|_{H^2(\tilde{K})} + h^2h^{-1}|v|_{H^2(\tilde{K})}) = Ch|v|_{H^1(\tilde{K})}|w|_{H^2(\tilde{K})}.
\end{equation}

Substituting this estimate in (6.6) and using the Cauchy-Schwarz inequality, we get

\begin{equation}
|a(v, w) - a_h(v, w)| \leq C h \sum_{K \in T_h} |v|_{H^1(K)} |w|_{H^2(K)} \leq Ch \left( \sum_{K \in T_h} |v|_{H^2(K)}^2 \right)^{1/2} |w|_{H^2(\Omega)}.
\end{equation}

Finally, in the last estimate, replacing $v$ by $\Pi_h u$ and using (6.4) with $m = 3$, we obtain estimate (6.5).

A similar analysis with a numerical integration based on the three-point Gaussian quadrature gives an optimal consistence error estimate of order $O(h^2)$. The following result gives the consistency error estimate of the approximate linear form $(f, \cdot)_h$ defined by (4.3) and (4.2).

**Lemma 6.2.** If $f \in C^2(\Omega)$ then

\begin{equation}
|(f, v) - (f, v)_h| \leq Ch^2 |f|_{C^2(\Omega)} |v|_{H^1(\Omega)}, \quad v \in V_h.
\end{equation}
Proof. The proof is similar to that of Theorem 4.1.5 in [11]. Take any \( v \in V_h \). Using (2.2), (4.3), (4.2), and (6.7), we get

\[
(f, v) - (f, v)_h = \sum_{K \in T_h} E_K(f v).
\]

Take any \( K \in T_h \), and let \( F : \hat{K} = (0, 1)^2 \to K \) be an invertible affine mapping. Changing variables by the transformation \( F \), we obtain

\[
E_K(f v) = h^2 E_{\hat{K}}(\hat{f}\hat{v}),
\]

where \( \hat{f} = f|_K \circ F \in C^2(\hat{K}) \) and \( \hat{v} = v|_K \circ F \in Q_3(\hat{K}) \). Linearity of \( E_K(\cdot) \) implies

\[
E_{\hat{K}}(\hat{f}\hat{v}) = \hat{v}_a \hat{E}(\hat{f}) + \hat{E}(\hat{f} \hat{v} - \hat{v}_a),
\]

where \( \hat{v}_a = \int_{\hat{K}} \hat{v}(s_1, s_2) \, ds_1 \, ds_2 \) is the average value of \( \hat{v} \) on \( \hat{K} \). Using the Cauchy-Schwarz inequality on \( L^2(\hat{K}) \), we obtain

\[
|\hat{v}_a| \leq \|\hat{v}\|_{L^2(\hat{K})}.
\]

Applying the Bramble-Hilbert Lemma and using the norm equivalence on \( Q(\hat{K}) \), we get

\[
|\hat{v} - \hat{v}_a|_{C(\hat{K})} \leq C|\hat{v}|_{C^1(\hat{K})} \leq C|\hat{v}|_{H^1(\hat{K})}.
\]

Let us estimate the first term in (6.14). A linear functional

\[
l(\hat{\phi}) = \hat{v}_a E_{\hat{K}}(\hat{\phi}), \quad \hat{\phi} \in C^2(\hat{K}),
\]

is bounded on \( C^2(\hat{K}) \) since, by (6.15),

\[
|l(\hat{\phi})| \leq C|\hat{v}_a| \|\hat{\phi}\|_{C(\hat{K})} \leq \left( C\|\hat{v}\|_{L^2(\hat{K})} \right) \|\hat{\phi}\|_{C^2(\hat{K})}, \quad \hat{\phi} \in C^2(\hat{K}),
\]

and \( l(\cdot) \) vanishes on polynomials of degree \( \leq 1 \). Applying the Bramble-Hilbert Lemma, we obtain

\[
|l(\hat{\phi})| \leq C\|\hat{v}\|_{L^2(\hat{K})} \|\hat{\phi}\|_{C^2(\hat{K})}, \quad \hat{\phi} \in C^2(\hat{K}).
\]

