MULTILEVEL PRECONDITIONERS FOR NON–SELF-ADJOINT OR INDEFINITE ORTHOGONAL SPLINE COLLOCATION PROBLEMS

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Abstract. Efficient numerical algorithms are developed and analyzed that implement symmetric multilevel preconditioners for the solution of an orthogonal spline collocation (OSC) discretization of a Dirichlet boundary value problem with a non–self-adjoint or an indefinite operator. The OSC solution is sought in the Hermite space of piecewise bicubic polynomials. It is proved that the proposed additive and multiplicative preconditioners are uniformly spectrally equivalent to the operator of the normal OSC equation. The preconditioners are used with the preconditioned conjugate gradient method, and numerical results are presented that demonstrate their efficiency.

Key words. orthogonal spline collocation, multilevel methods, preconditioner, non–self-adjoint or indefinite operator, elliptic boundary value problem

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1. Introduction. Let Ω be a unit square (0, 1) × (0, 1) with boundary ∂Ω, and let x = (x1, x2). We consider a Dirichlet boundary value problem (BVP)

\[ Lu = f \quad \text{in } \Omega \quad \text{and } u = 0 \quad \text{on } \partial \Omega, \]

where we let

\[ Lu(x) = \sum_{i,j=1}^{2} a_{ij}(x) u_{x_i x_j}(x) + \sum_{i=1}^{2} b_i(x) u_{x_i}(x) + c(x) u(x). \]

With respect to BVP (1.1), we make the following assumptions. The functions \( \{a_{ij}\}_{i,j=1}^{2}, \{b_i\}_{i=1}^{2}, c, \) and \( f \) are sufficiently smooth, and \( a_{12} = a_{21} \). The differential operator \( L \) satisfies the uniform ellipticity condition; that is, there is \( \nu > 0 \) such that

\[ \sum_{i,j=1}^{2} a_{ij}(x) \eta_i \eta_j \geq \nu (\eta_1^2 + \eta_2^2), \quad x \in \Omega, \ (\eta_1, \eta_2) \in \mathbb{R}^2. \]

For any \( f \in L^2(\Omega) \), BVP (1.1) has a unique solution \( u(x) \) in

\[ \tilde{H}_0^2(\Omega) = \{ v \in H^2(\Omega) : v = 0 \ \text{on} \ \partial \Omega \}, \]

where \( L^2(\Omega) \) and \( H^2(\Omega) \) are the Sobolev spaces.

We approximate BVP (1.1) by an orthogonal spline collocation (OSC) scheme, in which the discrete solution is sought in the Hermite space of piecewise bicubic polynomials, and it satisfies the differential equation exactly at the special set of collocation points. The primary advantages of the OSC method are as follows: it has low computational cost of forming a linear system of algebraic equations; it has relatively easy
The application of higher order finite elements; the OSC solution possesses optimal order error estimates [7]; and the solution exhibits the so-called superconvergence property [8, 15]. The matrix of a linear system resulting from the OSC discretization is sparse and can be expressed as a sum of tensor products of so-called almost block diagonal matrices [2].

The solution of the OSC equations by a banded Gaussian elimination requires \( O(N^2) \) arithmetic operations, where \( N \) is the number of unknowns [22, 23, 34]. If the differential operator is separable and the partition is uniform in one direction, then the OSC problem can be solved by a fast direct algorithm with the cost \( O(N \log N) \) [10].

Classical iterative methods, such as Jacobi, Gauss–Seidel, and SOR, for the OSC solution of Poisson’s equation on a uniform partition were studied in [21, 26, 37]. ADI methods for solving OSC problems with separable operators were investigated in [5, 14, 18].

Modern techniques are underdeveloped for the solution of OSC equations in comparison with finite element Galerkin or finite difference methods. Only a few optimal cost algorithms have been proposed to solve the OSC discretization of self-adjoint positive definite BVPs. In [19], the multigrid method was applied to the OSC problem and compared with a multigrid finite difference method. The author concluded that the proposed multigrid OSC method is less efficient than a multigrid finite difference method. A domain decomposition–based fast solver for the OSC discretization of the Dirichlet problem for Poisson’s equation was developed in [6] requiring \( O(N \log \log N) \) arithmetic operations. In [13], multigrid methods were developed and analyzed for quadratic spline collocation equations. Numerical results were presented indicating that a multigrid iteration is an efficient solver for the quadratic spline collocation equations.

It is well known that the primary issue in the efficient application of an iterative algorithm for solving a BVP is the construction of a good preconditioner. In [24], the authors studied preconditioning of a non–self-adjoint or an indefinite OSC operator by a finite element operator and investigated \( H^1 \) condition numbers and the distribution of singular values of the preconditioned matrices. Additive and multiplicative multilevel preconditioners were proposed in [9] for the iterative solution of the OSC discretization of a self-adjoint positive definite Dirichlet BVP. It was proved that the preconditioners are uniformly spectrally equivalent to the OSC operator corresponding to a BVP with the Laplacian and require \( O(N) \) arithmetic operations. An efficient two-level domain decomposition–type “edge” preconditioner was proposed in [29] that requires \( O(N) \) arithmetic operations. The preconditioner is applied with the GMRES method, and the number of iterations is independent of the partition stepsize \( h \).

Numerical techniques developed for self-adjoint positive definite BVPs are usually inefficient or even fail when applied to non–self-adjoint or indefinite BVPs, and hence, special, more sophisticated methods are required to obtain the solution [4, 11, 12, 27, 28]. A fast direct preconditioning algorithm for the solution of the normal OSC equation approximating non–self-adjoint or indefinite BVPs was proposed in [1].

In this work, we develop additive and multiplicative multilevel preconditioners for the computation of the solution of the normal OSC equation. Results and algorithms presented in this paper are closely related to those in [1, 7, 9, 31, 32, 39, 40]. To prove uniform spectral equivalence of our preconditioners, we use the approach described in [31] and [32] based on the equivalence of a norm of a certain Besov space with the Sobolev \( H^2 \)-norm. We note that the approach used in [39] and [40] requires higher regularity of the solution of BVP (1.1). Our main conclusion in this work is that the
general framework of multilevel methods can be applied to the OSC discretization of BVPs to construct efficient preconditioners. Rather general non–self-adjoint or indefinite OSC problems can be preconditioned quite well by the proposed multilevel OSC preconditioners.

The outline of this article is as follows. We introduce our notation and define the OSC problem in section 2. In section 3, we present auxiliary facts that are used to prove main results of this work. In section 4, we define additive and multiplicative OSC preconditioners and prove that they are uniformly spectrally equivalent to the operator of the normal OSC equation. In section 5, we introduce the matrix-vector form of the OSC problem in the standard Hermite finite element basis and obtain recurrence relations for the computation of the OSC approximations and other quantities required by the multilevel algorithms. In section 6, we describe implementations of the additive and the multiplicative preconditioners, and in section 7, we present numerical results of the application of the preconditioners with the preconditioned conjugate gradient (PCG) method to solve test problems.

2. OSC problem. In this section, we introduce our notation and define the OSC problem. Throughout this paper, $C$, $C_1$, and $C_2 \geq C_1$ denote generic positive constants independent of the partition stepsize, the number of partition levels, and other variables in the expressions where the constants appear. By $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{H^2(\Omega)}$ we denote the standard Sobolev norms.

**Construction of nested spaces.** We set $\pi_0 = \Omega$ and, for integer $K > 0$, we construct a sequence of partitions $\{\pi_k\}_{k=0}^K$ by subdividing each rectangular element of partition $\pi_{k-1}$ into four congruent rectangular elements of partition $\pi_k$. Let $h_k = 2^{-k}$ denote the stepsize of partition $\pi_k$. In what follows, if not stated otherwise, the index variable $k$ takes all values in $\{0, 1, \ldots, K\}$. We note that there are $K + 1$ partition levels, and integer $K$ is an important parameter in our analysis.

Let $V_k$ be the vector space of Hermite piecewise bicubic polynomials that vanish on $\partial\Omega$, which has the dimension $N_k = 4^{k+1}$ (see Chapter 3 in [35]). The sequence of vector spaces $\{V_k\}$ is nested as follows:

$$V_0 \subset V_1 \subset \cdots \subset V_K \subset \bar{H}_0^2(\Omega).$$

We denote $h = h_K$, $N = N_K$, $\pi_h = \pi_K$, and $V_h = V_K$.