In the last estimate, replacing \( \hat{\phi} \) by \( \hat{f} \) and using (6.17), we obtain

\[
|\hat{v}_a E_{\hat{K}}(\hat{f})| \leq C|\hat{f}|_{C^2(\hat{K})} \|\hat{v}\|_{L^2(\hat{K})}.
\]

Let us estimate the second term in (6.14). Consider now a linear functional

\[
l(\hat{\phi}) = E_{\hat{K}}(\hat{\phi} (\hat{v} - \hat{v}_a)), \quad \hat{\phi} \in C^1(\hat{K}).
\]

Using (6.16), we obtain boundedness of \( l(\cdot) \):

\[
|l(\hat{\phi})| \leq \|\hat{\phi}\|_{C(\hat{K})} \|\hat{v} - \hat{v}_a\|_{C(\hat{K})} \leq \left( C|\hat{v}|_{H^1(\hat{K})} \right) \|\hat{\phi}\|_{C^1(\hat{K})}, \quad \hat{\phi} \in C^1(\hat{K}).
\]
Since \( \hat{v} - \hat{u}_a \in Q_a(\hat{K}) \), functional \( l(\cdot) \) vanishes on polynomials of degree zero. Applying the Bramble-Hilbert Lemma, we get

\[
\|l(\hat{\phi})\| \leq C|\hat{v}|_{H^1(\hat{K})} |\hat{\phi}|_{C^1(\hat{K})}.
\]

In the last estimate, replacing \( \hat{\phi} \) by \( \hat{f} \) and using (6.19), we obtain

\[
|E_K(\hat{f} (\hat{v} - \hat{u}_a))| \leq C|\hat{f}|_{C^1(\hat{K})} |\hat{v}|_{H^1(\hat{K})}.
\]  

(6.20)

Relations (6.14), (6.18), and (6.20) imply

\[
|E_K(\hat{f}\hat{v})| \leq C||\hat{f}||_{C^2(\hat{K})} ||\hat{v}||_{L^2(\hat{K})} + C|\hat{f}|_{C^1(\hat{K})} |\hat{v}|_{H^1(\hat{K})}.
\]

Changing variables by the transformation \( F^{-1} \), we get

\[
|E_K(\hat{f}\hat{v})| \leq Ch^2 ||f||_{C^2(\Omega)} ||v||_{L^2(\Omega)} + Ch||f||_{C^1(\Omega)} ||v||_{H^1(\Omega)}
\]  

(6.21)

From (6.12), (6.13), and (6.21), using the Cauchy-Schwarz inequality, we obtain

\[
|\langle f, v \rangle - \langle f, v \rangle_h| \leq Ch^3 \sum_{K \in T_h} ||f||_{C^2(K)} ||v||_{H^1(K)} \leq Ch^3 ||f||_{C^2(\Omega)} \sum_{K \in T_h} ||v||_{H^1(K)}
\]

\[
\leq Ch^2 ||f||_{C^2(\Omega)} ||v||_{H^1(\Omega)} \left( \sum_{K \in T_h} 1 \right)^{1/2} \leq Ch^2 ||f||_{C^2(\Omega)} ||v||_{H^1(\Omega)},
\]

which is (6.11).

**Theorem 6.3.** Let \( u \) and \( u_h \) be the solutions of problems (2.1) and (4.6), respectively. If \( u \in H^4(\Omega) \) and \( f \in C^2(\Omega) \) then

\[
\|u - u_h\|_{H^2(\Omega)} \leq Ch^2 ||f||_{C^2(\Omega)} + Ch||u||_{H^4(\Omega)}.
\]  

(6.22)

**Proof.** The statement follows from the First Strang Lemma (Theorem 4.1.1 in [11]), \( V \)-ellipticity and boundedness of the approximate bilinear form \( a_h(\cdot, \cdot) \) proved in Lemma 5.3, and the consistency results obtained in Lemma 6.1 and Lemma 6.2.

The error estimate (6.22) has suboptimal order \( O(h) \). In section 8, using an auxiliary orthogonal spline collocation problem, we obtain optimal order error estimates in \( H^1 \) and \( H^2 \)-norms under higher than optimal regularity assumptions on the solution of the variational problem. An analysis in [17] shows that the solution of the quadrature scheme based on the three-point Gaussian quadrature has an optimal \( H^2 \)-norm error estimate (see Theorem 8.9).