**Original OSC equation.** Let $\mathcal{G}_h$ be the set of nodes of the two-dimensional composite Gaussian quadrature on partition $\pi_h$ with 4 nodes in each element of $\pi_h$. In the OSC discretization of BVP (1.1), we seek $u_h \in V_h$ that satisfies the OSC equations

$$(2.2) \quad Lu_h(\xi) = f(\xi), \quad \xi \in \mathcal{G}_h,$$

where the differential operator $L$ is defined in (1.2). Existence and uniqueness of a solution and a convergence analysis of problem (2.2) are given in [7].

The OSC problem (2.2) can be written as the operator equation

$$(2.3) \quad L_h u_h = f_h,$$

where the OSC operator $L_h$ from $V_h$ into $V_h$ and the vector $f_h \in V_h$ are defined by

$$(2.4) \quad (L_h v)(\xi) = L v(\xi), \quad \text{for any } \xi \in \mathcal{G}_h \text{ and for any } v \in V_h,$$

$$f_h(\xi) = f(\xi), \quad \text{for any } \xi \in \mathcal{G}_h.$$

Both $L_h$ and $f_h$ are well defined since a function in $V_h$ is uniquely determined by its values at $\mathcal{G}_h$ (see Lemma 5.1 in [33]). We call (2.3) the original OSC equation.
Normal OSC equation. The vector space $V_h$ is a Hilbert space with the inner product

$$ (v, w)_h = \frac{h^2}{4} \sum_{\xi \in \mathcal{G}_h} v(\xi) w(\xi), $$

which corresponds to the composite Gaussian quadrature on $\pi_h$. Let $L^*_h$ be the adjoint to $L_h$ with respect to the inner product $(\cdot, \cdot)_h$. Applying $L^*_h$ on (2.3), we obtain the normal OSC equation

$$ L^*_h L_h u_h = L^*_h f_h. $$

We introduce a bilinear form

$$ a_h(v, w) = (L^*_h L_h w, v)_h, \quad w, v \in V_h, $$

and consider the following variational form of problem (2.6): find $u_h \in V_h$ that satisfies

$$ a_h(u_h, v) = (L^*_h f_h, v)_h \quad \text{for all} \quad v \in V_h. $$

The problems (2.8) and (2.2) are equivalent; hence, problem (2.8) has a unique solution. In this work, we develop and analyze multilevel preconditioners for the iterative solution of (2.8).

Space decompositions. In what follows, we denote $\sum_k$ and $\sum_{k,i}$ for $\sum_{k=0}^K$ and $\sum_{k=0}^K \sum_{i=1}^{N_k}$, respectively, where $N_k$ is the dimension of $V_k$.

Let $\{\psi^k_i\}_{i=1}^{N_k}$ be a finite element basis of $V_k$ that satisfies

$$ C_1 h_k^{1-|\alpha|} \leq \|\partial^\alpha \psi^k_i\|_{L^2(\Omega)} \leq C_2 h_k^{1-|\alpha|}, \quad |\alpha| \leq 2, $$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index (see Theorem 5.7 in [3]). Let

$$ V_{ki} = \text{span}(\psi^k_i), \quad 1 \leq i \leq N_k, $$

be one-dimensional subspaces of $V_k$. Based on (2.1), we consider the following two space decompositions:

$$ \sum_k V_k = V_h \quad \text{and} \quad \sum_{k,i} V_{ki} = V_h. $$

For $v \in V_h$, let

$$ \mathcal{V}_1(v) = \left\{ \{v_k\} : \sum_k v_k = v, \ v_k \in V_k, \ 0 \leq k \leq K \right\}, $$

$$ \mathcal{V}_2(v) = \left\{ \{v_{ki}\} : \sum_{k,i} v_{ki} = v, \ v_{ki} \in V_{ki}, \ 1 \leq i \leq N_k, \ 0 \leq k \leq K \right\}. $$

We call an element in $\mathcal{V}_1(v)$ and in $\mathcal{V}_2(v)$ a representation of $v$. The sets $\mathcal{V}_1(v)$ and $\mathcal{V}_2(v)$ will be used to define auxiliary equivalent norms on $V_h$.

3. Auxiliary results. In this section we present auxiliary facts that are used to prove main results of this work. The following inequalities are proved in Theorem 3.1 in [1].

**Lemma 3.1.** For $h$ sufficiently small,

$$ C_1 \|v\|^2_{H^2(\Omega)} \leq a_h(v, v) \leq C_2 \|v\|^2_{H^2(\Omega)}, \quad v \in V_h. $$
Remark. In what follows, by sufficiently small $h$, we mean values of $h$ for which the statement of Lemma 3.1 holds.

It follows from (3.1) and (2.7) that, for $h$ sufficiently small, the bilinear form $a_h(\cdot,\cdot)$ is an inner product on $V_h$. The following is the estimate (8.24) in Chapter 3 of [25].

**Lemma 3.2.**

\[
\|v\|_{H^2(\Omega)} \leq C\|\Delta v\|_{L^2(\Omega)}, \quad v \in \tilde{H}_0^2(\Omega),
\]

(3.2)

Inequalities (3.1), (3.2), and

\[
\|\Delta v\|_{L^2(\Omega)}^2 \leq C \int_\Omega (v_{x_1}^2 + v_{x_2}^2) dx, \quad v \in \tilde{H}_0^2(\Omega),
\]

(3.3)

imply that, for $h$ sufficiently small,

\[
C_1\|\Delta v\|_{L^2(\Omega)}^2 \leq a_h(v,v) \leq C_2\|\Delta v\|_{L^2(\Omega)}^2, \quad v \in V_h.
\]

(3.4)

**Lemma 3.3.** Let $v \in V_k$, and let

\[
v = \sum_{i=1}^{N_k} c_i \psi_i^k \text{ and } \bar{c}_v = (c_1, \ldots, c_{N_k})^t.
\]

(3.5)

Then,

\[
C_1 h_k |\bar{c}_v| \leq \|v\|_{L^2(\Omega)} \leq C_2 h_k |\bar{c}_v|, \quad v \in V_k,
\]

(3.6)

where $|\cdot|$ is the 2-norm on $R^{N_k}$.

**Proof.** Using (3.5), we obtain

\[
\|v\|_{L^2(\Omega)}^2 = \int_\Omega \sum_{i=1}^{N_k} c_i \psi_i^k \sum_{j=1}^{N_k} c_j \psi_j^k dx = \sum_{i,j=1}^{N_k} c_i c_j \int_\Omega \psi_i^k \psi_j^k dx.
\]

The inequalities in (3.6) follow from the last identity and the fact that the eigenvalues of the mass matrix corresponding to the finite element basis \{\psi_i^k\}_{i=1}^{N_k} belong to the interval $[C_1 h_k^2, C_2 h_k^2]$ (see (5.103) in [3]).

Let

\[
\|v\|_{*,h} = \left( \inf_{V_1(v)} \sum_{k=0}^{K} h_k^{-4} \|v_k\|_{L^2(\Omega)}^2 \right)^{1/2}, \quad v \in V_h,
\]

(3.7)

where $\inf_{V_1(v)}$ denotes the infimum with respect to all representations \{v_k\} in $V_1(v)$. The following important statement is similar to Corollary 2.1 in [32] and Lemma 2 in [31].

**Lemma 3.4.** The norms $\|\cdot\|_{H^2(\Omega)}$ and $\|\cdot\|_{*,h}$ are uniformly equivalent on $V_h$; that is,

\[
C_1 \|v\|_{H^2(\Omega)} \leq \|v\|_{*,h} \leq C_2 \|v\|_{H^2(\Omega)}, \quad v \in V_h.
\]

(3.8)

**Proof.** First, we prove the second inequality in (3.8). It is known that the Besov space $B^2_{2,2}(\Omega)$ coincides, up to equivalent norms, with the Sobolev space $H^2(\Omega)$ (see
part (b) of Theorem 4.6.1 and (3) of section 4.2.1 in [38]). Since \( \tilde{H}_0^2(\Omega) \) is a closed subspace of \( H^2(\Omega) \), it is a closed subspace of \( B_{2,2}^2(\Omega) \). We note that functions in \( \tilde{H}_0^2(\Omega) \) are continuous on \( \overline{\Omega} \), and therefore, for any \( v \in \tilde{H}_0^2(\Omega) \), \( v(x) = 0 \) for all \( x \in \partial\Omega \); in particular, the trace of \( v \in \tilde{H}_0^2(\Omega) \) is continuous. In a natural way, we extend the definition of \( \{ V_k \} \) for \( k > K \). It follows from Theorem 5.1 in [16] that

\[
\| v \|_{A_{2,2}^2} = \left( \| v \|_{L^2_2(\Omega)}^2 + \sum_{k=0}^{\infty} 2^{4k} \left( \inf_{z \in V_k} \| v - z \|_{L^2_2(\Omega)} \right)^2 \right)^{1/2}
\]

is a norm on \( \tilde{H}_0^2(\Omega) \) equivalent to the standard norm in \( B_{2,2}^2(\Omega) \) (see [30] for a definition of the approximation space \( A_{2,2}^2 \)). Therefore, using the equivalence of norms \( \| \cdot \|_{H^2(\Omega)} \) and \( \| \cdot \|_{A_{2,2}^2} \) on \( \tilde{H}_0^2(\Omega) \) and the fact \( V_h \subset \tilde{H}_0^2(\Omega) \), we get

\[
\| v \|_{A_{2,2}^2} \leq C \| v \|_{H^2(\Omega)}, \quad v \in V_h.
\]

Let us prove

\[
\| v \|_{A_{2,2}^2} \leq C \| v \|_{H^2(\Omega)}, \quad v \in V_h.
\]

We note that

\[
\| v \|_{A_{2,2}^2} = \left( \| v \|_{L^2_2(\Omega)}^2 + \sum_{k=0}^{K-1} 2^{4k} \left( \inf_{z \in V_k} \| v - z \|_{L^2_2(\Omega)} \right)^2 \right)^{1/2}, \quad v \in V_h,
\]

since

\[
\inf_{z \in V_k} \| v - z \|_{L^2_2(\Omega)} = 0, \quad k \geq K, \quad v \in V_h.
\]

Take any \( v \in V_h \) and set \( v_0 = z_0 \) and \( v_k = z_k - z_{k-1} \) for \( 1 \leq k \leq K \), where \( z_k \) is the orthogonal \( L^2 \)-projection of \( v \) into \( V_k \). Note that \( v = z_K = \sum_{k=0}^{K} v_k \). Using \( \| v_0 \|_{L^2_2(\Omega)} \leq \| v \|_{L^2_2(\Omega)} \), the triangle inequality, the definition of \( \{ z_k \} \), and (3.11), we obtain

\[
\sum_{k=0}^{K} h_k^{-4} \| v_k \|_{L^2_2(\Omega)}^2 = \| v_0 \|_{L^2_2(\Omega)}^2 + \sum_{k=1}^{K} h_k^{-4} \| z_k - v + v - z_{k-1} \|_{L^2_2(\Omega)}^2
\]

\[
\leq \| v \|_{L^2_2(\Omega)}^2 + 4 \sum_{k=0}^{K-1} h_k^{-4} \| v - z_k \|_{L^2_2(\Omega)}^2
\]

\[
= \| v \|_{L^2_2(\Omega)}^2 + 4 \sum_{k=0}^{K-1} 2^{4k} \left( \inf_{z \in V_k} \| v - z \|_{L^2_2(\Omega)} \right)^2 \leq 4 \| v \|_{A_{2,2}^2}^2.
\]

Taking the infimum over \( V_1(v) \), we get (3.10). Inequalities (3.10) and (3.9) imply the second inequality in (3.8).

Let us prove the first inequality in (3.8). Take any \( v \in V_h \) and let \( \{ v_k \} \in V_1(v) \). Using the strengthened Cauchy–Schwarz inequality,

\[
\int_{\Omega} \Delta v_k \Delta v_l dx \leq C 2^{-|k-l|/2} (h_k h_l)^{-2} \| v_k \|_{L^2_2(\Omega)} \| v_l \|_{L^2_2(\Omega)}, \quad v_k \in V_k, \quad v_l \in V_l,
\]
(see Lemma 5.1 in [40]) and the fact that the spectral radius of matrix $B = (b_{kl})$ with the entries

$$b_{kl} = 2^{-|k-l|/2}, \quad 0 \leq k, l \leq K,$$

is bounded by the maximum norm $\|B\|_{\infty} \leq 3 + 2\sqrt{2}$, we get

$$\|\Delta v\|_{L^2(\Omega)}^2 = \int_{\Omega} (\Delta v)^2 dx = \int_{\Omega} \left( \sum_{k=0}^{K} \Delta v_k \right)^2 dx = \sum_{k,l=0}^{K} \int_{\Omega} \Delta v_k \Delta v_l dx$$

$$\leq C \sum_{k,l=0}^{K} 2^{-|k-l|/2} h_k h_l^{-2} \|v_k\|_{L^2(\Omega)} \|v_l\|_{L^2(\Omega)}$$

$$\leq C(3 + 2\sqrt{2}) \sum_{k} h_k^{-2} \|v_k\|_{L^2(\Omega)}^2.$$

From the last estimate, using (3.2) and taking the infimum over $V_1(v)$, we obtain the first inequality in (3.8).

To finish the proof of the lemma, we establish that $\| \cdot \|_{*,h}$ is a norm on $V_h$. It follows from the inequalities in (3.8) that $\|v\|_{*,h} \geq 0$ for any $v \in V_h$, and $\|v\|_{*,h} = 0$ if and only if $v = 0$. Let us show that $\|cv\|_{*,h} = |c| \|v\|_{*,h}$ for any $v \in V_h$ and $c \in \mathbb{R}$. Since the case $c = 0$ is trivial, assume $c \neq 0$. Using the fact that the infimum over $V_1(cv)$ equals the infimum over $V_1(v)$, we obtain

$$\|cv\|_{*,h} = \inf_{\{v_k\} \in V_1(cv)} \sum_{k=0}^{K} h_k^{-4} \|v_k\|_{L^2(\Omega)}^2 = \inf_{\{v_k\} \in V_1(v)} \sum_{k=0}^{K} h_k^{-4} \|cv_k\|_{L^2(\Omega)}^2 = c^2 \|v\|_{*,h}^2,$$

which implies the required identity.

To prove the triangle inequality for $\| \cdot \|_{h,*}$, using the triangle inequality for the $L^2$-norm and the Minkowski inequality, we obtain, for any $v$ and $w$ in $V_h$,

$$\|v + w\|_{*,h}^2 = \inf_{\{z_k\} \in V_1(v+w)} \sum_{k=0}^{K} h_k^{-4} \|z_k\|_{L^2(\Omega)}^2$$

$$\leq \inf_{V_1(v)} \inf_{V_1(w)} \sum_{k=0}^{K} h_k^{-4} (\|v_k\|_{L^2(\Omega)} + \|w_k\|_{L^2(\Omega)})^2 \leq (\|v\|_{*,h} + \|w\|_{*,h})^2.$$

Thus, $\| \cdot \|_{*,h}$ is a norm on $V_h$ which is equivalent to the $H^2$-norm. □

**Remark.** The result of Lemma 3.4 is analogous to those formulated in [32, Corollary 2.1] and [31, Lemma 2], where relations similar to (3.8) were proved first for Sobolev spaces with equivalent norms involving infinite number series, and then the versions for finite-dimensional subspaces were obtained. Our proof of Lemma 3.4 is somewhat different since it is based on the representation (3.11).

Let

$$\|v\|_{\Sigma,\Delta} = \left( \inf_{V_2(v)} \sum_{k,l} \|\Delta v_{kl}\|_{L^2(\Omega)}^2 \right)^{1/2}, \quad v \in V_h.$$

**Lemma 3.5.**

$$C_1 \|v\|_{H^2(\Omega)} \leq \|v\|_{\Sigma,\Delta} \leq C_2 \|v\|_{H^2(\Omega)}, \quad v \in V_h.$$
Proof. We call nonnegative quantities \( A(h) \) and \( B(h) \) uniformly equivalent with respect to \( h \) and write \( A(h) \approx B(h) \) if

\[
C_1 B(h) \leq A(h) \leq C_2 B(h).
\]

Our proof consists of establishing the following sequence of equivalence relations:

\[
\| v \|^2_{\Sigma, \Delta} \approx \inf_{v_2(v)} \sum_{k,i} h_{ki}^{-4} \| v_{ki} \|_{L^2(\Omega)}^2 \approx \| v \|^2_{\psi, h} \approx \| v \|_{H^2(\Omega)}^2. 
\]

The last equivalence relation in (3.14) is stated and proved in Lemma 3.4.