**7. Additional auxiliary results.** In this section, we obtain auxiliary results that will be used to analyze an auxiliary orthogonal spline collocation scheme for problem (2.1). The following lemma is a generalization of Lemma 4.2 in [6].

**Lemma 7.1.** If \( v \in H^4(\Omega) \) then

\[
\|\partial^\alpha (v - \Pi_h v)\|_h \leq Ch^{4-|\alpha|} ||v||_{H^4(\Omega)}, \quad |\alpha| \leq 2.
\]  

(7.1)

Moreover, if \( v \in H^5(\Omega) \) then

\[
\|\partial^\alpha (v - \Pi_h v)\|_h \leq Ch^3 ||v||_{H^5(\Omega)}, \quad |\alpha| = 2.
\]  

(7.2)
Proof. We prove (7.1) first. Take any \( v \in H^4(\Omega) \) and let \( \alpha \) be a multi-index such that \( |\alpha| \leq 2 \). By (4.3),
\begin{equation}
\| \partial^{\alpha} (v - \Pi_h v) \|_h^2 = \sum_{K \in T_h} \sum_{G_K} \| \partial^{\alpha} (v - \Pi_K v) \|_{G_K}^2.
\end{equation}
Take any \( K \in T_h \), let \( F : \hat{K} = (0,1)^2 \to K \) be an invertible affine mapping, and let \( \hat{v} = v|_K \circ F \). Changing variables by the transformation \( F \), we obtain
\begin{equation}
\sum_{G_K} \| \partial^{\alpha} (v - \Pi_K v) \|_{G_K}^2 \leq Ch^{2-2|\alpha|} \sum_{\xi \in G_K} |l_\xi(\hat{v})|^2,
\end{equation}
where, for any \( \xi \in G_K \),
\begin{equation}
l_\xi(\hat{w}) = \partial^{\alpha} (\hat{w} - \Pi_K \hat{w})(\xi), \quad \text{for all } \hat{w} \in C^2(\hat{K}).
\end{equation}
Let us show that \( |l_\xi(\hat{w})| \leq C|\hat{w}|_{H^4(\hat{K})}, \xi \in G_K \). Linear functional \( l_\xi(\cdot) \) is bounded on \( H^4(\hat{K}) \) since \( H^4(\hat{K}) \subset C^2(\hat{K}) \) and \( |\alpha| \leq 2 \), and it vanishes on polynomials of degree \( \leq 3 \). Applying the Bramble-Hilbert Lemma, we obtain
\begin{equation}
|l_\xi(\hat{w})| \leq C|\hat{w}|_{H^4(\hat{K})}, \hat{w} \in H^4(\hat{K}),
\end{equation}
which implies (7.6).
Substituting (7.6) in (7.4) and changing the variables by the transformation \( F^{-1} \), we get
\begin{equation}
\sum_{G_K} \| \partial^{\alpha} (v - \Pi_K v) \|_{G_K}^2 \leq Ch^{2(4-|\alpha|)} |v|_{H^4(\hat{K})}^2.
\end{equation}
The last estimate along with (7.3) gives
\begin{equation}
\| \partial^{\alpha} (v - \Pi_h v) \|_h^2 \leq Ch^{2(4-|\alpha|)} \sum_{K \in T_h} |v|_{H^4(\hat{K})}^2 = Ch^{2(4-|\alpha|)} |v|_{H^4(\Omega)}^2,
\end{equation}
which implies (7.1). Similarly, we obtain estimate (7.2) using the fact that \( l_\xi(\cdot) \) in (7.5) vanishes on polynomials of degree \( \leq 4 \) when \( |\alpha| = 2 \).

The following is a discrete form of Green’s formula.