Let us prove the first equivalence relation in (3.14). Take any \( v \in V_h \) and consider a representation \( \{ v_{ki} \} \in V_2(v) \). Using (2.10), (3.3), (3.2), and (2.9) with \( |\alpha| = 2 \) and \( \alpha = (0, 0) \), we obtain

\[
C_1 \| \Delta v_{ki} \|_{L^2(\Omega)}^2 \leq h_{ki}^{-4} \| v_{ki} \|_{L^2(\Omega)}^2 \leq C_2 \| \Delta v_{ki} \|_{L^2(\Omega)}^2.
\]

Summing these inequalities with respect to \( k \) and \( i \) and taking the infimum over \( V_2(v) \), we obtain the first equivalence relation in (3.14).

We now prove the second equivalence relation in (3.14):

\[
C_1 \inf_{V_2(v)} \sum_{k,i} h_{ki}^{-4} \| v_{ki} \|_{L^2(\Omega)}^2 \leq \| v \|_{\psi, h}^2 \leq C_2 \inf_{V_2(v)} \sum_{k,i} h_{ki}^{-4} \| v_{ki} \|_{L^2(\Omega)}^2, \quad v \in V_h.
\]

Take any \( v \in V_h \). Using uniqueness of the representation

\[
v_k = \sum_{i=1}^{N_k} v_{ki}, \quad v_{ki} \in V_{ki}, \quad 1 \leq i \leq N_k,
\]

for \( v_k \in V_k \), we define injection mappings \( V_1(v) \to V_2(v) \) and \( V_2(v) \to V_1(v) \) by

\[
\sum_k v_k = \sum_{ki} v_{ki}.
\]

The Schroeder–Bernstein theorem implies that there is a bijection \( V_1(v) \to V_2(v) \).

Consider any representation \( \{ v_k \} \in V_1(v) \) and the representation \( \{ v_{ki} \} \in V_2(v) \) given by the bijection \( V_1(v) \to V_2(v) \). Let \( c_{ki} \) be such that \( v_{ki} = c_{ki} v_k^h \). Using (2.9) with \( \alpha = (0, 0) \), we have

\[
C_1 c_{ki}^2 h_k^2 \leq \| v_{ki} \|_{L^2(\Omega)}^2 \leq C_2 c_{ki}^2 h_k^2.
\]

Summing the last inequalities with respect to \( i \) and using (3.6), we obtain

\[
C_1 \sum_{i=1}^{N_k} \| v_{ki} \|_{L^2(\Omega)}^2 \leq \| v_k \|_{L^2(\Omega)}^2 \leq C_2 \sum_{i=1}^{N_k} \| v_{ki} \|_{L^2(\Omega)}^2.
\]

Multiplying (3.16) by \( h_{ki}^{-4} \), summing for \( k = 0, 1, \ldots, K \), taking the infimum over \( V_1(v) \), and using (3.7), we obtain

\[
C_1 \inf_{V_1(v)} \sum_{k,i} h_{ki}^{-4} \| v_{ki} \|_{L^2(\Omega)}^2 \leq \| v \|_{\psi, h}^2 \leq C_2 \inf_{V_1(v)} \sum_{k,i} h_{ki}^{-4} \| v_{ki} \|_{L^2(\Omega)}^2.
\]

Since there is a bijection \( V_1(v) \to V_2(v) \), the infimum over \( V_1(v) \) is equal to the infimum over \( V_2(v) \); hence, (3.17) implies (3.15). \( \Box \)
4. Uniform spectral equivalence. In this section, we define additive and multiplicative OSC preconditioners and prove that they are uniformly spectrally equivalent to the OSC operator \( L_h^* L_h \).

Additive preconditioner. For \( 0 \leq k \leq K \) and \( 1 \leq i \leq N_k \), let \( T_i^k \) be a linear operator from \( V_h \) into \( V_{ki} \) defined as follows: for any \( w \in V_h \), \( T_i^k w \) satisfies

\[
 a_h(T_i^k w, v) = a_h(w, v) \quad \text{for all} \quad v \in V_{ki}.
\]

The following is our main result for the additive preconditioner.

**Theorem 4.1.** Assume that \( h \) is sufficiently small. Linear operator

\[
 T_A = \sum_{k,i} T_i^k
\]

is self-adjoint positive definite on \( V_h \) in the inner product \( a_h(\cdot, \cdot) \). Linear operator

\[
 B_A = L_h^* L_h T_A^{-1}
\]

is self-adjoint positive definite on \( V_h \) in the inner product \( (\cdot, \cdot)_h \), and

\[
 C_1(B_A v, v)_h \leq (L_h^* L_h v, v)_h \leq C_2(B_A v, v)_h, \quad v \in V_h.
\]

**Proof.** Let

\[
 \|v\|_{\Sigma,a_h}^2 = \inf_{\nu_2(v)} \sum_{k,i} a_h(v_{ki}, v_{ki}), \quad v \in V_h.
\]

First, let us prove the equivalence relation

\[
 C_1 a_h(v, v) \leq \|v\|_{\Sigma,a_h}^2 \leq C_2 a_h(v, v), \quad v \in V_h.
\]

Using (3.4), (3.12), and (4.5), we obtain inequalities

\[
 C_1 \|v\|_{\Sigma,\Delta} \leq \|v\|_{\Sigma,a_h} \leq C_2 \|v\|_{\Sigma,\Delta}, \quad v \in V_h,
\]

which, by Lemma 3.5, imply

\[
 C_1 \|v\|_{H^2(\Omega)} \leq \|v\|_{\Sigma,a_h} \leq C_2 \|v\|_{H^2(\Omega)}, \quad v \in V_h.
\]

The last inequalities and Lemma 3.1 imply (4.6). We note that the second inequality in (4.6) is one of the key assumptions in the abstract theory of Schwarz methods (see Assumption 1 in section 5.2 of [36]).

Using (4.2), (4.1), the second inequality in (4.6), and (3.1), we obtain

\[
 a_h(T_A v, v) \geq C a_h(v, v) \geq C \|v\|_{H^2(\Omega)}^2, \quad v \in V_h
\]

(see Theorem 1 in [17]). Therefore, operator \( T_A \) is positive definite. Operators \( T_i^k \) are self-adjoint since \( a_h(\cdot, \cdot) \) is a symmetric bilinear form; hence, \( T_A \) is self-adjoint (see Lemma 2 in section 5.2 of [36]). Thus, operator \( T_A^{-1} \) is self-adjoint positive definite. It follows from (4.3), (2.7), and Lemma 1 in section 5.2 of [36] that

\[
 (B_A v, v)_h = a_h(T_A^{-1} v, v) = \|v\|_{\Sigma,a_h}^2, \quad v \in V_h.
\]

The last relation, along with (4.6) and (2.7), gives (4.4).
Multiplicative preconditioner. Let
\begin{equation}
T_M = I_h - \left[ \prod_{k=K}^{0} \prod_{i=1}^{N_k} (I_h - T_i^k) \right] \left[ \prod_{k=0}^{K} \prod_{i=N_k}^{1} (I_h - T_i^k) \right],
\end{equation}
where \(I_h\) is the identity operator on \(V_h\).

**Theorem 4.2.** Assume that \(h\) is sufficiently small. Linear operator \(T_M\) is self-adjoint positive definite on \(V_h\) in the inner product \(a_h(\cdot, \cdot)\). Linear operator
\begin{equation}
B_M = L_h^* L_h T_M^{-1}
\end{equation}
is self-adjoint positive definite on \(V_h\) in the inner product \((\cdot, \cdot)\), and
\begin{equation}
C_1(B_M v, v)_h \leq (L_h^* L_h v, v)_h \leq C_2(B_M v, v)_h, \quad v \in V_h.
\end{equation}

**Proof.** Since operators \(\{T_i^k\}\) are self-adjoint on \(V_h\) with respect to the inner product \(a_h(\cdot, \cdot)\), it is easy to see that \(T_M\) is also a self-adjoint operator. Hence, \(B_M\) is self-adjoint in the inner product \((\cdot, \cdot)_h\). Inequalities in (4.9) follow from Lemma 4 in section 5.2 of [36] with \(\omega = 1\), the second inequality in (4.6), and Lemma 6.1 in [40] formulated for \(\{V_k\}\).