**Lemma 7.2.**
\begin{equation}
(\Delta v, w)_h = (v, \Delta w)_h, \quad v \in V_h, \ w \in X_h.
\end{equation}

**Proof.** Take any \( v \in V_h \) and \( w \in X_h \). We prove
\begin{equation}
(v_{x_1 x_1}, w)_h = (v, w_{x_1 x_1})_h,
\end{equation}
and the proof of \( (v_{x_2 x_2}, w)_h = (v, w_{x_2 x_2})_h \) is similar.
Using (4.3), (4.2), (4.1) in $x_1$-direction, the integration by parts in $x_1$-variable, continuity of $v_x, w,$ and the fact $v_x|_{\partial \Omega} = 0,$ we get

$$
(v_{x1};1, w)_h = \sum_{I_1 \times I_2 \in T_h} \sum_{G_{I_2}} \sum_{G_{I_1}} v_{x11} w \eta_{I_2}
$$

(7.9)

$$
= \sum_{I_1 \times I_2 \in T_h} \sum_{G_{I_2}} \left( \int_{I_1} v_{x11} w(x_1, \cdot) dx_1 - C|I_1|^5 (\partial^{3,0} v \partial^{3,0} w)(t_{I_1 \times I_2, \cdot}) \right) = S,
$$

where

$$
S = - \sum_{I_1 \times I_2 \in T_h} \left( \sum_{G_{I_2}} \int_{I_1} v_{x11} w_1(x_1, \cdot) dx_1 + C|I_1|^5 (\partial^{3,0} v \partial^{3,0} w)(t_{I_1 \times I_2, \cdot}) \right).
$$

Similarly, using continuity of $w_{x1}, v|_{\partial \Omega} = 0,$ and the fact that $(\partial^{3,0} w \partial^{3,0} w)|_{I_1 \times I_2}$ is constant in $x_1$-variable, we obtain $(w_{x1,1}, v)_h = S,$ which, along with (7.9), implies (7.8).

The following lemma states that a bicubic polynomial $p$ is uniquely determined by the values of $p$ and $\Delta p$ at the Gauss points in a square and the values of $p$ at any two connected vertices of the square.

**Lemma 7.3.** Let $K = (-1, 1)^2,$ and let $\eta_1$ and $\eta_2$ be connected vertices of $K.$ Let $p(x_1, x_2)$ be a bicubic polynomial such that

$$
(7.10) \quad p(\xi) = \Delta p(\xi) = 0, \quad \text{for all} \quad \xi \in \mathcal{G}_K, \quad p(\eta_i) = p_{x1}(\eta_i) = p_{x2}(\eta_i) = p_{x1x2}(\eta_i) = 0, \quad i = 1, 2.
$$

Then, $p = 0.$

**Proof.** Let the following set of 16 collocation points

$$
\{\mathcal{G}_K, \mathcal{G}_K, \eta_1, \eta_1, \eta_1, \eta_1, \eta_2, \eta_2, \eta_2, \eta_2\}
$$

and the set of the coefficients of the polynomial $p$ be ordered in a certain way, and let $M \in \mathbb{R}^{16 \times 16}$ be the matrix corresponding to the equations in (7.10). Using a computer algebra system, we computed

$$
\det(M) = \pm 268435456/59049,
$$

which implies $p = 0.$

We note that, if the vertices $\eta_1$ and $\eta_2$ are not connected, then matrix $M$ is singular.

**8. An equivalent OSC problem.** In this section, we introduce an auxiliary orthogonal spline collocation scheme, determine its relation with the quadrature scheme, and obtain optimal order error estimates. As in [16], we consider the following coupled form of problem (1.1):

$$
(8.1) \quad \Delta u = v, \quad \Delta v = f \quad \text{in} \ \Omega, \quad \text{and} \quad u = \partial_n u = 0 \ \text{on} \ \partial \Omega.
$$

Let $\mathcal{G}_h = \bigcup_{K \in T_h} \mathcal{G}_K$ be the set of all Gauss points in $\Omega.$ We note that the number of points in $\mathcal{G}_h$ is greater than the dimension of $V_h$ since the number of elements in $T_h$ is greater than the number of interior vertices of $T_h.$ We consider the following
orthogonal spline collocation scheme for problem (8.1): find $u_h \in V_h$ and $v_h \in X_h$, such that

\begin{align}
(8.2a) \quad & \Delta u_h(\xi) = v_h(\xi), \quad \text{for all } \xi \in \mathcal{G}_h, \\
(8.2b) \quad & \Delta v_h(\xi) = f(\xi), \quad \text{for all } \xi \in \mathcal{G}_h.
\end{align}