**Remark.** Using the multigrid terminology, we note that the multiplicative preconditioner \(B_M\) corresponds to the V-cycle multigrid algorithm with the Gauss–Seidel smoother.

**Iterative method.** Since the operator of (2.6) is self-adjoint positive definite, we can use our multilevel preconditioners with the PCG algorithm to compute the OSC solution (see Algorithm 9.4.14 in [20]).

Let \(\lambda_{\text{min},h}\) and \(\lambda_{\text{max},h}\) be, respectively, the smallest and the largest eigenvalues of the preconditioned operator
\[
\tilde{A}_h = M_h^{-1/2} L_h^* L_h M_h^{-1/2},
\]
where \(M_h\) is a preconditioner. It is well known that the convergence rate of the PCG is bounded from above by
\[
(\sqrt{\kappa_h} - 1) / (\sqrt{\kappa_h} + 1),
\]
where \(\kappa_h = \lambda_{\text{max},h}/\lambda_{\text{min},h}\) is the spectral condition number of \(\tilde{A}_h\) (see Theorem 9.4.14 in [20]). It follows from Theorems 4.1 and 4.2 that
\begin{equation}
\kappa_h \leq C_2/C_1 < \infty \quad \text{as} \quad h \to \infty.
\end{equation}
The estimate (4.10) implies that it takes \(O(\ln \varepsilon)\) iterations of the PCG algorithm with the multilevel OSC preconditioners to approximate the solution of (2.6) with tolerance \(\varepsilon\); that is, the number of iterations is bounded by a constant independent of \(h\) and \(K\).

**5. OSC matrix-vector representation.** In this section, we introduce the matrix-vector representation of the OSC problem in the standard Hermite finite element basis and obtain recurrence relations for the computation of the OSC approximations and other required quantities.

**Representation of Hermite piecewise cubic polynomials.** Let \(V_k^1\) denote a vector space of Hermite piecewise cubic polynomials vanishing at \(t = 0\) and \(t = 1\) and
corresponding to the partition \( \{ t_i^k = i/2^k \}_{i=0}^{2^k} \) of the interval \([0, 1]\). The dimension of \( V_k^1 \) is \( M_k = 2^{k+1} \). Let
\[
\Phi_k = \{ s_{0}^k, s_{1}^k, s_{2}^k, \ldots, s_{2^{k-1}}^k, s_{2^k}^k \} = \{ \phi_i \}_{i=1}^{M_k}
\]
be the standard basis of \( V_k^1 \) consisting of nodal value and nodal slope basis functions \( v_i(t) \) and \( s_i(t) \), respectively, defined for \( 0 \leq i \leq 2^k \) by
\[
v_i^k(t) = \delta_{ij}, \quad (v_i^k)'(t_j^k) = 0, \quad 0 \leq j \leq 2^k,
\]
\[
s_i^k(t_j^k) = 0, \quad (s_i^k)'(t_j^k) = h_k^{-1} \delta_{ij}, \quad 0 \leq j \leq 2^k,
\]
where \( \delta_{ij} \) is the Kronecker delta. We note that the basis functions \( v_i^k \) and \( v_i^k \) corresponding to the boundary points \( t = 0 \) and \( t = 1 \) are not included in \( \Phi_k \).

The value and the slope basis functions in \( \Phi_k \) corresponding to an interior partition node are uniquely expressed as a linear combination of five basis functions in \( \Phi_{k+1} \) as follows:
\[
v_i^k = \frac{1}{2} v_{2i-1}^k + \frac{3}{2} v_{2i}^k + \frac{1}{4} v_{2i+1}^k - \frac{3}{4} v_{2i+2}^k,
\]
\[
s_i^k = -\frac{1}{8} v_{2i-1}^k - \frac{1}{8} v_{2i}^k + \frac{1}{2} v_{2i+1}^k + \frac{1}{8} v_{2i+2}^k - \frac{1}{8} v_{2i+3}^k.
\]
Let
\[
P = \begin{pmatrix} 4 & -1 \\ 6 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 8 & 0 \\ 0 & 4 \end{pmatrix}, \quad \text{and} \quad R = \begin{pmatrix} 4 & 1 \\ -6 & 1 \end{pmatrix}
\]
be matrices corresponding to the representation (5.3). Let \( P_k^1 \) be a \( 2M_k \times M_k \) matrix obtained from
\[
\frac{1}{8} \begin{pmatrix} Q & O & O & \ldots & O \\ R & P & O & \ldots & \ldots \\ O & Q & P & \ldots & \ldots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ O & \ldots & O & Q \end{pmatrix} \in \mathbb{R}^{(2M_k+2) \times (M_k+2)},
\]
where \( O \) is the \( 2 \times 2 \) zero matrix, by removing the first and next-to-last rows and columns. For \( v \in V_k^1 \), let \( [v]_{\kappa,k} \) denote the vector representation of \( v \) in the basis \( \Phi_k \).

It follows from (5.3), (5.4), and (5.5) that
\[
[v]_{\kappa,k+1} = P_k^1[v]_{\kappa,k}, \quad v \in V_k^1, \quad k \in \{0, 1, 2, \ldots, K-1\}.
\]

**Representation of Hermite piecewise bicubic polynomials.** We note that \( V_k = V_k^1 \otimes V_k^1 \), where the symbol \( \otimes \) denotes a vector space tensor product. Set
\[
(5.7) \quad \psi_{M_k(i-1)+j}^k(x) = \phi_i(x_1) \phi_j(x_2) \quad \text{for} \quad 1 \leq i, j \leq M_k,
\]
where basis functions \( \phi_i \), for \( 1 \leq i \leq M_k \), are defined by (5.1) and (5.2). The set \( \Psi_k = \{ \psi_{M_k(i-1)+j}^k \}_{i,j=1}^{N_k} \), where \( N_k = M_k^2 = 4^{k+1} \), is the standard basis for \( V_k \) in the standard ordering.

It follows from (5.3) and (5.7) that a basis function in \( \Psi_k \) corresponding to an interior partition node is uniquely expressed as a linear combination of 25 basis functions in \( \Psi_{k+1} \). Let \([v]_{\kappa,k} \) denote the vector representation of \( v \in V_k \) in the basis \( \Psi_k \), and let \([v]_{\kappa} = [v]_{\kappa,K} \) for \( v \in V_k \). It is obvious that
\[
(5.8) \quad [\psi_{M_k(i-1)+j}^k]_{\kappa,k} = \epsilon_j^k, \quad 1 \leq j \leq N_k,
\]
where \( \vec{e}_j^k \) is the \( j \)th standard basis vector in \( \mathbb{R}^{N_k} \).

We set

\[
P_k = P^1_k \otimes P^1_k \in \mathbb{R}^{2N_k \times N_k},
\]

where \( P^1_k \) is the one-dimensional interpolation matrix in (5.6) and the symbol \( \otimes \) now denotes the matrix tensor product. Matrix \( P_k \) corresponds to the piecewise bicubic Hermite interpolation in \( V_{k+1} \), and we call \( P_k \) the interpolation matrix from level \( k \) to level \( k+1 \). It follows from (5.6), (5.7), and (5.9) that

\[
[v]_{\mathcal{H},k+1} = P_k[v]_{\mathcal{H},k} \quad \text{for} \quad 0 \leq k \leq K - 1 \quad \text{and} \quad v \in V_k.
\]

Applying formula (5.10) recurrently, we obtain

\[
[v]_{\mathcal{H}} = P_{K-1} \cdots P_1[v]_{\mathcal{H},0}, \quad v \in V_0.
\]

In particular, replacing \( v \) in (5.11) by \( \psi_j^k \) and using (5.8), we get

\[
[\psi_j^k]_{\mathcal{H}} = P_{K-1} \cdots P_1 \vec{e}_j^k, \quad 1 \leq j \leq N_k.
\]

**Matrix-vector form of the OSC problem.** Assume that the set of Gauss points \( \mathcal{G}_h = \{ \xi_i \}_{i=1}^N \) is ordered, and let

\[
[v]_{\mathcal{G}} = [v(\xi_1), \ldots, v(\xi_N)]^t
\]

for any function \( v \) defined on \( \mathcal{G}_h \). From (2.5), we have

\[
(v, w)_h = (h^2/4)[w]_{\mathcal{G}}^t[v]_{\mathcal{G}}, \quad v, w \in V_h.
\]