**Lemma 8.1.** Let $f \in C(\Omega)$ and let $\{\eta_1, \eta_2\}$ be connected vertices of any element $K_\eta \in T_h$. The system of OSC equations (8.2) is under-determined by eight constraints. The system (8.2) with the additional equations

\begin{equation}
(8.3) \quad v_h(\eta_i) = (v_h)_{x_i}(\eta_i) = (v_h)_{x_1x_2}(\eta_i) = 0, \quad i = 1, 2,
\end{equation}

has a unique solution $\{u_h, v_h\}$, where $u_h$ is the solution of the quadrature scheme (4.6).

**Proof.** Let us prove that the OSC system (8.2) is under-determined by eight constraints. Let $N_e, N_i,$ and $N_b$ denote, respectively, the numbers of elements, internal nodes, and boundary nodes in the triangulation $T_h$. The set $\mathcal{G}_h$ has $4N_e$ points; hence, there are $8N_e$ equations in (8.2). Using a mathematical induction argument, we verified that $N_b = 2(N_e - N_i + 1)$. Thus, the system (8.2) has

$$8N_e = 8N_i + 4N_b - 8 \quad \text{constraints.}$$

Since $u_h \in V_h$ and $v_h \in X_h$, the system (8.2) has

$$4N_i + 4(N_i + N_b) = 8N_i + 4N_b \quad \text{degrees of freedom;}$$

that is, the OSC system (8.2) is under-determined by eight constraints.

Let us prove that the problem (8.2)–(8.3) has a unique solution $\{u_h, v_h\}$. Since the numbers of degrees of freedom and constraints are equal, it suffices to show that the homogeneous system consisting of the equations (8.2a),

\begin{equation}
(8.4) \quad \Delta v_h(\xi) = 0, \quad \text{for all } \xi \in \mathcal{G}_h,
\end{equation}

and (8.3) has only the zero solution. Let $u_h$ and $v_h$ be solutions of $\{(8.2a),(8.4), (8.3)\}$. Using (4.5), (8.2a), Lemma 7.2, and (8.4), we obtain

$$a_h(u_h, u_h) = (\Delta u_h, \Delta u_h)_h = (v_h, \Delta u_h)_h = (\Delta v_h, u_h)_h = 0.$$\n
Hence, by (5.23), we obtain $u_h = 0$.

Let us prove that $v_h = 0$. Using (8.2a) with $u_h = 0$, (8.4), (8.3), and Lemma 7.3, we obtain $v_h|_{K_{\eta}} = 0$. By a similar argument, $v_h|_{K_{\eta}} = 0$ for any element $K \in T_h$ adjacent to $K_\eta$. By recursion, $v_h|_{K} = 0$ for any $K \in T_h$, and, hence, $v_h = 0$.

Thus, problem (8.2)–(8.3) has a unique solution $\{u_h, v_h\}$. Using (4.5), (8.2a), Lemma 7.2, and (8.2b), we obtain, for any $w \in V_h$,

$$a_h(u_h, w) = (\Delta u_h, \Delta w)_h = (v_h, \Delta w)_h = (\Delta v_h, w)_h = (f, w)_h,$$

that is, $u_h$ is a solution of problem (4.6), which is unique by Theorem 5.4. \Box

In [16], for $\Omega = (0, 1)^2$, problem (8.1) was approximated by the OSC scheme (8.2) with additional eight constraints

\begin{equation}
(8.5) \quad v_h(a, b) = \frac{\partial v_h}{\partial x_2}(a, b) = 0, \quad \text{for } a, b = 0, 1.
\end{equation}
The authors proved existence and uniqueness of the solution and obtained optimal order error estimates in the Sobolev $H^k$-norms for $k = 0, 1, 2$. In the following theorem, we state optimal order convergence estimates for the quadrature scheme (4.6) in the $H^1$- and $H^2$-norms.

**Theorem 8.2.** Let $f \in C(\Omega)$ and let $u$ and $u_h$ be the solutions of the problems (2.1) and (4.6), respectively. If $u \in H^{3-k}(\Omega)$, then

$$
\|u - u_h\|_{H^k(\Omega)} \leq Ch^{4-k}\|u\|_{H^{6-k}(\Omega)}, \quad k = 1, 2.
$$

*Proof.* It follows from Theorem 5.4 and Lemma 8.1, that the quadrature scheme (4.6) has a unique solution $u_h$, which is also the solution of the OSC scheme (8.2)–(8.3). The proof of (8.6) is similar to that of Theorem 3.1 in [16]. Since $u \in H^6(\Omega)$, $u$ and $v = \Delta u \in C^2(\Omega)$ are solutions of the coupled problem (8.1). Let

$$
U = u_h - \Pi_h u \quad \text{and} \quad V = v_h - \Pi_h v,
$$

where $v_h$ is the solution of the OSC scheme (8.2)–(8.3). We note that

$$
U \in V_h \quad \text{and} \quad V \in X_h.
$$

It follows from (8.2a) and $\Delta u = v$ that

$$
\Delta U(\xi) - V(\xi) = -\Delta(\Pi_h u)(\xi) + (\Pi_h)v(\xi)
$$

$$
= \Delta(u - (\Pi_h)u)(\xi) - (v - (\Pi_h)v)(\xi), \quad \text{for any } \xi \in G_h.
$$

Similarly, using (8.2b) and $\Delta v = f$, we obtain

$$
\Delta V(\xi) = \Delta(v - \Pi_h v)(\xi), \quad \text{for any } \xi \in G_h.
$$

Taking the product $(\cdot, \cdot)_h$ of (8.9) and (8.10) with $\Delta U$ and $U$, respectively, we obtain

$$
\|\Delta U\|_h^2 = (V, \Delta U)_h = (\Delta(u - \Pi_h u), \Delta U)_h = (v - \Pi_h v, \Delta U)_h,
$$

$$
(\Delta V, U)_h = (\Delta(v - \Pi_h v), U)_h.
$$

Summing the last two identities and using (8.8) and Lemma 7.2, we obtain

$$
\|\Delta U\|_h^2 = (\Delta(u - \Pi_h u), \Delta U)_h - (v - \Pi_h v, \Delta U)_h + (\Delta(v - \Pi_h v), U)_h.
$$

Using the Cauchy-Schwarz inequality for $(\cdot, \cdot)_h$, estimate (5.22) with $\alpha = (0,0)$ and (5.23), we get

$$
\|\Delta U\|_h \leq \|\Delta(u - \Pi_h u)\|_h + \|v - \Pi_h v\|_h + C\|\Delta(v - \Pi_h v)\|_h.
$$

Using (5.23), (8.11), Lemma 7.1, and $v = \Delta u$, we obtain

$$
\|U\|_{H^2(\Omega)} \leq C\|\Delta U\|_h \leq Ch^{4-k}\|\Delta u\|_{H^{6-k}(\Omega)} + Ch^4\|v\|_{H^{4}(\Omega)} + Ch^{4-k}\|v\|_{H^{6-k}(\Omega)},
$$

$$
\|U\|_{H^2(\Omega)} \leq Ch^{4-k}\|u\|_{H^{6-k}(\Omega)}, \quad k = 1, 2.
$$

Finally, using the triangle inequality for the $H^k$-norm, (6.3) with $m = 1, 2$, (8.7), and (8.12), we get

$$
\|u - u_h\|_{H^k(\Omega)} \leq \|u - \Pi_h u\|_{H^k(\Omega)} + \|\Pi_h u - u_h\|_{H^2(\Omega)} \leq Ch^{4-k}\|u\|_{H^{6-k}(\Omega)}, \quad k = 1, 2,
$$

which is (8.6).  \[ \square \]
Table 9.1

Maximum nodal errors and approximate convergence orders.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$e_h$</th>
<th>$(e_h)_{x_1}$</th>
<th>$(e_h)_{x_2}$</th>
<th>$(e_h)_{x_1,x_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/8$</td>
<td>1.3E-3</td>
<td>7.8E-4</td>
<td>4.6</td>
<td>7.8E-4</td>
</tr>
<tr>
<td>$1/16$</td>
<td>7.9E-5</td>
<td>4.2E-5</td>
<td>4.2</td>
<td>9.4E-4</td>
</tr>
<tr>
<td>$1/32$</td>
<td>4.9E-5</td>
<td>2.5E-6</td>
<td>4.1</td>
<td>5.8E-5</td>
</tr>
<tr>
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<td>3.0E-7</td>
<td>1.5E-7</td>
<td>4.0</td>
<td>3.6E-6</td>
</tr>
<tr>
<td>$1/128$</td>
<td>1.1E-8</td>
<td>3.3E-8</td>
<td>2.2</td>
<td>3.1E-7</td>
</tr>
</tbody>
</table>