Let \( [L_h] \) be the OSC matrix of size \( N \times N \), corresponding to the differential operator \( L \) in (1.2), with entries \( L\psi_j^K(\xi_i) \), where \( i \) is the row index. For a continuous function \( g(x) \), let

\[
D(g) = \text{diag}(g(\xi_1), \ldots, g(\xi_N))
\]

be a diagonal matrix. We note that

\[
[L_h] = D(a_{11})(\hat{A} \otimes \hat{B}) + 2D(a_{12})(\hat{C} \otimes \hat{C}) + D(a_{22})(\hat{B} \otimes \hat{A}) + D(b_1)(\hat{C} \otimes \hat{B}) + D(b_2)(\hat{B} \otimes \hat{C}) + D(c)(\hat{B} \otimes \hat{B}),
\]

and the matrices \( \hat{A}, \hat{C}, \) and \( \hat{B} \) have the following almost block diagonal structure:

\[
\begin{pmatrix}
\tilde{W} & Z & O & O & \cdots & O & \tilde{O} \\
\tilde{O} & W & Z & O & \cdots & O & \tilde{O} \\
\tilde{O} & O & W & Z & \cdots & O & \tilde{O} \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\
\tilde{O} & O & \cdots & W & \tilde{Z}
\end{pmatrix} \in \mathbb{R}^{M_K \times M_K},
\]

where \( W, Z, O \) and \( \tilde{W}, \tilde{Z}, \tilde{O} \) are, respectively, \( 2 \times 2 \) and \( 2 \times 1 \) blocks; \( O \) and \( \tilde{O} \) are zero matrices. The \((i, j)\) entries of matrices \( \hat{A}, \hat{C}, \) and \( \hat{B} \) are \( \phi_i''(\eta_j), \phi_i'(\eta_j), \) and \( \phi_i(\eta_j) \), respectively, where \( \{ \eta_j \}_{j=1}^{M_K} \) is the set of Gauss points in interval \([0,1]\) corresponding to partition \( \pi_h \).
It is easy to see that
\begin{equation}
[L_h v]_\varnothing = [L_h][v]_N, \quad v \in V_h.
\end{equation}

Using (2.7), (5.13), and (5.14), we obtain, for any \(v\) and \(w\) in \(V_h\),
\begin{equation}
(a_h(v, w) = (L_h v, L_h w)_h = (h^2/4)[L_h w]_\varnothing[L_h v]_\varnothing = (h^2/4)[w]_N[L_h]_t[L_h][v]_N.
\end{equation}

Let \(A_k = (a_{ij}^k)\) be an \(N_k \times N_k\) matrix with entries
\begin{equation}
a_{ij}^k = (4/h^2)a_h(\psi_j^k, \psi_i^k),
\end{equation}
and let \(A = A_K\). From (5.16), using (5.15), we get
\begin{equation}
a_{ij}^k = [\psi_j^k]^t[L_h]^t[L_h][\psi_i^k]_N.
\end{equation}

From (5.17) for \(k = K\), using (5.8), we obtain
\begin{equation}
A = [L_h]^t[L_h].
\end{equation}

Similarly, using (5.13), (5.14), the relation \([f]_\varnothing = [f_h]_\varnothing\), and (5.8), we obtain
\begin{equation}
(4/h^2)(L_h f_h, \psi_i^K)_h = (4/h^2)(f_h, L_h \psi_i^K)_h = [f_h]_\varnothing[L_h \psi_i^K]_\varnothing = [f_h]_\varnothing^t[L_h]^t[L_h][\psi_i^K]_N = ([L_h]^t[L_h][f]_\varnothing)^t.
\end{equation}

Thus, the variational OSC equation (2.8) has the matrix-vector form
\begin{equation}
A[u_h]_N = [L_h]^t[f]_\varnothing.
\end{equation}

6. Implementation. In this section, we describe implementations of both the additive and the multiplicative OSC preconditioners.

**Additive preconditioner.** Let us describe the computation of \(w = B_A^{-1}v\) for \(v \in V_h\), where \(B_A\) is defined by (4.3). Let
\begin{equation}
w_k = \sum_{i=1}^{N_k} w_{ki}, \quad \text{where} \quad w_{ki} = T_i^k(L_h^* L_h)^{-1}v.
\end{equation}

Using (4.3), (4.2), and (6.1), we get
\begin{equation}
w = T_A(L_h^* L_h)^{-1}v = \left(\sum_{k,i} T_i^k\right)(L_h^* L_h)^{-1}v = \sum_{k,i} w_{ki} = \sum_k w_k.
\end{equation}

Thus, to compute \(w\), we need to compute and sum \(w_k\) for \(k = 0, 1, \ldots, K\).

Using (4.1) and (2.7), we obtain, for \(1 \leq i \leq N_k\),
\begin{equation}
(a_h(w_{ki}, \psi_i^k) = a_h(T_i^k(L_h^* L_h)^{-1}v, \psi_i^k) = a_h((L_h^* L_h)^{-1}v, \psi_i^k) = (v, \psi_i^k)_h.
\end{equation}

Let
\begin{equation}
w_{ki} = c_{ki}\psi_i^k, \quad c_{ki} \in R,
\end{equation}
\begin{equation}
[v]_k = (4/h^2)(v, \psi_1^k)_h, \ldots, (v, \psi_{N_k}^k)_h \in R^{N_k}.
\end{equation}
Substituting (6.4) into (6.3) and using (6.5) and (5.16), we rewrite (6.3) in the form
\begin{equation}
\text{diag}(A_k)\tilde{w}_k = [v]_k,
\end{equation}
where $\tilde{w}_k = (c_{k1}, \ldots, c_{kN_k})^t = [w_k]_{\gamma, k}$ and $\text{diag}(A_k) = \text{diag}(a_{11}, \ldots, a_{N_k N_k})$.

Let $[I_h]$ be an $N \times N$ matrix with entries $\psi^K_j(\xi)$, where $i$ is the row index. Matrix $[I_h]$ is nonsingular and maps $[v]_\gamma$ to $[v]_\varphi$, that is,
\begin{equation}
[v]_\varphi = [I_h][v]_\gamma, \quad v \in V_h.
\end{equation}

For $[v]_k$ defined by (6.5), let us show
\begin{equation}
[v]_k = P_k^t[v]_{k+1} \text{ for } k = K - 1, K - 2, \ldots, 0,
\end{equation}
where the interpolation matrix $P_k$ is defined by (5.9) and (5.5). Using (6.5), (5.13), and (5.18), we obtain
\begin{equation}
(\vec{e}_k)^t [v]_k = \left(\frac{4}{h^2}\right)[v]_\gamma = [\psi_j^K]^t [v]_\varphi = [\psi_j^K]^t [I_h]^t [v]_\varphi, \quad 1 \leq j \leq N_k.
\end{equation}
Relation (6.8) follows from (6.10) with $k = K$ and (5.8). Using (6.10), (5.12), and (6.8), we obtain
\begin{equation}
[v]_k = P_k^t \cdots P_{K-1}^t [v]_K,
\end{equation}
which implies (6.9).

The multiplication by $P_k^t$ is carried out using the representation
\begin{equation}
P_k^t = ((P_k^t)^t \otimes I_k)(I_k \otimes (P_k^t)^t),
\end{equation}
where $I_k$ is the identity matrix of size $M_k \times M_k$. Matrices $\{A_k\}$ can be precomputed using the recurrence formula
\begin{equation}
A_k = P_k^t A_{k+1} P_k \text{ for } k = K - 1, K - 2, \ldots, 0,
\end{equation}
which follows from (5.17), (5.12), and (5.18). Finally, to compute $w = \sum_k w_k$, that is, $[w]_\gamma$, we implement
\begin{equation}
\tilde{w}_{k+1} \leftarrow \tilde{w}_{k+1} + P_k \tilde{w}_k, \quad k = 0, 1, \ldots, K - 1.
\end{equation}

The additive preconditioning algorithm is presented in Figure 6.1. It is easy to see that the computational cost of the additive algorithm is $O(h^{-2}) = O(4^K)$.