Table 9.2

Sobolev norm errors and convergence orders.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$L^2$</th>
<th>$H^1$</th>
<th>$H^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/8$</td>
<td>6.9E-4</td>
<td>2.2E-2</td>
<td>1.1E+0</td>
</tr>
<tr>
<td>$1/16$</td>
<td>4.2E-5</td>
<td>2.7E-3</td>
<td>2.8E-1</td>
</tr>
<tr>
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<td>2.6E-6</td>
<td>3.4E-4</td>
<td>6.9E-2</td>
</tr>
<tr>
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<td>1.6E-7</td>
<td>4.2E-5</td>
<td>1.7E-2</td>
</tr>
<tr>
<td>$1/128$</td>
<td>1.1E-8</td>
<td>5.2E-6</td>
<td>4.3E-3</td>
</tr>
</tbody>
</table>

9. Numerical results. In this section, we present numerical results for a test problem, which was used by several authors, in particular, as Example 1 in [5] and as Problem 2 in [2]. A test problem is considered on $\Omega = (0,1)^2$ with the exact solution $u(x) = (1 - \cos 2\pi x_1)(1 - \cos 2\pi x_2)$. We carried out a series of computations for decreasing values of $h$ obtained by halving to determine convergence orders of the numerical solution. Linear systems were solved by an LU decomposition.

Let $\tilde{e}_h$ be an error defined on the set $N_h$ of the interior nodes of the triangulation $T_h$. Let $\|\tilde{e}_h\|_{C_h} = \max_{x \in N_h} |\tilde{e}_h(x)|$ be the maximal nodal error of $\tilde{e}_h$. If $\|\tilde{e}_h\|_{C_h} = O(h^p)$ for $p > 0$, then the quantity $\log_2(\|\tilde{e}_{2h}\|_{C_h}/\|\tilde{e}_h\|_{C_h})$, which we call an approximate convergence order, approximates $p$.

In Table 9.1, we present the maximum nodal errors and the corresponding approximate convergence orders of the quadrature Galerkin solution and its first-order and mixed second-order derivatives. We observe that all approximate convergence orders are close to 4. The higher convergence orders for the derivatives indicate to a superconvergence property of the quadrature solution. The approximate convergence orders for $h = 1/128$ decrease due to the effect of round-off errors. We note that the condition number of the stiffness matrix is of order $O(h^{-4})$, and for $h = 1/128$, the condition number $\approx 10^7$.

In Table 9.2, we present errors in Sobolev norms and their approximate convergence orders. The approximate convergence orders of the error in the $H^2$-norm, $k = 0, 1, 2$, are close to the optimal values 4, 3, and 2, respectively. We note that, for $h = 1/128$, in contrast with the maximum nodal norm, the convergence order corresponding to the $L^2$-norm deteriorated insignificantly due to the average nature of the norm.

10. Conclusion. A quadrature finite element Galerkin scheme for a biharmonic Dirichlet problem on a rectangular polygonal domain is proposed and analyzed. A finite element theory suggests to use a Gaussian quadrature with at least three points. Our quadrature scheme uses a numerical integration based on the product two-point Gaussian quadrature and the finite element space of Bogner-Fox-Schmit rectangles. Forming a stiffness matrix for the proposed scheme is faster since fewer function evalu-
A quadrature scheme for a biharmonic problem

propositions are required. The scheme has a unique solution, which converges to the solution of the boundary value problem with optimal rates in the Sobolev $H^1$- and $H^2$-norms. The solution of the quadrature scheme is also the solution of an equivalent orthogonal spline collocation problem. Numerical results confirm our theoretical estimates and demonstrate a superconvergence property of the quadrature Galerkin solution.

REFERENCES