**Multiplicative preconditioner.** We now consider the computation of $w = B_M^{-1}v$ for $v \in V_h$, where $B_M$ is defined in (4.8). Let
\begin{equation}
u = (L_h^* L_h)^{-1} v.
\end{equation}
Using (4.8) and (6.12), we get
\begin{equation}
w = B_M^{-1} v = T_M (L_h^* L_h)^{-1} v = T_M u,
\end{equation}
which, by (4.7), implies
\begin{equation}
u - w = \left[ \prod_{k=K}^0 \prod_{i=1}^{N_k} (I_h - T_i^k) \right] \left[ \prod_{k=0}^{K} \prod_{i=N_k}^{1} (I_h - T_i^k) \right] u.
\end{equation}
**Fig. 6.1. Additive preconditioning algorithm.**

```
input: K, [v]_K, \{\text{diag}(A_k)\}_{k=0}^K
output: [w]_\infty
\bar{v}_K \leftarrow [v]_K
for k = K, K - 1, \ldots, 0
    if (k < K) \bar{v}_k = P_k \bar{v}_{k+1}
    solve diag(A_k) \bar{w}_k = \bar{v}_k
end
for k = 0, 1, \ldots, K - 1
    \bar{w}_{k+1} \leftarrow \bar{w}_{k+1} + P_k \bar{w}_k
end
[w]_\infty \leftarrow \bar{w}_K
```

Let S be an ordered set of pairs (k, i) with the ordering corresponding to that of factors in (6.13) from right to left. Setting \(y = u - w\), we see that \(u - w\) can be computed by

\[
y \leftarrow u; \quad y \leftarrow (I_h - T_i^k) y \quad \text{for} \quad (k, i) \in S,
\]

which is equivalent to

\[
w \leftarrow 0; \quad w \leftarrow w + T_i^k(u - w) \quad \text{for} \quad (k, i) \in S.
\]

Let us develop an efficient implementation of the algorithm in (6.14). Using (4.1), (2.7), and (6.12), we obtain

\[
a_h(T_i^k(u - w), \psi_i^k) = a_h(u - w, \psi_i^k) = (v, \psi_i^k)_h - a_h(w, \psi_i^k), \quad \psi_i^k \in V_{ki}.
\]

Substituting \(T_i^k(u - w) = c_{ki} \psi_i^k\) into (6.15) and (6.14), we get

\[
c_{ki} = g_{ki}^k(w)/a_{ii}^k,
\]

\[
w \leftarrow w + c_{ki} \psi_i^k,
\]

where \(a_{ii}^k\) is as defined in (5.16), and

\[
g_{ki}^k(w) = \left(\frac{4}{h^2}\right) [(v, \psi_i^k)_h - a_h(w, \psi_i^k)], \quad 1 \leq i \leq N_k.
\]

Let \(\bar{g}_k(w) = (g_{k1}(w), \ldots, g_{KN_k}(w))^t\). We note that, each time the value of w is changed by (6.17), all entries of vector \(\bar{g}_k(w)\) should be updated by (6.18), and such computation requires a matrix-vector product. It is more efficient to use a recurrent assignment

\[
\bar{g}_k(w) \leftarrow \bar{g}_k(w) - c_{ki}(a_{ii}^k, \ldots, a_{N_k,i}^k)^t,
\]

which is obtained by multiplying (6.17) by \(\psi_i^k\) in the \(a_h(\cdot, \cdot)\) inner product and subtracting \((v, \psi_i^k)_h\) from both sides of the resulting assignment.

For \(k = K - 1, K - 2, \ldots, 0\), let \(w_k\) be the value of w after implementing

\[
w \leftarrow w + T_i^l(u - w), \quad i = 1, \ldots, N_i, \quad l = K, K - 1, \ldots, k + 1,
\]
and let
\begin{equation}
\bar{g}_k = \bar{g}_k(w_k). 
\end{equation}
We note that (6.16) followed by (6.19) for \( i = 1, \ldots, N_k \) is the column-oriented algorithm of solving a lower triangular linear system \( L_k \bar{w}_k = \bar{g}_k \), where matrix \( L_k \) contains the lower triangular part of \( A_k \) and \( \bar{w}_k = (c_{k1}, \ldots, c_{kN_k})^t \).

Let us show that vectors \( \{\bar{g}_k\} \) can be computed by the recurrence formula
\begin{equation}
\bar{g}_{k-1} = P_{k-1}^t (\bar{g}_k - A_k \bar{w}_k). 
\end{equation}
Using (6.19), the definition of \( \bar{w}_{k-1} \), and (6.20), we obtain
\begin{equation}
\bar{g}_k(w_{k-1}) = \bar{g}_k - A_k \bar{w}_k. 
\end{equation}
Applying (6.18), (6.5), (5.15), (5.12), we get
\begin{equation}
\bar{g}_k(w) = [v]_k - P_k^t \cdots P_{K-1}^t [L_h]^t [L_h][w]_H. 
\end{equation}
From (6.23) with \( k \) replaced by \( k - 1 \) and (6.9), we have
\[ \bar{g}_{k-1}(w) = P_{k-1}^t ([v]_k - P_k^t \cdots P_{K-1}^t [L_h]^t [L_h][w]_H) = P_{k-1}^t \bar{g}_k(w), \]
which, by (6.22), implies (6.21).

In the ascend phase, for \( k = 0, \ldots, K \), an upper triangular linear system with the matrix \( L_k^t \) is solved. The algorithm implementing the multiplicative preconditioning is given in Figure 6.2. It is easy to see that the cost of the multiplicative algorithm is \( O(h^{-2}) = O(4^K) \).

**7. Numerical results.** In this section, we present numerical results for solving test problems by the PCG method with the multilevel OSC preconditioners developed in this work. For a chosen exact solution \( u(x) \) of BVP (1.1), we set \( f = Lu \). PCG iterations are stopped when the relative residual norm, that is, the ratio of the 2-norm of the residual of (5.19) to the 2-norm of the right-hand side, becomes less than tolerance \( \epsilon \).
Table 7.1
Comparison of multilevel preconditioners to solve the normal and the original OSC equations by PCG ($L = \Delta$, $\epsilon = 10^{-12}$).

<table>
<thead>
<tr>
<th>(h)</th>
<th>(\kappa_h) Additive</th>
<th>(\kappa_h) Iter.</th>
<th>(\kappa_h)</th>
<th>(\kappa_h) Iter.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/16</td>
<td>4.490</td>
<td>26</td>
<td>4.490</td>
<td>26</td>
</tr>
<tr>
<td>1/32</td>
<td>5.016</td>
<td>29</td>
<td>4.490</td>
<td>26</td>
</tr>
<tr>
<td>1/64</td>
<td>5.488</td>
<td>32</td>
<td>4.516</td>
<td>29</td>
</tr>
<tr>
<td>1/128</td>
<td>5.845</td>
<td>34</td>
<td>4.548</td>
<td>32</td>
</tr>
<tr>
<td>1/256</td>
<td>6.162</td>
<td>35</td>
<td>4.584</td>
<td>34</td>
</tr>
</tbody>
</table>

The spectral condition number \(\kappa_h\) of the preconditioned OSC operator satisfies (4.10), that is, \(\kappa_h < C_2/C_1\) as \(h \to 0\). To demonstrate this fact numerically, we compute quantities that approximate \(\kappa_h\) on a sequence of nested partitions using the PCG iteration parameters and present these approximations in the following tables under \(\kappa_h\). Under "Iter.", we report the numbers of iterations to reduce the relative residual norm within the specified tolerance.

In the first set of experiments, we compare our preconditioners with those proposed in [9], where both additive and multiplicative multilevel preconditioners were developed to solve the original OSC equation (2.3) for a self-adjoint \(L\). As in [9], we take \(L = \Delta\), the Laplacian, and

\[
 u(x) = 10x_1^2(1-x_1)x_2^3(1-x_2).
\]

Since \(u(x)\) is a bicubic polynomial, it is also the solution of the OSC problem; hence, the discretization error is zero. The numerical results are presented in Table 7.1, and they indicate that, for \(L = \Delta\), PCG with multilevel preconditioners for the original OSC equation is slightly more efficient than that for the normal OSC equation. For smaller values of \(h\), the approximations to \(\kappa_h\) and the numbers of iterations are relatively small and change insignificantly.

In the next set of experiments, we solve the same problems as in [1], where the PCG algorithm was tested with a direct solver preconditioner. The operator \(L\) in (1.2) is taken with the coefficients

\[
 a_{11}(x) = e^{x_1x_2}, \quad b_1(x) = x_2e^{x_1x_2} + \beta_1 \cos[\pi(x_1 + x_2)], \\
 a_{12}(x) = \alpha/(1 + x_1 + x_2), \quad b_2(x) = -x_1e^{-x_1x_2} + \beta_2 \sin(2\pi x_1 x_2), \\
 a_{22}(x) = e^{-x_1x_2}, \quad c(x) = \gamma[1 + 1/(1 + x_1 + x_2)],
\]

where \(\alpha, \beta_1, \beta_2,\) and \(\gamma\) are parameters, and the exact solution of BVP (1.1) is set to

\[
 u(x) = e^{x_1x_2}x_1x_2(1-x_1)(1-x_2).
\]

Using the PCG with the multiplicative preconditioner, we solve the following four test problems corresponding to the differential operator \(L\), which is defined in each problem:

- **P1** – self-adjoint negative definite, \(\alpha = \beta_1 = \beta_2 = \gamma = 0\).
- **P2** – self-adjoint indefinite, \(\alpha = \beta_1 = \beta_2 = 0\) and \(\gamma = 100\).
- **P3** – non-self-adjoint indefinite, \(\beta_2 = 100\) and \(\alpha = \beta_1 = \gamma = 0\).
- **P4** – non-self-adjoint indefinite, \(\alpha = 0.5, \beta_1 = 10, \beta_2 = \gamma = 50\).
### Table 7.2

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\kappa_h$</th>
<th>Iter.</th>
<th>$\kappa_h$</th>
<th>Iter.</th>
<th>$\kappa_h$</th>
<th>Iter.</th>
<th>$\kappa_h$</th>
<th>Iter.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/16</td>
<td>2.570</td>
<td>15 (26)</td>
<td>363.0</td>
<td>68 (46)</td>
<td>624.5</td>
<td>43 (34)</td>
<td>516.1</td>
<td>51 (68)</td>
</tr>
<tr>
<td>1/32</td>
<td>3.646</td>
<td>17 (30)</td>
<td>363.2</td>
<td>70 (51)</td>
<td>499.8</td>
<td>42 (38)</td>
<td>457.8</td>
<td>61 (75)</td>
</tr>
<tr>
<td>1/64</td>
<td>4.922</td>
<td>20 (33)</td>
<td>361.6</td>
<td>71 (54)</td>
<td>459.3</td>
<td>46 (40)</td>
<td>402.9</td>
<td>56 (81)</td>
</tr>
<tr>
<td>1/128</td>
<td>6.189</td>
<td>23 (34)</td>
<td>361.5</td>
<td>72 (55)</td>
<td>398.6</td>
<td>52 (42)</td>
<td>382.0</td>
<td>58 (84)</td>
</tr>
<tr>
<td>1/256</td>
<td>7.253</td>
<td>25</td>
<td>360.3</td>
<td>73</td>
<td>376.0</td>
<td>54</td>
<td>377.8</td>
<td>59</td>
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<tr>
<td>1/16</td>
<td>30.3</td>
<td>43</td>
<td>5835.0</td>
<td>103</td>
<td>9764.6</td>
<td>71</td>
<td>6961.2</td>
<td>67</td>
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<tr>
<td>1/32</td>
<td>50.9</td>
<td>58</td>
<td>5502.3</td>
<td>117</td>
<td>7396.7</td>
<td>79</td>
<td>5999.5</td>
<td>83</td>
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<td>1/64</td>
<td>75.6</td>
<td>70</td>
<td>5801.5</td>
<td>128</td>
<td>6562.9</td>
<td>98</td>
<td>5131.5</td>
<td>104</td>
</tr>
<tr>
<td>1/128</td>
<td>99.9</td>
<td>81</td>
<td>5803.4</td>
<td>140</td>
<td>5414.9</td>
<td>122</td>
<td>5008.0</td>
<td>121</td>
</tr>
<tr>
<td>1/256</td>
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<td>89</td>
<td>5805.2</td>
<td>147</td>
<td>5025.1</td>
<td>139</td>
<td>5213.8</td>
<td>137</td>
</tr>
</tbody>
</table>

In the top part of Table 7.2, we report results with the multiplicative preconditioner, and, in parentheses, we reproduce results reported in [1].

For all four problems, the numbers of iterations increase slowly with decreasing $h$. It takes the least number of iterations to solve the self-adjoint negative definite problem P1, and the most number of iterations to solve P2, the “most indefinite” problem of P1–P4. For P1, approximations to the spectral ratio $\kappa_h$ are much smaller than those for P2–P4, where the approximations to $\kappa_h$ are about the same for smaller values of $h$. It is interesting to note that $\kappa_h$ monotonically increases for P1 and decreases for P2–P4 as $h$ decreases. We observe that the approximations to $\kappa_h$ for P2–P4 are significantly larger than those for P1. The numbers of iterations for P1 and P4 favor the multilevel multiplicative preconditioner rather than the direct solver preconditioner developed in [1], although the preconditioner in [1] produces smaller numbers of iterations for P2 and P3.

Results for the additive preconditioner are presented in the bottom part of Table 7.2, and they suggest that, for the tested problems, the additive preconditioner is less efficient, in terms of numbers of iterations, than the multiplicative preconditioner. The approximations of $\kappa_h$ computed with the additive preconditioner are approximately 15 times larger than those computed with the multiplicative preconditioner, and the indicated difference is reflected by a larger number of iterations for the additive preconditioner. This result is intuitively expected based on known properties of Jacobi and Gauss–Seidel smoothers for finite difference operators.

In Figure 7.1, we display plots of residual curves for $h = 1/256$. We observe monotone convergence only for the self-adjoint negative definite problem P1; for P2–P4, the residual curves are plotted relatively close to each other.

In the last set of experiments, we solve the Helmholtz equation $\Delta u + k^2 u = f$ for several values of $k^2$. Numerical results were obtained using the multiplicative preconditioner, and they are presented in Table 7.3. We see that the approximations to $\kappa_h$ change insignificantly when $h$ is decreased for all taken values of $k^2$. For $k^2 = 1000$, the approximations to $k_h$ are very large; however, for $k^2 = 1400$ they are the smallest. The approximations to $k_h$ are large because of small values of the smallest eigenvalue $\lambda_{\min,h}$ of the preconditioned operator. In Figure 7.2, we plot the eigenvalues of the OSC matrix $[L_h]$ with $h = 1/32$ ($N = 4,096$) for the Helmholtz equation with $k^2 = 1400$. We note that the eigenvalues are widely spread over the complex plane.
Fig. 7.1. Logarithmic plots of the relative residual norm versus iteration number \((h = 1/256)\). Top figure: multiplicative preconditioner. Bottom figure: additive preconditioner.

Table 7.3
Approximations to the spectral condition number \(\kappa_h\) and PCG iteration numbers for the Helmholtz equation \((\epsilon = 10^{-10})\).

<table>
<thead>
<tr>
<th>(h)</th>
<th>(k^2 = 100)</th>
<th>(k^2 = 500)</th>
<th>(k^2 = 1000)</th>
<th>(k^2 = 1400)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\kappa_h)</td>
<td>Iter.</td>
<td>(\kappa_h)</td>
<td>Iter.</td>
</tr>
<tr>
<td>1/16</td>
<td>1468.6</td>
<td>51</td>
<td>1808.6</td>
<td>108</td>
</tr>
<tr>
<td>1/32</td>
<td>1446.7</td>
<td>51</td>
<td>1465.0</td>
<td>117</td>
</tr>
<tr>
<td>1/64</td>
<td>1442.1</td>
<td>51</td>
<td>1437.6</td>
<td>125</td>
</tr>
<tr>
<td>1/128</td>
<td>1441.2</td>
<td>51</td>
<td>1435.0</td>
<td>128</td>
</tr>
<tr>
<td>1/256</td>
<td>1441.0</td>
<td>51</td>
<td>1434.8</td>
<td>144</td>
</tr>
</tbody>
</table>

Fig. 7.2. Eigenvalues of the OSC matrix \([L_h]\) for the Helmholtz equation on the complex plane \((k^2 = 1400, \ h = 1/32)\).
In Figure 7.3, we plot the eigenvalues of the symmetric matrix $[I_h]^t L_h$, which are the eigenvalues of the OSC operator $L_h$. We note that $L_h$ has large numbers of both positive and negative eigenvalues.

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REFERENCES